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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Inverse limits of certain interval mappings as attractors in two dimensions

by

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Abstract. Let p be a continuous piecewise monotonic transitive map of the unit interval into itself such that, for positive iterations, the orbit of every critical point is finite and does not contain critical points. It is proved that for each two-dimensional manifold M, the inverse limit map of p is conjugate to an attractor of some C^1-diffeomorphism of M into itself, which is of class C^∞ outside some finite invariant set and can be chosen from an arbitrary diffeotopy class.

1. Introduction. Consider a topological space X and a continuous map $p: X \to X$, and look upon the pair (p, X) as a dynamical system (d.s.). By a *quasiattractor* of (p, X) we mean a d.s. (p, A), where $A \subseteq X$, such that there exists an open set $U \subseteq X$, called a *domain of attraction*, with $\mathrm{cl}(p(U))$ compact, $\mathrm{cl}(p(U)) \subseteq U$ and $\bigcap_{n=0}^{\infty} p^n(U) = A$. If, in addition, the restriction $p|A$ is transitive then (p, A) is called an *attractor*. We also refer to A as a (quasi) attractor of p.

Now let the space X be compact. Let $\underleftarrow{\lim}(p, X)$ denote the limit of the inverse system $X \overset{p}{\leftarrow} X \overset{p}{\leftarrow} X \overset{p}{\leftarrow} \ldots$, that is, the subspace of the infinite product X^N defined by

$$\underleftarrow{\lim}(p, X) = \{v \in X^N: v_n = p(v_{n+1}) \text{ for } n = 0, 1, 2, \ldots\}.$$

The *inverse limit* of (p, X) is the d.s. $\underleftarrow{\mathrm{Lim}}(p, X) = (\hat{p}, \underleftarrow{\lim}(p, X))$, where the mapping $\hat{p}: \underleftarrow{\lim}(p, X) \to \underleftarrow{\lim}(p, X)$ is given by $(\hat{p}(v))_n = p(v_n) = v_{n-1}$ (here $v_{-1} = p(v_0)$), or simply $\hat{p}(v) = p \circ v$. Note that $\underleftarrow{\lim}(p, X)$ is a compact space and $\hat{p}$ is a homeomorphism.

Now let I denote the interval $[0, 1]$. Given a mapping $p: I \to I$ and a smooth manifold M we search for a homeomorphic embedding $h: \underleftarrow{\lim}(p, I) \to M$ and a diffeomorphism $f: M \to M$ satisfying $h \circ \hat{p} = f \circ h$, with the additional property that $h(\underleftarrow{\lim}(p, I))$ is a quasiattractor of f. This means that $\underleftarrow{\mathrm{Lim}}(p, I)$ is conjugate to a quasiattractor of (f, M). Notice that $h(\underleftarrow{\lim}(p, I))$ will be an attractor if and only if p is transitive because, as it can be easily checked, $\hat{p}$ is transitive if and only if p is.

The study of the problem was started in [3] by M. Misiurewicz, who proved the following.

[3, Theorem A]. *If the map* $p\colon I\to I$ *is given by* $p(t) = 1-|2t-1|$ *then every smooth manifold of dimension at least three admits a* C^∞*-diffeomorphism possessing an attractor conjugate to the inverse limit of* (p, I).

A question arises whether a similar statement is true for dimension 2 or for other mappings of the interval. To this we give a partial affirmative answer.

2. Results. Let $p\colon J\to J$ be a mapping of an interval J. By a *critical point* we mean one that does not possess any neighbourhood in J, on which p is one-to-one. We will say that p is *of finite type* if it satisfies the following conditions:

(i) the number of the critical points is finite (that is, p is piecewise monotonic);

(ii) the orbit of each critical point is finite;

(iii) if a and b are critical points and $p^n(a) = b$ then $n = 0$.

Recall that a *homterval* is proper interval $H\subseteq J$ such that $p^n|H$ is a homeomorphic embedding for each n. Our main result is the following.

Theorem A. *Let* $p\colon I\to I$ *be a mapping of finite type having no homtervals and let* M *be a two-dimensional manifold. Then, in every diffeotopy class there exists a* C^1 *diffeomorphism* $f\colon M\to M$ *such that*:

(a) *the inverse limit of* (p, I) *is conjugate to a quasiattractor of* (f, M);

(b) *there is a finite invariant set* Z *such that* $f|\mathbf{R}^2\setminus Z$ *is of class* C^∞.

Notice that the supposition of the absence of homtervals can be replaced by transitivity, and then we obtain attractors.

Theorem A will be established as a consequence of the following result.

Theorem B. *Let* $p\colon I\to I$ *be a mapping of finite type having no homtervals. Then there exists a* C^1 *diffeomorphic embedding* $f\colon \mathbf{R}^2\to\mathbf{R}^2$ *such that*:

(a) *the inverse limit of* (p, I) *is conjugate to a quasiattractor of* (f, M), *for which* $\mathbf{R}^2$ *is a domain of attraction*;

(b) *there is a finite invariant set* Z *such that* $f|\mathbf{R}^2\setminus Z$ *is of class* C^∞;

(c) f *preserves (resp. reverses) orientation.*

In Appendix we show that our method cannot give a C^2-diffeomorphism unless p is monotonic.

Questions. 1. Does there exist a diffeomorphic embedding of class C^2 of a 2-dimensional manifold into itself, which possesses an attractor conjugate to the inverse limit of an interval mapping?

2. Does Theorem A hold true without condition (iii) (possibly, after deleting conclusion (b) or assuming that there are no periodic critical points)?

3. Proof of Theorems A and B. Most of this section concerns the proof of Theorem B, which is in six parts. In Part 1 we impose on the map p some additional conditions which are shown not to affect the generality, and then p is fixed. In Parts 2, 3 and 4 we construct the asserted diffeomorphism f. In Parts 5 and 6 we prove the properties of f. The construction consists in factorizing one discontinuous map of the plane through another; a general description will be given at the end of Part 2.

Part 1. *Refinement of* p. In the first place, we may assume that p maps I onto I. Indeed, consider the d.s. (p, J), where $J = \bigcap_{n=0}^{\infty} p^n(I)$. Then J is a closed interval (possibly a point) and p maps J onto J as I is compact. The mapping $p|J$ is of finite type and homterval-free provided p is so, and we have $\varprojlim(p, I) = \varprojlim(p, J)$.

For every mapping $\varphi\colon I\to I$ write $A(\varphi)$ for the set consisting of 0, 1 and the critical points of φ, and let $A'(\varphi) = \bigcup_{n=0}^{\infty}\varphi^n(A(\varphi))$.

Lemma 1. *If* $\varphi, \psi\colon I\to I$ *are two piecewise monotonic homterval-free mappings such that* $A(\varphi) = A(\psi)$ *and* $\psi|A'(\varphi) = \varphi|A'(\varphi)$ *then* φ *is conjugate with* ψ.

Proof. For each $x\in A(\varphi) = A(\psi)$ the orbits of x under φ and ψ are identical. Hence φ and ψ have the same kneading invariant [2]. Since both maps are homterval-free, it follows [2] that they are conjugate. ∎

Lemma 2. *If* φ *is a homterval-free mapping of* I *onto* I *satisfying conditions* (i) *and* (ii) *then the set* $A'(\varphi)$ *is finite and* φ *is conjugate to a piecewise stretching map.*

(We call a monotonic map $f\colon J\to\mathbf{R}$, where J is an interval, *stretching*, if for every distinct points $a, b\in J$ we have $|a-b| < |f(a)-f(b)|$.)

Proof. By (ii), 0 and 1 have finite orbits since they have preimages in $A(\varphi)$. Hence, by (i) and (ii), the set $A'(\varphi)$ is finite.

Let $A'(\varphi) = \{d_0, d_1, \ldots, d_M\}$ where $0 = d_0 < d_1 < \ldots < d_M = 1$ and let $J_k = [d_{k-1}, d_k]$ for $k = 1, \ldots, M$. Then each interval J_k is mapped homeomorphically onto an interval of the form $\bigcup_{i=m}^{n} J_i$. We define

$$w_k = 2M-\sup\{r\colon \varphi^s(J_k)\in\{J_i\}_{i=1,\ldots,M} \text{ for } s = 0, 1, \ldots, r\} \quad (k = 1, \ldots, M).$$

Then

$$2M \geqslant w_k \geqslant M+1, \tag{2.1}$$

otherwise $\varphi^n(J_k)\in\{J_i\}_{i=1,\ldots,M}$ for every natural n, and hence J_k would be a homterval.

Now choose the points $a_0, a_1, \ldots, a_M\in I$ with $a_k-a_{k-1} = w_k(w_1+\ldots+w_M)^{-1}$ for $k = 1, \ldots, M$, and take a homeomorphism $h\colon I\to I$ such that $h(d_k) = a_k$ for $k = 1, \ldots, M$. Define the map $\psi\colon I\to I$ by letting $\psi(a_k) = h\circ\varphi\circ h^{-1}(a_k)$ and taking the linear extension over each interval $[a_{k-1}, a_k]$. We claim that ψ is piecewise stretching and $\varphi\approx\psi$. Consider an interval $[a_{k-1}, a_k]$. It is mapped linearly by ψ onto $[a_{i-1}, a_j]$, where $0 < i\leqslant j\leqslant M$. If $i<j$ then, by (2.1), $(a_j-a_{i-1})(a_k-a_{k-1})^{-1} \geqslant 2(M+1)/2M > 1$. If $i = j$ then $w_j = w_k+1$, whence $(a_j-a_{j-1})(a_k-a_{k-1})^{-1} = w_j w_k^{-1} > 1$. Consequently, ψ is piecewise stretching, and hence, homterval-free. Since

$$A'(h\circ\varphi\circ h^{-1}) = \{a_0, a_1, \ldots, a_M\},$$

we have $A(\psi) = A(h\circ\varphi\circ h^{-1})$ and

$$\psi|A'(h\circ\varphi\circ h^{-1}) = \varphi|A'(h\circ\varphi\circ h^{-1}),$$

so it follows from Lemma 1 that $\psi\approx h\circ\varphi\circ h^{-1}$, that is, $\psi\approx\varphi$. ∎

Suppose that a mapping φ of I onto I is of finite type and homterval-free. By Lemma 2, there is a piecewise stretching map $\psi: I\to I$ such that $\psi\approx\varphi$ and $A'(\psi)$ is finite. Let $A'(\psi) = \{d_0, d_1, \ldots, d_M\}$, where $0 = d_0 < d_1 < \ldots < d_M = 1$. Take a map $p: \mathbf{R}\to(-1, 2)$ satisfying the following conditions:

1°. $p|A'(\psi) = \psi|A'(\psi)$.

2°. For $k = 1, \ldots, M$, the restriction $p|[d_{k-1}, d_k]$ is a stretching C^∞-immersion extendable to a C^∞-diffeomorphism $p_i: \mathbf{R}\to\mathbf{R}$ such that $|p_i'(x)| = 1$ for $x\in\mathbf{R}\setminus[d_{k-1}, d_k]$.

3°. $p|(-\infty, d_1]$ and $p|[d_{M-1}, \infty)$ are C^∞-immersions into $(-1, 2)$ such that $|p'(x)| < 1$ for $x\in\mathbf{R}\setminus I$.

Lemma 1 implies that $p|I$ is conjugate to ψ. As inverse limits of conjugate maps are conjugate, instead of proving Theorem B for φ we may, and will, prove it for $p|I$.

From now on the map p will be fixed. Let C denote the set of the critical points of p, let $C = \{c_1, \ldots, c_{N-1}\}$ and $c_1 < c_2 < \ldots < c_{N-1}$. Let $I_k = [c_{k-1}, c_k]$ for $k = 2, \ldots, N-1$ and $I_1 = (-\infty, c_1]$, $I_N = [c_{N-1}, \infty)$. We define D to be $\bigcup_{n=0}^{\infty} p^n(C)$ and E to be $\bigcap_m \bigcup_{n=m}^{\infty} p^n(C)$. By (i), E consists of those images of critical points under iterates of p, which are periodic. Finally, let $\chi: \mathbf{R}\to\{1, \ldots, N\}$ be the address function: $\chi(x) = k$ if $x\in\operatorname{int}(I_k)$ or $x = c_k$.

Henceforward we will not need the fact that $p(I) = I$, nor that $p|I$ is homterval-free or piecewise stretching. We will use only conditions (i), (ii), (iii) and the following consequences of 1°, 2° and 3°:

(iv) For $k = 1, \ldots, N$, the map $p|I_k$ is a C^∞-immersion into $\mathbf{R}$.

(v) For each $x\in C$, the map $t\mapsto(p(t)-p(x))\operatorname{sign}(t-x)$ is a C^∞-immersion of some neighbourhood of x.

(vi) If $x\in D\setminus C$ and $\varepsilon\in\{1, -1\}$ then the map given by

$$t\mapsto(p(\varepsilon|t|+x)-p(x))\operatorname{sign}(t)$$

is a C^∞-immersion of some neighbourhood of 0.

(vii) For each $x\in E$, $|p'(x)| = 1$.

(viii) The set $p(\mathbf{R})$ is bounded and $\bigcap_{n=0}^{\infty} p^n(I)\subseteq I$.

Part 2. *Basic auxiliary maps*. Denote $J_k = ((k-\frac{2}{3})/N, (k-\frac{1}{3})/N)$ for $k = 1, 2, \ldots, N$, and $J = \bigcup_{k=1}^{N} J_k$.

PROPOSITION 3. *There exists a map $Q: J\to\mathbf{R}$ with the restrictions $Q_k = Q|J_k$ such that:*

(ix) *Q_k is a diffeomorphism of J_k onto $\mathbf{R}$ ($k = 1, \ldots, N$);*

(x) *Q_k is stretching ($k = 1, \ldots, N$);*

(xi) *$Q(t) = Q(2kN^{-1}-t)$ for $t\in J_k$ ($k = 1, 2, \ldots, N-1$);*

(xii) *if $x\in E$, $p^s(x) = x$ and*

$$Q^{-1}_{\chi(p^s(x))}\circ\ldots\circ Q^{-1}_{\chi(p^2(x))}\circ Q^{-1}_{\chi(p(x))}(y) = y$$

then $|Q'(y)| = 1$ and $Q'''(y)\neq 0$.

In addition, the sign of Q_1' can be chosen arbitrarily.

Proof. Fix $\varepsilon\in\{1, 0\}$ and define $\bar{Q}_k: J\to J$ ($k = 1, \ldots, N$) by

$$\bar{Q}_k(x) = \operatorname{cotan}(3N(-1)^{k+\varepsilon}\pi x).$$

For each $x\in E$ with $p^s(x) = x$ ($s\neq 0$) let x_* be the unique fixed point of the contraction $\bar{Q}^{-1}_{\chi(p^s(x))}\circ\bar{Q}^{-1}_{\chi(p^{s-1}(x))}\circ\ldots\circ\bar{Q}^{-1}_{\chi(p(x))}$. Clearly, this definition is independent of the choice of s. Denote $E_* = \{x_*\}_{x\in E}$ and let $E_1 = \bar{Q}^{-1}(E_*)$. Since $Q(p(x)_*) = x_*$ for every $x\in E$, we have $\bar{Q}(E_*) = E_*$ and thus $E_*\subseteq E_1$. By (i) and (ii), E_1 is finite. Let $E_1\cap J_1 = \{a_1, a_2, \ldots, a_m\}$ where $a_1 < a_2 < \ldots < a_m$. Consider the mapping $f: J_1\to\mathbf{R}$ of the form

$$f(x) = \bar{Q}_1(a_1)+(-1)^\varepsilon\int_{a_1}^{x}\Big(1-(\sin 3\pi Nt)^{-1}\varphi(t)\prod_{k=1}^{m}(t-a_k)^2\Big)dt$$

where $\varphi: J_1\to\mathbf{R}$ is of class C^∞. Using the fact that $\bar{Q}_1$ is stretching, we can choose the function φ so that $\inf\varphi > 0$ and $f(a_k) = \bar{Q}_1(a_k)$ for $k = 1, \ldots, m$. Now define $Q: J\to\mathbf{R}$ by

$$Q(x) = \begin{cases} f(x-(k-1)/N) & \text{if } x\in J_k \text{ where } k \text{ is odd},\\ f(-x+k/N) & \text{if } x\in J_k \text{ where } k \text{ is even}.\end{cases}$$

Then, conditions (ix), (x) and (xi) are satisfied,

$$Q|E_* = \bar{Q}|E_* \quad\text{and} \tag{3.1}$$

$$|Q'(x_*)| = 1 \quad\text{and}\quad Q'''(x_*)\neq 0 \quad\text{for each } x\in E. \tag{3.2}$$

Let x, s, y satisfy the hypothesis of condition (xii). By (x), (3.1) and since $\bar{Q}(E_*) = E_*$, we have $y = x_*$, so the conclusion of (xii) follows from (3.2). ∎

Henceforward the map Q satisfying (ix)–(xii) will be fixed. If the diffeomorphism f is required to preserve orientation then we assume $\operatorname{sign}(Q_1') = \operatorname{sign}(p')|\operatorname{int}(I_1)$ which, by (xi), implies $\operatorname{sign}(Q_k') = \operatorname{sign}(p')|\operatorname{int}(I_k)$ for each k. Otherwise we assume $\operatorname{sign}(Q_k') = -\operatorname{sign}(p')|\operatorname{int}(I_k)$.

For any $x\in E$ let x_* be defined by the equality

$$Q^{-1}_{\chi(p^{s-1}(x))}\circ Q^{-1}_{\chi(p^{s-2}(x))}\circ\ldots\circ Q^{-1}_{\chi(x)}(x_*) = x_*$$

where s is the period of x. For every sequence $u = (u_n)_{n=0}^{\infty}\in\{1, \ldots, N\}^{\mathbf{N}}$ and every natural number n we write

$$Q^{-1}_{u_0}\circ Q^{-1}_{u_1}\circ\ldots\circ Q^{-1}_{u_n}(I) = [u_0, u_1, \ldots, u_n].$$

Condition (ix) implies that $[u_0, u_1, \ldots, u_n]$ is a closed interval.

LEMMA 4. *For any sequence* $u \in \{1, \ldots, N\}^N$ *the set* $\bigcap_{n=0}^{\infty} [u_0, u_1, \ldots, u_n]$ *is a singleton.*

Proof. For $m \in N$ let $A_m = \bigcap_{n=m}^{\infty} [u_m, u_{m+1}, \ldots, u_n]$. Each set A_m is a closed interval and for every $i, j \in N$ either $A_i = A_j$ or $A_i \cap A_j = \emptyset$. Suppose A_0 has more than one point. Since $A_m = Q_{u_m}^{-1}(A_{m+1})$, conditions (x) implies that for each m the interval A_{m+1} is longer than A_m. Hence the intervals A_m must be disjoint but that is now impossible for they are contained in $[0, 1]$. ■

For every sequence $u \in \{1, \ldots, N\}^N$, the unique element of $\bigcap_{n=0}^{\infty} [u_0, u_1, \ldots, u_n]$ will be denoted by $[u]$ or $[u_0, u_1, \ldots]$. Define X to be $\bigcap_{n=0}^{\infty} Q^{-n}(I)$. Obviously, $Q^{-1}(X) = X = Q(X)$ and $X = \{[u]\colon u \in \{1, \ldots, N\}^N\}$.

PROPOSITION 5. *There exists a map* $\Psi\colon \mathbf{R}^2 \to \mathbf{R}^2$ *with the following properties.*

(xiii) $\operatorname{int}\{z \in \mathbf{R}^2\colon \Psi(z) = z\} \supseteq \mathbf{R}^2 \setminus (-1, 1)^2$;

(xiv) Ψ *maps the set* $\mathbf{R}^2 \setminus H_0$ C^∞*-diffeomorphically onto* $\mathbf{R}^2 \setminus V_0$, *where*

$$H_0 = \{0\} \times [-\tfrac{4}{5}, \tfrac{4}{5}] \quad \text{and} \quad V_0 = [-\tfrac{4}{5}, \tfrac{4}{5}] \times \{0\};$$

(xv) $\Psi(t, x) = (\frac{4}{5} - |x|, 0)$ *for* $(t, x) \in H_0$ *and*

$$\Psi(t, x) = ((\tfrac{4}{5} - |x|)\operatorname{sign}(t), |t| \cdot \operatorname{sign}(x)) \quad \text{for } (t, x) \in W \setminus H_0,$$

where

$$W = \{(t, x)\colon |t| + |x - \tfrac{2}{5}| \leqslant \tfrac{2}{5} \text{ or } |t| + |x + \tfrac{2}{5}| \leqslant \tfrac{2}{5}\}.$$

The action of Ψ is schematically illustrated in Figure 1.

Proof. Define $\Psi | W$ according to condition (xv). Take a C^∞-diffeomorphism $g\colon \mathbf{R} \to \mathbf{R}$ such that $g\left(\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]\right) = \left[-\frac{3}{4}\pi, \frac{3}{4}\pi\right]$ and $g'(x) = 1$ for any $x \in \mathbf{R} \setminus \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$. Let D and E be C^∞-disks with $W \subseteq \operatorname{int}(E)$, $E \subseteq \operatorname{int}(D)$ and $D \subseteq (-1, 1)^2$. Let D be close to W enough that we can extend Ψ over D by letting

$$\Psi(r \cdot \cos u, r \cdot \sin u) = \left(\frac{4}{5}\operatorname{sign}(r) + r \cdot \cos g(u), r \cdot \sin g(u)\right)$$

$$\text{for } r \neq 0,\ u \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \quad \text{and}$$

$$\Psi(r \cdot \sin u, \tfrac{4}{5}\operatorname{sign}(r) + r \cdot \cos u) = (r \cdot \sin g^{-1}(u), r \cdot \cos g^{-1}(u))$$

$$\text{for } r \neq 0,\ u \in \left[-\frac{3}{4}\pi, \frac{3}{4}\pi\right],$$

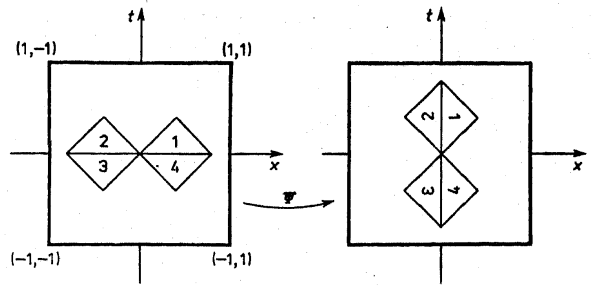


Fig. 1

to obtain a C^∞-embedding $\Psi | D \setminus H_0\colon D \setminus H_0 \to (-1, 1)^2$. Then, $D \setminus H_0$ is carried onto $D' \setminus V_0$, and $E \setminus H_0$ onto $E' \setminus V_0$, where D' and E' are disks, $E' \subseteq \operatorname{int}(D')$, $V_0 \subseteq \operatorname{int}(E')$ and $D' \subseteq (-1, 1)^2$. By extending $\Psi | \operatorname{bd}(E)$ to a diffeomorphism of E onto E' and using [1] (Ch. 8, Th. 1.9), one gets a diffeomorphism $\Psi'\colon D \to D'$ which coincides with Ψ near $\operatorname{bd}(D)$. By [1] (Ch. 8, Th. 3.1), Ψ' extends to a diffeomorphism of $\mathbf{R}^2$ having the support contained in $(-1, 1)^2$. Extending Ψ in the same way provides conditions (xiii) and (xiv). ■

Now let the map $F\colon \mathbf{R}^2 \to \mathbf{R}^2$ be defined as follows:

$$F(t, x) = (p(t), Q_{\chi(t)}^{-1}(x)). \tag{1}$$

By (iv) and (ix), F is one-to-one and $F | \mathbf{R}^2 \setminus C \times \mathbf{R}$ is a diffeomorphic embedding of class C^∞. However, F is discontinuous at each point of $C \times \mathbf{R}$. For $\varepsilon > 0$ small enough, the stripe $[c_k - \varepsilon, c_k] \times \mathbf{R}$ is carried onto $[p(c_k), p(c_k - \varepsilon)] \times J_k$ or $[p(c_k - \varepsilon), p(c_k)] \times J_k$ (with $F(\{c_k\} \times \mathbf{R}) = \{p(c_k)\} \times J_k$), and the stripe $(c_k, c_k + \varepsilon] \times \mathbf{R}$ onto $(p(c_k), p(c_k + \varepsilon)] \times J_{k+1}$ or $[p(c_k + \varepsilon), p(c_k)) \times J_{k+1}$, respectively. By (xi), the points $F(c_k, x)$ and $\lim_{t \to c_k^+} F(t, x) \in \{p(c_k)\} \times J_{k+1}$ are symmetric with respect to the line $x = \frac{k}{N}$.

The required diffeomorphism f will be the unique continuous map satisfying $G \circ \tilde{F} = f \circ G$, where $G\colon \mathbf{R}^2 \to \mathbf{R}^2$ will be constructed in Part 3 (formula (9)) and $\tilde{F}\colon \mathbf{R}^2 \to \mathbf{R}^2$ in Part 4 (formula (11)). The map $\tilde{F}$ is a slight modification of F, needed in order that f be smooth (cf. Claim 11, Proof, 3). If we worked with F instead of $\tilde{F}$ then f would still be continuous and have the required quasiattractor. (In this case, the present proof goes through after obvious simplification.) Intuitively, the map G is obtained by cutting the plane along countably many parallel segments and rejoining the edges of the incisions with the aid of suitable maps like Ψ of Proposition 5, where the incision has been made along H_0. This operation is applied to the segments $\{p(c_k)\} \times \left[\frac{k}{N} - \frac{4}{5N}, \frac{k}{N} + \frac{4}{5N}\right]$ and their images under $\tilde{F}$ (or F) so that G cancels the discontinuities on $C \times \mathbf{R}$.

Part 3. Define the closed intervals A_n^k recursively by

$$(2)\qquad A_k^1 = \left[\frac{k-1}{N}, \frac{k+1}{N}\right] \quad (k = 1, \ldots, N-1), \qquad A_k^{n+1} = Q^{-1}_{\chi(p^n(c_k))}(A_k^n)$$

and let $a_{k,n}$ be the length of A_n^k.

Choose a positive number d such that

$$(3)\qquad (2+\sup_{[-1,2]}|p'|)d < \inf_{x\in D} \operatorname{dist}(x, D\setminus\{x\}) \text{ and}$$

$$(4)\qquad 2d\sup_{[-1,2]}|p''| < \inf_{x\in I}|(Q_1^{-1})'(x)|.$$

Define the rectangles B_k^n by

$$(5)\qquad B_k^n = [p^n(c_k)-da_{k,n}, p^n(c_k)+da_{k,n}]\times A_k^n.$$

Inequality (3) guarantees in particular, together with condition (iii), that $(C\times \boldsymbol{R})\cap \operatorname{cl}(\bigcup_{k,n} B_k^n \cup F(B_k^n)) = \emptyset$. We note that

$$F(\{c_k\}\times \boldsymbol{R}) = \{p(c_k)\}\times J_k \quad \text{and}$$
$$\{p(c_k)\}\times \operatorname{cl}(J_k)\subseteq \{p(c_k)\}\times \operatorname{int}(A_k^1)\subseteq \operatorname{int}(B_k^1),$$

and that

$$F(\{p^n(c_k)\}\times A_k^n) = \{p^{n+1}(c_k)\}\times A_k^{n+1}.$$

For $x\in E$ write $x^* = (x, x_*)$, and let $Z = \{x^*\colon x\in E\}$. We have $F(x^*) = p(x)^*$, therefore $F(Z) = Z$. The set Z will later satisfy conclusion (b).

LEMMA 6. (a) *If* $B_k^n\cap \operatorname{int} B_l^m \neq \emptyset$ *then* $k = l$ *and* $m = n$.

(b) *If* $F(B_k^n)\cap \operatorname{int} B_l^m \neq \emptyset$ *then* $k = l$ *and* $m = n+1$.

(c) $Z\cap \bigcup_{n,k} B_k^n = \emptyset$.

(d) *For each* k, $\lim_{n\to\infty} \max_{z\in B_k^n} \operatorname{dist}(z, p^n(c_k)^*) = 0$.

Proof. (a) By (3) and (5) (where $a_{k,n}\leqslant 1$) we have $p^n(c_k) = p^m(c_l)$. Hence

$$(6.1)\qquad \{p^n(c_k)\}\times A_k^n \cap \{p^m(c_l)\}\times \operatorname{int}(A_l^m) \neq \emptyset.$$

We may assume $m\geqslant n$. By (1) and (2),

$$\{p^n(c_k)\}\times A_k^n = F^{n-1}(\{p(c_k)\}\times A_k^1) \quad \text{and}$$
$$\{p^m(c_l)\}\times \operatorname{int}(A_l^m) = F^{n-1}(\{p^{m-n+1}(c_l)\}\times \operatorname{int}(A_l^{m-n+1})).$$

Since F is one-to-one, (6.1) implies

$$\{p(c_k)\}\times A_k^1 \cap \{p^{m-n+1}(c_l)\}\times \operatorname{int}(A_l^{m-n+1}) \neq \emptyset,$$

that is

$$(6.2)\qquad p(c_k) = p^{1+m-n}(c_l) \quad \text{and}$$

$$(6.3)\qquad A_k^1\times \operatorname{int}(A_l^{m-n+1}) \neq \emptyset.$$

First suppose $m = n$. By (6.3) we have $|k-l|\leqslant 1$, but if $|k-l| = 1$ then (6.2) cannot hold, so that $k = l$. Suppose then $m > n$. Since $A_l^{m-n+1} = Q^{-1}_{\chi(p^{m-n}(c_l))}(A_l^{m-n})$, by (6.3) we have that $p^{m-n}(c_l)\in I_k\cup I_{k+1}$. This and (6.2) imply $p^{m-n}(c_l) = c_k$, contrary to condition (iii).

(b) Since $B_k^n\cap (C\times \boldsymbol{R}) = \emptyset$, it follows from (1) that

$$F(B_k^n)\subseteq [p^{n+1}(c_k)-d\sup_{[-1,2]}|p'|, p^{n+1}(c_k)+d\sup_{[-1,2]}|p'|]\times A_k^{n+1}.$$

This implies $A_k^{n+1}\cap \operatorname{int}(A_l^m) \neq \emptyset$ and, by (3), $p^{n+1}(c_k) = p^m(c_l)$. Hence $B_k^{n+1}\cap \operatorname{int}(B_l^m) \neq \emptyset$, so that (b) follows from (a).

(c) Suppose $Z\cap B_k^n \neq \emptyset$. Then, by (3),

$$Z\cap (\{p^n(c_k)\}\times A_k^n) \neq \emptyset.$$

Since $F(Z) = Z$ this implies

$$Z\cap F^{-n}(\{p^n(c_k)\}\times A_k^n) \neq \emptyset.$$

But since

$$F^{-n}(\{p^n(c_k)\}\times A_k^n) = \{c_k\}\times \boldsymbol{R},$$

this means $c_k\in E$, which is impossible by virtue of (iii).

(d) Let k be fixed and let $p^n(c_k)\in E$ for every $n\geqslant m$. Denote

$$i_n = \chi(p^n(c_k)) \quad \text{and} \quad p^n(c_k)_* = [i_0^n, i_1^n, i_2^n, \ldots] \quad \text{for } n\geqslant m.$$

Then,

$$A_k^n = Q^{-1}_{i_{n-1}}\circ Q^{-1}_{i_{n-2}}\circ \ldots \circ Q^{-1}_{i_0}(A_k^1)\subseteq [i_{n-1}, i_{n-2}, \ldots, i_0]$$
$$\subseteq [i_{n-1}, \ldots, i_{m+1}, i_m] = [i_0^n, i_1^n, \ldots, i_{n-m-1}^n],$$

whence

$$\max_{z\in B_k^n} \operatorname{dist}(z, p^n(c_k)^*) \leqslant (1+d^2)^{1/2} \operatorname{diam}([i_0^n, i_1^n, \ldots, i_{n-m-1}^n]).$$

As the sequence $(i_n)_{n=m}^{\infty}$ is periodic, Lemma 4 implies that $\operatorname{diam}([i_0^n, i_1^n, \ldots, i_{n-m-1}^n])$ tends to 0. ∎

Let the mappings $F_{k,n}\colon \boldsymbol{R}^2\to \boldsymbol{R}^2$ $(k = 1, \ldots, N-1, n = 1, 2, \ldots)$ be given by

$$(6)\qquad F_{k,n}(t, x) = (p^{n+1}(c_k)+a_{k,n+1}a_{k,n}^{-1}\varepsilon_{k,n}(t-p^n(c_k)), Q^{-1}_{\chi(p^n(c_k))}(x)),$$

where $\varepsilon_{k,n} = \operatorname{sign}(p'(p^n(c_k)))$. Define the maps $\varphi_{k,n}\colon \boldsymbol{R}^2\to \boldsymbol{R}^2$ recursively by

$$(7)\qquad \varphi_{k,1}(t, x) = (p(c_k)+2\varepsilon_k dN^{-1}t, (x+k)N^{-1}), \qquad \varphi_{k,n+1} = F_{k,n}\circ \varphi_{k,n}$$

where ε_k is the sign of the right-hand-side derivative of p at c_k. Let $B = [-1, 1]^2$. Note that $F_{k,n}$ settles diffeomorphisms of $\boldsymbol{R}\times A_k^n$ onto $\boldsymbol{R}\times A_k^{n+1}$ and of B_k^n onto B_k^{n+1} taking the segment $\{p^n(c_k)\}\times A_k^n$ onto $\{p^{n+1}(c_k)\}\times A_k^{n+1}$. Thus, $\varphi_{k,n}$ maps B C^∞-diffeomorphically onto B_k^n and takes $\{0\}\times[-1, 1]$ onto $\{p^n(c_k)\}\times A_k^n$. By induction we immediately obtain the formulas

$$\varphi_{k,n}(t,x) = \left(p^n(c_k)+\varepsilon_k^n d a_{k,n} t,\ Q^{-1}_{\chi(p^{n-1}(c_k))}\circ\ldots\circ Q^{-1}_{\chi(p(c_k))}((k+x)N^{-1})\right)$$
$$\varphi^{-1}_{k,n}(t,x) = \left(\varepsilon_k^n d^{-1} a^{-1}_{k,n}(t-p^n(c_k)),\ NQ^{n-1}(x)-k\right),$$

where $\varepsilon_k^n = \varepsilon_k\varepsilon_{k,1}\ldots\varepsilon_{k,n-1}$.

Now let us take a map $\Psi\colon \boldsymbol{R}^2\to\boldsymbol{R}^2$ satisfying conditions (xiii), (xiv) and (xv) of Proposition 5 and define the map $G\colon \boldsymbol{R}^2\to\boldsymbol{R}^2$ by

(9) $$G(z) = \begin{cases}\varphi_{k,n}\circ\Psi\circ\varphi^{-1}_{k,n}(z) & \text{for } z\in \operatorname{int}(B_k^n),\\ z & \text{for } z\notin \bigcup_{k,n}\operatorname{int}B_k^n.\end{cases}$$

By (xiii), (xiv) and Lemma 6(a), G is well defined. Under the notation of Proposition 5, let

$$V_k^n = \varphi_{k,n}(V_0),\quad H_k^n = \varphi_{k,n}(H_0)\quad\text{and}\quad V = \bigcup_{k,n} V_k^n,\quad H = \bigcup_{k,n} H_k^n.$$

By Lemma 6(a), the segments H_k^n as well as the segments V_k^n, are disjoint. It follows from Lemma 6(a)(d), (xiii) and (xiv) that the set $\boldsymbol{R}^2\setminus(H\cup Z)$ is open and mapped C^∞-diffeomorphically by G onto $\boldsymbol{R}^2\setminus(V\cup Z)$.

Part 4. We take functions $\beta\colon \boldsymbol{R}^2\to\boldsymbol{R}^2$ and $\gamma\colon \boldsymbol{R}^2\to\boldsymbol{R}$ of class C^∞ satisfying the following conditions:

(10a) $$\beta(t,x)\equiv(\gamma(t,x),x);$$

(10b) $$\gamma(t,x)\equiv-\gamma(-t,x)\equiv\gamma(t,-x);$$

(10c) $$t\gamma(t,x)\geqslant 0\quad\text{for } t,x\in\boldsymbol{R};$$

(10d) $$\operatorname{int}(\gamma^{-1}(\{0\}))\supseteq(\boldsymbol{R}^2\setminus\operatorname{int}(B))\cup\boldsymbol{R}\times\left((-\tfrac{2}{3},-\tfrac{1}{3})\cup(\tfrac{1}{3},\tfrac{2}{3})\right);$$

(10e) there is a neighbourhood U of the set $\{0\}\times\{-\frac{4}{5},0,\frac{4}{5}\}$ such that $\beta|U=\mathrm{id}_U$;

(10f) $$\sup(\partial\gamma/\partial x_1)=1\quad\text{and}\quad\sup(-\partial\gamma/\partial x_1)<\inf_{[-1,2]}|p'|\cdot\inf_{[-1,2]}|p'|^{-1}.$$

The proof of the existence of β, being a standard exercise, is omitted. Figure 2 displays a possible shape of the graph of the function $t\mapsto\gamma(t,x)$ for x close to $-\frac{4}{5}$, 0 or $-\frac{4}{5}$.

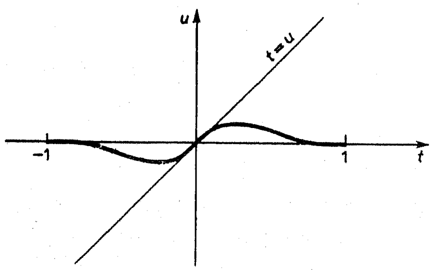


Fig. 2

Now define the map $\tilde{F}\colon \boldsymbol{R}^2\to\boldsymbol{R}^2$ as follows:

(11) $$\tilde{F}(z) = \begin{cases}F(z)+F_{k,n}\circ\varphi_{k,n}\circ\beta\circ\varphi^{-1}_{k,n}(z)-F\circ\varphi_{k,n}\circ\beta\circ\varphi^{-1}_{k,n}(z) & \text{for } z\in B_k^n,\\ F(z) & \text{for } z\notin\bigcup_{k,n}B_k^n.\end{cases}$$

LEMMA 7. (a) $\pi_2\circ\tilde{F}=\pi_2\circ F$ *(π_2 denotes the projection to the second coordinate).*

(b) *There exists a neighbourhood U of the set $\{0\}\times\{-\frac{4}{5},0,\frac{4}{5}\}$ such that*

$$\tilde{F}\circ\varphi_{k,n}|U = F_{k,n}\circ\varphi_{k,n}|U\quad\textit{for every } k \textit{ and } n.$$

(c) *There exists a set $U\subseteq B$ with $\{0\}\times[-1,1]\subseteq U$ and*

$$\operatorname{int}_B(U)\supseteq\operatorname{bd}(B)\cup[-1,1]\times\left((-\tfrac{2}{3},-\tfrac{1}{3})\cup(\tfrac{1}{3},\tfrac{2}{3})\right),$$

such that

$$\tilde{F}|\bigcup_{k,n}\varphi_{k,n}(U) = F|\bigcup_{k,n}\varphi_{k,n}(U).$$

(d) $\tilde{F}|\boldsymbol{R}\times X = F|\boldsymbol{R}\times X$.

Proof. (a) By the definitions we have $\pi_2\circ F_{k,n}|B_k^n=\pi_2\circ F|B_k^n$, so it is enough to use (11).

(b) follows from (10e) and (11).

(c) Let $U=\gamma^{-1}(\{0\})\cap B$. By (10b), (10c) and (10d) we have $\{0\}\times[-1,1]\subseteq U$ and $\operatorname{int}_B(U)\supseteq\operatorname{bd}(B)\cup[-1,1]\times((-\frac{2}{3},\frac{1}{3})\cup(\frac{1}{3},\frac{2}{3}))$, so we only have to check that $F_{k,n}\circ\varphi_{k,n}\circ\beta|U=F\circ\varphi_{k,n}\circ\beta|U$, and this is a direct consequence of (1), (6), (8) and the equality $\beta|U=(0,\mathrm{id})$.

(d) Let $(t,x)\in B_k^n\cap(\boldsymbol{R}\times X)$. Then

$$Q^{n-1}(x)\in Q^{n-1}(X)\cap Q^{n-1}(A_k^n) = X\cap A_k^1\subseteq J_k\cup J_{k+1}$$
$$=((k-\tfrac{2}{3})/N,(k-\tfrac{1}{3})/N)\cup((k+\tfrac{1}{3})/N,(k+\tfrac{2}{3})/N)$$

so, by (8),

$$\varphi^{-1}_{k,n}(t,x)\in(-1,1]\times\left((-\tfrac{2}{3},-\tfrac{1}{3})\cup(\tfrac{1}{3},\tfrac{2}{3})\right).$$

Hence $\tilde{F}(t,x)=F(t,x)$ by (c). ∎

LEMMA 8. *The restrictions $\tilde{F}|B_k^n$ are C^∞-diffeomorphisms onto $F(B_k^n)$.*

Proof. Fix n and k and let $F|B_k^n=(F_1,F_2)$, $\tilde{F}|B_k^n=(\tilde{F}_1,\tilde{F}_2)$. The map $\tilde{F}|B_k^n$ is of class C^∞. By Lemma 7(a) and (1) we have $F_2(t,x)=\tilde{F}_2(t,x)$ and $\partial F_2/\partial x_1=0$, $\partial F_2/\partial x_2\neq 0$. Moreover, $F|B_k^n$ is one-to-one and $\tilde{F}|\operatorname{bd}(B_k^n)=F|\operatorname{bd}(B_k^n)$ by Lemma 7(c). Consequently, it suffices to show that

(*) $$(\partial\tilde{F}_1/\partial x_1)(t,x)\neq 0\quad\text{for every } (t,x)\in B_k^n.$$

The derivative of the first coordinate with respect to the first coordinate will be denoted by a prime. Take a point $(t,x)\in B_k^n$ and denote $u=\varphi^{-1}_{k,n}(t,x)\in B$, $(v_1,v_2)=v=\varphi_{k,n}\circ\beta(u)\in B_k^n$. By (1), (6), (10a) and Lemma 7(a), all the functions which occur in the definition of $\tilde F|B_k^n$ preserve the foliation parallel to the first coordinate. Hence we compute

$$\begin{aligned}\tilde F'(t,x) &= F'(t,x)+(F_{k,n}\circ\varphi_{k,n}\circ\beta\circ\varphi_{k,n}^{-1})'(t,x)-(F\circ\varphi_{k,n}\circ\beta\circ\varphi_{k,n}^{-1})'(t,x)\\ &= F'(t,x)+F'_{k,n}(v)\cdot\varphi'_{k,n}(\beta\circ\varphi_{k,n}^{-1}(t,x))\cdot\beta'(u)\cdot(\varphi_{k,n}^{-1})'(t,x)\\ &\quad -F'(v)\cdot\varphi'_{k,n}(\beta\circ\varphi_{k,n}^{-1}(t,x))\cdot\beta'(u)\cdot(\varphi_{k,n}^{-1})'(t,x)\\ &= F'(t,x)+F'_{k,n}(v)\cdot\beta'(u)-F'(v)\cdot\beta'(u)\\ &= p'(t)+(a_{k,n+1}a_{k,n}^{-1}\varepsilon_{k,n}-p'(v_1))\gamma'(u).\end{aligned}$$

Note that $\operatorname{sign}(p'(t)) = \operatorname{sign}(p'(v_1)) = \varepsilon_{k,n}$. Now if $\gamma'(u)\leqslant 0$ then, by (10f).

$$\begin{aligned}|\tilde F'(t,x)| &\geqslant |p'(t)|-|(a_{k,n+1}a_{k,n}^{-1}\varepsilon_{k,n}-p'(v_1))\gamma'(u)|\\ &> \inf_{[-1,2]}|p'|-|\sup_{[-1,2]}|p'|\gamma'(u)|>0.\end{aligned}$$

If $\gamma'(u)>0$ and $\operatorname{sign}(p'(t)) = \operatorname{sign}(a_{k,n+1}a_{k,n}^{-1}\varepsilon_{k,n}-p'(v_1))$ then $|\tilde F'(t,x)|\geqslant|p'(t)|>0$. Finally, if $\gamma'(u)>0$ and $\operatorname{sign}(p'(t)) = -\operatorname{sign}(a_{k,n+1}a_{k,n}^{-1}\varepsilon_{k,n}-p'(v_1))$ then

$$\begin{aligned}\varepsilon_{k,n}F'(t,x) &= \varepsilon_{k,n}p'(t)+(a_{k,n+1}a_{k,n}^{-1}-\varepsilon_{k,n}p'(v_1))\gamma'(u)\\ &\geqslant \varepsilon_{k,n}p'(t)+a_{k,n+1}a_{k,n}^{-1}-\varepsilon_{k,n}p'(v_1) \qquad \text{(by (10f))}\\ &\geqslant a_{k,n+1}a_{k,n}^{-1}-|t-v_1|\sup_{[-1,2]}|p''|\\ &\geqslant \inf_{[0,1]}|(Q_1^{-1})'|-2d\sup_{[-1,2]}|p''|>0 \qquad \text{(by (4))}. \blacksquare\end{aligned}$$

We can now define the map $f\colon \mathbf{R}^2\to\mathbf{R}^2$. Let

$$f(z)=\begin{cases} G\circ\tilde F\circ G^{-1}(z) & \text{for } z\in\mathbf{R}^2\setminus V,\\ F_{k,n}(z) & \text{for } z\in V_k^n.\end{cases}\tag{12}$$

LEMMA 9. $f\circ G = G\circ\tilde F$.

Proof. If $z\in\mathbf{R}^2\setminus H$ then $G(z)\in\mathbf{R}^2\setminus V$, so

$$f\circ G(z) = G\circ\tilde F\circ G^{-1}(G(z)) = G\circ\tilde F(z).$$

If $z\in H_k^n$ then Lemma 7(c) implies that $\tilde F(z) = F_{k,n}(z)$, whence

$$\begin{aligned}G\circ\tilde F(z) &= G\circ F_{k,n}(z) = \varphi_{k,n+1}\circ\Psi\circ\varphi_{k,n+1}^{-1}\circ F_{k,n}(z) = F_{k,n}\circ\varphi_{k,n}\circ\Psi\circ\varphi_{k,n}^{-1}(z)\\ &= F_{k,n}\circ G(z) = f\circ G(z). \blacksquare\end{aligned}$$

LEMMA 10. *The map f is one-to-one.*

Proof. By Lemma 8, the map $\tilde F$ is one-to-one. By Lemma 8 and 6(a), we have

$$\tilde F(\mathbf{R}^2\setminus H)\subseteq(\mathbf{R}^2\setminus H)\cup\bigcup_{k=1}^{N-1}H_k^1.\tag{10.1}$$

Suppose $f(a)=f(b)$. If $a\in V$ or $b\in V$ then $a=b$ since each map $F_{k,n}$ takes V_k^n one-to-one onto V_k^{n+1} and $f(\mathbf{R}^2\setminus V)\subseteq(\mathbf{R}^2\setminus V)\cup\bigcup_{k=1}^{N-1}V_k^1$ by (10.1). Suppose that $a,b\in\mathbf{R}^2\setminus V$. Then $a=G(c)$ and $b=G(d)$ for some $c,d\in\mathbf{R}^2\setminus H$ and Lemma 9 implies

$$G\circ\tilde F(c) = G\circ\tilde F(d).$$

From this and (10.1) it follows that $\tilde F(c),\tilde F(d)\in H_k^1$ for some k. This and Lemma 8 imply that $c,d\notin\bigcup_{k,n}B_k^n$, therefore $\tilde F(c)=F(c)$, $\tilde F(d)=F(d)$ and $c,d\in\{c_k\}\times\mathbf{R}$. Hence, by formula (1), $\tilde F(c),\tilde F(d)\subseteq\{p(c_k)\}\times J_k$, which equals either $\varphi_{k,1}(\{0\}\times(-\frac{2}{3},-\frac{1}{3}))$ or $\varphi_{k,1}(\{0\}\times(\frac{1}{3},\frac{2}{3}))$. On these sets G is one-to-one, so that $\tilde F(c)=\tilde F(d)$, which means $c=d$. $\blacksquare$

Part 5. *Smoothness of f.* The proof of Claim 11, stated below, is based mainly on conditions (xi), (v), (vi) and Lemma 7(b). Smoothness near Z involves conditions (vii) and (xii), and passing to the limit of the derivative.

CLAIM 11. *The restriction $f|\mathbf{R}^2\setminus Z$ is a diffeomorphic embedding of class C^∞.*

Proof. In view of Lemma 10 it suffices to show that $F|\mathbf{R}^2\setminus Z$ is of class C^∞ and the derivative $Df(z)$ is nonsingular for every $z\in\mathbf{R}^2\setminus Z$. We will show this by examining f on four open sets separately.

1. The first set is $U=\mathbf{R}^2\setminus(Z\cup V\cup C\times\mathbf{R})$. By (9), (xiii), (xiv) and since $(C\times\mathbf{R})\cap\bigcup_{k,n}B_k^n=\emptyset$, the set U is mapped C^∞-diffeomorphically by G^{-1} onto $U_1=\mathbf{R}^2\setminus(Z\cup H\cup(C\times\mathbf{R}))$. By (1), Lemma 8 and Lemma 7(c), the map $\tilde F|U_1$ is a C^∞-embedding into $U_2=\mathbf{R}^2\setminus(Z\cup H)$. By (xiii), (xiv) and Lemma 6(a)(d), the map $G|U_2$ is a C^∞-embedding. It follows that the restriction $f|U = G\circ\tilde F\circ G^{-1}|U$ is also a C^∞-embedding.

2. Let U be a neighbourhood of $\{c_k\}\times\mathbf{R}$ such that

$$U\cap\Big(\bigcup_{k,n}B_k^n\cup(c\setminus\{c_k\})\times\mathbf{R}\Big)=\emptyset \quad\text{and}\quad F(U)\subseteq\varphi_{k,1}(W)$$

and let $z=(t,x)\in U$. From (7) and (xv) we compute

$$\begin{aligned}f(z) &= G\circ\tilde F\circ G^{-1}(z) = G\circ\tilde F(z) = G\circ F(z) = \varphi_{k,1}\circ\Psi\circ\varphi_{k,1}^{-1}\circ F(z)\\ &= \varphi_{k,1}\circ\Psi\circ\varphi_{k,1}^{-1}(p(t),Q_{\chi(t)}^{-1}(x))\\ &= \varphi_{k,1}\circ\Psi(\varepsilon_k d^{-1}a_{k,1}^{-1}(p(t)-p(c_k)),NQ_{\chi(t)}^{-1}(x)-k)\\ &= \begin{cases}\varphi_{k,1}(NQ_k^{-1}(x)-k+\frac{4}{5},-\varepsilon_k d^{-1}a_{k,1}^{-1}(p(t)-p(c_k))) & \text{for } t\leqslant c_k,\\ \varphi_{k,1}(k-NQ_{k+1}^{-1}(x)+\frac{4}{5},\varepsilon_k d^{-1}a_{k,1}^{-1}(p(t)-p(c_k))) & \text{for } t>c_k.\end{cases}\end{aligned}$$

It follows from condition (xi) that $Q_{k+1}^{-1}(x)=2kN^{-1}-Q_k^{-1}(x)$, and so

$$f(t,x)=\varphi_{k,1}(NQ_k^{-1}(x)-k+\tfrac{4}{5},\varepsilon_k d^{-1}a_{k,1}^{-1}(p(t)-p(c_k))\operatorname{sign}(t-c_k)).$$

Now conditions (ix) and (v) imply that $f|U$ is an immersion.

3. Let $U_1 \subseteq \operatorname{int}(B)$ be a neighbourhood of the set $\{0\}\times\{0, -\frac{4}{5}, \frac{4}{5}\}$ such that $\tilde{F}|\varphi_{k,n}(U_1) = F_{k,n}|\varphi_{k,n}(U_1)$ for all indices k and n. Its existence is guaranteed by Lemma 7(b). Considering (xiii), (xiv) and (xv), it is easy to see that there is a neighbourhood U of the set $\{-\frac{4}{5}, 0, \frac{4}{5}\}\times\{0\}$ such that $\Psi^{-1}(U) \subseteq U_1$. We will show that $f|\varphi_{k,n}(U) = F_{k,n}|\varphi_{k,n}(U)$. We have $f|V_k^n = F_{k,n}|V_k^n$, and for every $z\in\varphi_{k,n}(U)\setminus V_k^n$.

$$\begin{aligned} f(z) &= G\circ\tilde{F}\circ G^{-1}(z) = G\circ\tilde{F}\circ(\varphi_{k,n}\circ\Psi^{-1}\circ\varphi_{k,n}^{-1})(z)\\ &= G\circ F_{k,n}\circ\varphi_{k,n}\circ\Psi^{-1}\circ\varphi_{k,n}^{-1}(z)\\ &= (\varphi_{k,n+1}\circ\Psi\circ\varphi_{k,n+1}^{-1})\circ F_{k,n}\circ\varphi_{k,n}\circ\Psi^{-1}\circ\varphi_{k,n}^{-1}(z)\\ &= \varphi_{k,n+1}\circ\Psi\circ(\varphi_{k,n}^{-1}\circ F_{k,n}^{-1})\circ F_{k,n}\circ\varphi_{k,n}\circ\Psi^{-1}\circ\varphi_{k,n}^{-1}(z)\\ &= F_{k,n}(z).\end{aligned}$$

4. We will show that for an arbitrary index (k, n), the map f is a C^∞-immersion of some neighbourhood of the set $V_k^n\setminus\varphi_{k,n}(\{-\frac{4}{5}, 0, \frac{4}{5}\}\times\{0\})$, and that will complete the proof.

Let U be a neighbourhood of $V_0\setminus\{-\frac{4}{5}, 0, \frac{4}{5}\}\times\{0\}$ such that $U\subseteq\operatorname{int}(\operatorname{cl}(\Psi(W)))$ and $\tilde{F}\circ\varphi_{k,n}\circ\Psi^{-1}(U)\subseteq\varphi_{k,n+1}(W)$. By (9) and (12) we have

$$f(z) = \begin{cases} F_{k,n}(z) & \text{for } z\in V_k^n\cap U,\\ \varphi_{k,n+1}\circ\Psi\circ\varphi_{k,n+1}^{-1}\circ\tilde{F}\circ\varphi_{k,n}\circ\Psi^{-1}\circ\varphi_{k,n}^{-1}(z) & \text{for } z\in U\setminus V_k^n.\end{cases}\tag{11.1}$$

By using (10a) and the formulas for $F_{k,n}$ and $\varphi_{k,n}$, we compute:

$$\begin{aligned}&\varphi_{k,n+1}^{-1}\circ\tilde{F}\circ\varphi_{k,n}(z)\\ &\quad= \varphi_{k,n+1}^{-1}\circ(F+F_{k,n}\circ\varphi_{k,n}\circ\beta\circ\varphi_{k,n}^{-1}-F\circ\varphi_{k,n}\circ\beta\circ\varphi_{k,n}^{-1})\circ\varphi_{k,n}(z)\\ &\quad= (\varphi_{k,n+1}^{-1}\circ F\circ\varphi_{k,n}+\varphi_{k,n+1}^{-1}\circ F_{k,n}\circ\varphi_{k,n}\circ\beta-\varphi_{k,n+1}^{-1}\circ F\circ\varphi_{k,n}\circ\beta)(z)\\ &\quad= (S_{k,n}(t, x), x)\quad \text{for } z = (t, x)\in B.\end{aligned}\tag{11.2}$$

where

$$\begin{aligned} S_{k,n}(t, x) &= L\big(p(Kt+p^n(c_k))-p^{n+1}(c_k)\big)\\ &\quad+\gamma(t, x)-L\big(p(K\gamma(t, x)+p^n(c_k))-p^{n+1}(c_k)\big),\end{aligned}\tag{11.3}$$

where, in turn, $K = \varepsilon_k^n d a_{k,n}$ and $L = \varepsilon_k^{n+1}d^{-1}a_{k,n+1}^{-1}$.

From (11.2), (xv) and the equality $\operatorname{sign}(S_{k,n}(t, x)) = \operatorname{sign}(t)$ (Lemmata 7(c) and 8), for $(t, x) = z\in\Psi(W)\setminus V_0$ we have

$$\begin{aligned}&\Psi\circ\varphi_{k,n+1}^{-1}\circ\tilde{F}\circ\varphi_{k,n}\circ\Psi^{-1}(t, x)\\ &\qquad= \big(t, S_{k,n}(|x|\operatorname{sign}(t)), (\tfrac{4}{5}-|t|\operatorname{sign}(x))\operatorname{sign}(x)\cdot\operatorname{sign}(t)\big).\end{aligned}\tag{11.4}$$

By expanding this formula according to (11.3) noting that, by (10b, c),

$$\gamma(|x|\operatorname{sign}(t), (\tfrac{4}{5}-|t|)\operatorname{sign}(x)) = |\gamma(x, \tfrac{4}{5}-|t|)|\operatorname{sign}(t),$$

we obtain

$$\begin{aligned}\varphi_{k,n+1}^{-1}\circ f\circ\varphi_{k,n}(t, x) &= \Big(t,\\ &\quad\big(p(K|x|\operatorname{sign}(t)+p^n(c_k))-p^{n+1}(c_k)\big)L\cdot\operatorname{sign}(x)\operatorname{sign}(t)\\ &\quad+\gamma(x, \tfrac{4}{5}-|t|)\\ &\quad-\big(p(K|\gamma(x, \tfrac{4}{5}-|t|)|\operatorname{sign}(t)+p^n(c_k))-p^{n+1}(c_k)\big)L\cdot\operatorname{sign}(\gamma(x, \tfrac{4}{5}-|t|))\operatorname{sign}(t)\Big)\\ &\qquad\qquad\text{for } (t, x)\in U\setminus V_0.\end{aligned}$$

By (11.1), this holds for $(t, x)\in V_0$ as well (for, in this case, $x = \gamma(x, \frac{4}{5}-|t|) = 0$). Since $\operatorname{sign}(t)$ is locally constant for $(t, x)\in U$ (as $\operatorname{int}\operatorname{cl}\Psi(W)\cap\{0\}\times\mathbf{R} = \emptyset$), it follows from condition (vi) that the map $\varphi_{k,n+1}^{-1}\circ f\circ\varphi_{k,n}|U$ is of class C^∞. By 1, the derivative $D\varphi_{k,n+1}^{-1}\circ f\circ\varphi_{k,n}$ is nonsingular on $U\setminus V_0$. Since $U\subseteq\operatorname{cl}(U\setminus V_0)$ and

$$\begin{aligned}\sup_{U\setminus V_0}\|D^{-1}\varphi_{k,n+1}^{-1}\circ f\circ\varphi_{k,n}\| &= \sup_{U\setminus V_0}\|D^{-1}\Psi\circ\varphi_{k,n+1}^{-1}\circ\tilde{F}\circ\varphi_{k,n}\circ\Psi^{-1}\|\\ &\leqslant \sup_{\operatorname{int}W}\|D\Psi\|\cdot\sup_{B}\|D^{-1}\varphi_{k,n+1}^{-1}\circ\tilde{F}\circ\varphi_{k,n}\|\cdot\sup_{\operatorname{int}W}\|D^{-1}\Psi\|\\ &= 1\cdot\sup_{B}\|D^{-1}\varphi_{k,n+1}^{-1}\circ\tilde{F}\circ\varphi_{k,n}\|\cdot 1<\infty,\end{aligned}$$

it follows that it is also nonsingular on $V_0\cap U$. ■

We go on to prove that f is of class C^1 in a neighbourhood of Z.

LEMMA 12. *If $a<b<c$ and f: $[a, c]\to[a, c]$ is a function of class C^2 such that $f(a) = a$, f' is decreasing and $0<f'(x)<1$ for $x\in(a, c)$ then the sequence of functions g_n: $[b, c]\to\mathbf{R}$ given by*

$$g_n(x) = (f^n(c)-f^n(b))^{-1}(f^n)'(x)$$

uniformly converges to a function g: $[b, c]\to(0, \infty)$.

Proof. Consider the functions h_n: $[b, c]\to\mathbf{R}$ given by

$$h_n(x) = \ln\big(((f^n)'(c))^{-1}(f^n)'(x)\big).$$

As

$$h_n(c) = 0\quad\text{and}\quad g_n(x) = \Big(\int_b^c\exp(h_n(x))\,dx\Big)^{-1}\exp(h_n(x)),$$

it suffices to show that the sequence (h_n') is uniformly convergent. We have

$$\begin{aligned}h_n'(x) &= ((f^n)'(x))^{-1}(f^n)''(x) = \Big(\prod_{k=0}^{n-1}f'\circ f^k(x)\Big)^{-1}\Big(\prod_{k=0}^{n-1}f'\circ f^k(x)\Big)'\\ &= \sum_{k=0}^{n-1}(f'\circ f^k(x))^{-1}(f'\circ f^k(x))'\\ &= \sum_{k=0}^{n-1}(f'\circ f^k(x))^{-1}(f''\circ f^k(x))(f^k)'(x).\end{aligned}$$

Hence the problem reduces to the uniform absolute convergence of the series $\sum_{k=0}^{\infty}(f^k)'(x)$ on $[b, c]$. Since $(f^k)'$ is positive and decreasing,

$$\sum_{k=0}^{\infty}|(f^k)'(x)| \leqslant \sum_{k=0}^{\infty}(f^k)'(b) \leqslant \sum_{k=0}^{\infty}\big(f^k(b)-f^{k+1}(b)\big)\big(b-f(b)\big)^{-1} = b\big(b-f(b)\big)^{-1} < \infty . \ \blacksquare$$

For $n \in N$ and $k = 1, \ldots, N-1$ let $\psi_{k,n}$ be the affine map satisfying $\psi_{k,n}|\{1, -1\}^2 = \varphi_{k,n}|\{1, -1\}^2$.

Lemma 13. *For $k = 1, \ldots, N-1$ there exist C^1-diffeomorphisms*

$$\psi_k\colon \boldsymbol{R}\times[-1, 1]\to\boldsymbol{R}\times[-1, 1]$$

such that $\lim_{n\to\infty}\psi_{k,n}^{-1}\circ\varphi_{k,n} = \psi_k$ in the space $C^1(\boldsymbol{R}\times[-1, 1], \boldsymbol{R}^2)$.

Proof. By (8),

$$D\psi_{k,n}^{-1}\circ\varphi_{k,n}(t, x) = \begin{bmatrix}1 & 0\\ 0 & |G_n(x)|\end{bmatrix},$$

where

$$G_n(x) = \big(f_n(1)-f_n(-1)\big)^{-1}f_n'(x)$$

and

$$f_n(x) = Q_{\chi(p^{n-1}(c_k))}^{-1}\circ\ldots\circ Q_{\chi(p(c_k))}^{-1}\big((x+k)/N\big).$$

It is sufficient to show that $|G_n|$ uniformly converges to a function $G_\infty\colon[-1, 1]\to(0, \infty)$. Assume $y = p^m(c_k) = p^{m+s}(c_k)$, where m and s are fixed positive integers, and let

$$f = Q_{\chi(p^{m+2s-1}(c_k))}^{-1}\circ\ldots\circ Q_{\chi(p^m(c_k))}^{-1}.$$

Then $f(y_*) = y_*$, and condition (xii) implies that

$$Q'\big(Q^k(y_*)\big) = 1 \quad\text{and}\quad Q'''\big(Q^k(y_*)\big) \neq 0 \quad (k = 0, 1, \ldots, 2s-1).$$

It follows from condition (xi) that $|Q'| \geqslant 1$, so we also have

$$Q''\big(Q^k(y_*)\big) = 0 \quad\text{and}\quad \operatorname{sign}\big(Q'(Q^k(y_*))\big) = \operatorname{sign}\big(Q'''(Q^k(y_*))\big).$$

Since $Q^s(y_*) = y_*$ this implies

$$(Q^{2s})'(y_*) = 1 \ \&\ (Q^{2s})''(y_*) = 0 \ \&\ (Q^{2s})'''(y_*) > 0,$$

whence

$$f'(y_*) = 1 \ \&\ f''(y_*) = 0 \ \&\ f'''(y_*) < 0.$$

Let U be an open interval with $y_* \in U$ and $f'''|U < 0$. By Lemma 6(d)(c), for some integer M we have $A_k^{m+2Ms} = f_{m+2Ms}([-1, 1]) \subseteq U\setminus\{y_*\}$. Lemma 12, applied either to f or to $-f\circ(-\mathrm{id})$, then yields

$$\lim_n\big(f^n\circ f_{m+2Ms}(1)-f^n\circ f_{m+2Ms}(-1)\big)^{-1}(f^n)' = g,$$

where $g\colon A_k^{m+2Ms}\to\boldsymbol{R}\setminus\{0\}$ is continuous. Since we have the identity $f_{m+2ks} = f^k\circ f_m$, it follows that

$$\lim_n\big(f_{m+2ns+2Ms}(1)-f_{m+2ns+2Ms}(-1)\big)^{-1}f'_{m+2ns+2Ms} = f'_{m+2Ms}\cdot g\circ f_{m+2Ms}.$$

Moreover, by Lemma 6(d) and condition (xii).

$$\lim_n|f_{n+1}\circ f_n^{-1})'|\,|A_k^n = \lim_n|(Q_{\chi(p^n(c_k))}^{-1})'|\,|A_k^n = \lim_n|(Q_{\chi(p^n(c_k))}^{-1})'(p^n(c_k)_*)| = 1$$

and thus $|G_n|$ tends to $|f'_{m+2Ms}\cdot g\circ f_{m+2Ms}|$. $\blacksquare$

Lemma 14. *For every sequence (z_n) such that $z_n = (t_n, x_n) \in B_k^n$,*

$$\lim_{n\to\infty}DF_{k,n}^{-1}\circ\tilde{F}(z_n) = \mathbf{1}.$$

Proof. First note that, under the same hypothesis,

$$\lim_{n\to\infty}DF_{k,n}^{-1}\circ F(z_n) = \mathbf{1}. \tag{14.1}$$

Indeed, we have

$$DF_{k,n}^{-1}\circ F(z_n) = \begin{bmatrix}a_{k,n}a_{k,n+1}^{-1}\varepsilon_{k,n}p'(t_n) & 0\\ 0 & 1\end{bmatrix}$$

by (1) and (6), and also

$$\lim_{n\to\infty}\varepsilon_{k,n}p'(t_n) = \lim_{n\to\infty}\varepsilon_{k,n}p'\big(p^n(c_k)\big) = \lim_{n\to\infty}\varepsilon_{k,n}^2 = 1$$

by Lemma 6(d) and condition (vii).

According to (11),

$$\tilde{F}(z) = F(z)+F_{k,n}\circ\varphi_{k,n}\circ\beta\circ\varphi_{k,n}^{-1}(z)-F\circ\varphi_{k,n}\circ\beta\circ\varphi_{k,n}^{-1}(z) \quad (z \in B_k^n).$$

This implies

$$F_{k,n}^{-1}\circ\tilde{F}(z) = F_{k,n}^{-1}\circ F(z)+\varphi_{k,n}\circ\beta\circ\varphi_{k,n}^{-1}(z)- -F_{k,n}^{-1}\circ F\circ\varphi_{k,n}\circ\beta\circ\varphi_{k,n}^{-1}(z) \quad (z \in B_k^n) \tag{14.2}$$

because

$$\pi_2\circ F(t, x) = \pi_2\circ F_{k,n}(t, x) = Q_{\chi(p^n(c_k))}^{-1}(x) \quad \big((t, x) \in B_k^n\big),$$
$$\pi_2\circ\beta(t, x) = x \quad \big((t, x) \in B_k^n\big),$$

the first component of $F_{k,n}$ is affine and because, as it is easy to check, both sides coincide for $z \in \{p^n(c_k)\}\times A_k^n$. To deduce the lemma from (14.2) and (14.1), it is thus enough to prove the estimate

$$\sup_n\sup_{z\in B_k^n}\|D\varphi_{k,n}\circ\beta\circ\varphi_{k,n}^{-1}(z)\| < \infty.$$

For the norms taken over B or B_k^n it holds that

$$\begin{aligned}\|D\varphi_{k,n}\circ\beta\circ\varphi_{k,n}^{-1}\| &\leqslant \|D\varphi_{k,n}\|\cdot\|D\beta\|\cdot\|D^{-1}\varphi_{k,n}\|\\ &= \|D\varphi_{k,n}\|\cdot\|D^{-1}\varphi_{k,n}\| \quad \text{(by (10a, f))}\\ &= \|a_{k,n}^{-1}D\varphi_{k,n}\|\cdot\|a_{k,n}D^{-1}\varphi_{k,n}\|\\ &\leqslant \|a_{k,n}^{-1}D\psi_{k,n}\|\cdot\|D\psi_{k,n}^{-1}\circ\varphi_{k,n}\|\cdot\|a_{k,n}D^{-1}\psi_{k,n}\|\\ &\quad\times\|D^{-1}\psi_{k,n}^{-1}\circ\varphi_{k,n}\|.\end{aligned}$$

By Lemma 13, $\sup_n\|D\psi_{k,n}^{-1}\circ\varphi_{k,n}\|\cdot\sup_n\|D^{-1}\psi_{k,n}^{-1}\circ\varphi_{k,n}\|<\infty$. Also

$$\sup_n\|a_{k,n}^{-1}D\psi_{k,n}\|\cdot\sup_n\|a_{k,n}D^{-1}\psi_{k,n}\|<\infty$$

since

$$D\psi_{k,n}=a_{k,n}\begin{bmatrix}\pm d & 0\\ 0 & \pm 1\end{bmatrix}. \ \blacksquare$$

CLAIM 15. *The map f is a diffeomorphic embedding of class C^1.*

Proof. In view of Lemma 6(a)(c)(d) and Lemma 8 it is clear that f is continuous at any point of Z and hence, by Claim 11, continuous everywhere. Taking conditions (xiii) and (vii) into account we see that for some neighbourhood U of Z we have $f|U\setminus\bigcup_{k,n}B_k^n=F|U\setminus\bigcup_{k,n}B_k^n$. By Lemma 6(d) the set $\bigcup_{k,n}B_k^n\cup Z$ is closed, so

$$Df(z)=DF(z)\quad\text{for } z\in U\setminus(\bigcup_{k,n}B_k^n\cup Z).$$

By virtue of Claim 11 and the continuity of f, the goal is to prove that for any $z_0\in Z$ and for every sequence (z_n) tending to z_0 of elements of $\boldsymbol{R}^2\setminus Z$ the limit $\lim Df(z_n)$ exists. We will show that it equals $DF(z_0)$. To this end it suffices to consider the sequences with $z_n\in\operatorname{int}(B_{i_n}^{j_n})\setminus V_{i_n}^{j_n}$ for some i_n, j_n. We may assume that i_n is constant, $i_n=k$. By Lemma 6(c) we have $\lim j_n=\infty$.

Lemma 8, Lemma 6(b) and the definitions (9), (12) and (7), yield

$$f(z)=F_{k,n}\circ\varphi_{k,n}\circ\Psi\circ\varphi_{k,n}^{-1}\circ F_{k,n}^{-1}\circ\tilde F\circ\varphi_{k,n}\circ\Psi^{-1}\circ\varphi_{k,n}^{-1}(z)\quad(z\in B_k^n\setminus V_k^n).$$

We begin with the middle part of this formula. Denote $G_{k,n}=\varphi_{k,n}^{-1}\circ F_{k,n}^{-1}\circ\tilde F\circ\varphi_{k,n}$. Using the fact that

$$\psi_{k,n}^{-1}\circ F_{k,n}^{-1}\circ\tilde F\circ\psi_{k,n}(0,0)=(0,0)\quad\text{and}\quad D\psi_{k,n}(z)=\begin{bmatrix}\pm d & 0\\ 0 & \pm 1\end{bmatrix}$$

we deduce from Lemma 14 that

$$\lim_{n\to\infty}\psi_{k,n}^{-1}\circ F_{k,n}^{-1}\circ\tilde F\circ\psi_{k,n}=\mathrm{id}_B\quad\text{in the space } C^1(B,\boldsymbol{R}^2).$$

This and Lemma 13 imply

$$\lim_{n\to\infty}G_{k,n}=\mathrm{id}_B\quad\text{in } C^1(B,\boldsymbol{R}^2).$$

Also, observe that $G_{k,n}|U=\mathrm{id}_U$, where U is some neighbourhood of $\{0\}\times\{-\frac45,0,\frac45\}$ (Lemma 7(b)), and $G_{k,n}$ preserves the sign of the first coordinate. This along with conditions (xiv) and (xv) imply that the derivative

$$D\Psi\circ G_{k,n}\circ\Psi^{-1}(\Psi(z))=D\Psi(G_{k,n}(z))\circ DG_{k,n}(z)\circ D^{-1}\Psi(z)$$

tends uniformly to $\mathrm{id}_{B\setminus H_0}$. Since $\Psi\circ G_{k,n}\circ\Psi^{-1}(1,1)=(1,1)$ and $\Psi(B\setminus H_0)=B\setminus V_0$, it follows that

$$\lim_{n\to\infty}\Psi\circ G_{k,n}\circ\Psi^{-1}=\mathrm{id}_{B\setminus V_0}\quad\text{in } C^1(B\setminus V_0,\boldsymbol{R}^2).$$

From this and Lemma 13, by expressing $\varphi_{k,n}$ in the form $\psi_{k,n}\circ(\psi_{k,n}^{-1}\circ\varphi_{k,n})$, we infer that

$$\lim_{n\to\infty}DF_{k,j_n}^{-1}\circ f(z_n)=\lim_{n\to\infty}D\varphi_{k,j_n}\circ\Psi\circ\varphi_{k,j_n}^{-1}\circ F_{k,j_n}^{-1}\circ\tilde F\circ\varphi_{k,j_n}\circ\Psi^{-1}\circ\varphi_{k,j_n}^{-1}(z_n)=1.$$

Since

$$Df(z_n)=DF_{k,j_n}^{-1}(F_{k,j_n}^{-1}\circ f(z_n))\circ DF_{k,j_n}^{-1}\circ f(z_n),$$

the proof reduces to showing that

$$(*)\qquad\lim_{n\to\infty}DF_{k,j_n}(F_{k,j_n}^{-1}\circ f(z_n))=DF(z_0).$$

Let $u_n=\big(p^{j_n}(c_k),\pi_2(F_{k,j_n}^{-1}\circ f(z))\big)$. We have $F_{k,n}^{-1}\circ f(B_k^n)\subseteq\boldsymbol{R}\times A_k^n$ for every n and $p^{j_n}(c_k)=\pi_1(z_0)$ for every sufficiently great n. Also, $DF_{k,j_n}(F_{k,j_n}^{-1}\circ f(z_n))=DF_{k,j_n}(u_n)$ since $DF_{k,j_n}(z)$ is independent of the first coordinate of z. Thus, the partial derivatives of F_{k,j_n} at $F_{k,j_n}^{-1}\circ f(z_n)$ and of F at u_n are equal except one:

$$(\partial(\pi_1\circ F)/\partial x_1)(u_n)=p'(p^{j_n}(c_k))\quad\text{and}$$
$$(\partial(\pi_1\circ F_{k,j_n})/\partial x_1)(F_{k,j_n}^{-1}\circ f(z_n))=a_{k,j_n+1}a_{k,j_n}^{-1}\operatorname{sign}\big(p'(p^{j_n}(c_k))\big).$$

Now $(*)$ follows from conditions (vii) and (xii). $\blacksquare$

Part 6. The quasiattractor. In this part we prove conclusion (a) of Theorem B.

Define Λ to be $\bigcap_{n=0}^{\infty}f^n(\boldsymbol{R}^2)$.

CLAIM 16. *The set Λ is a quasiattractor of f and $\boldsymbol{R}^2$ is its domain of attraction.*

Proof. Since f is continuous (Claim 15), we only need to show that $\mathrm{cl}(f(\boldsymbol{R}^2))$ is compact, that is, $f(\boldsymbol{R}^2)$ is bounded. Since $\mathrm{cl}(G(\boldsymbol{R}^2))=\boldsymbol{R}^2$ this reduces to boundedness of the set $f\circ G(\boldsymbol{R}^2)$ which, by Lemma 9, equals $G\circ\tilde F(\boldsymbol{R}^2)$. By Lemma 8 we have $\tilde F(\boldsymbol{R}^2)=F(\boldsymbol{R}^2)=p(\boldsymbol{R})\times Q^{-1}(\boldsymbol{R})$. It follows from condition (viii) that the last set is bounded, and so is its image under G since G displaces points within the sets B_k^n only. $\blacksquare$

We define K to be $\bigcap_{n=0}^{\infty}F^n(\boldsymbol{R}^2)$ and $\tilde K$ to be $\bigcap_{n=0}^{\infty}\tilde F^n(\boldsymbol{R}^2)$.

LEMMA 17. $K = \tilde{K}$.

Proof. We will prove the following sequence of equalities:

$$K = \bigcap_{n=0}^{\infty} F^n(R \times X) = \bigcap_{n=0}^{\infty} \tilde{F}^n(R \times X) = \tilde{K}.$$

The second one follows from Lemma 7(d). By definition, F is one-to-one and $K \subseteq R \times \bigcap_{n=0}^{\infty} Q^{-n}(R) = R \times X$, so we have the first. Lemma 8 implies that $\tilde{F}$ is also one-to-one, Lemma 7(a) implies that $\tilde{K} \subseteq R \times \bigcap_{n=0}^{\infty} Q^{-n}(R)$, and these facts together give the third equality. ■

LEMMA 18. *The map G takes K onto Λ.*

Proof. We will verify the following sequence of equalities:

$$G(K) = G(\tilde{K}) = G\Big(\bigcap_{n=0}^{\infty} \tilde{F}^n(R^2)\Big) = \bigcap_{n=0}^{\infty} G(\tilde{F}^n(R^2))$$

$$= \bigcap_{n=0}^{\infty} f^n(G(R^2)) = \bigcap_{n=0}^{\infty} f^n(R^2) = \Lambda.$$

The first equality follows from Lemma 17. The third one holds since the sets $\tilde{F}^n(R^2)$ form a descending sequence and each element has a finite number of preimages under G (at most two). The fourth holds because $G \circ \tilde{F}^n = f^n \circ G$ by Lemma 9. The fifth equality holds because $R^2 = G(R^2) \cup V$, $f(G(R^2)) \subseteq G(R^2)$ (since $G \circ \tilde{F} = f \circ G$), $f(V) \subseteq V$ by (12) and $\bigcap_{n=0}^{\infty} f^n(V) = \emptyset$ by (12). ■

LEMMA 19. *There exists a continuous map ψ: $R^2 \to R^2$ such that $\psi \circ G = \pi_1$.*

Proof. Define

$$\psi(z) = \begin{cases} p^n(c_k) & \text{for } z \in V_k^n, \\ \pi_1 \circ G^{-1}(z) & \text{for } z \in R^2 \setminus V. \end{cases}$$

If $z \in H_k^n$ then $G(z) \in V_k^n$, so $\psi(z) = p^n(c_k) = \psi \circ G(z)$. If $z \in R^2 \setminus H$ then $G(z) \in R^2 \setminus V$, so $\psi \circ G(z) = \pi_1 \circ G^{-1} \circ G(z) = \pi_1(z)$. Now we will check the continuity. If $U \subseteq R \setminus D$ is an open set (recall that $D = \bigcup_{k,n} \{p^n(c_k)\}$) then the preimage $\psi^{-1}(U) = G \circ \pi_1^{-1}(U) = G(U \times R)$ is also open since G settles a homeomorphism between the open sets $R^2 \setminus (H \cup Z)$ and $R^2 \setminus (V \cup Z)$, and $H \cup Z \subseteq D \times R$. Now let $y \in D$ and $U = (a, b)$ where $a < y < b$ and $b - a < d$ (see (3)). We will show that the preimages $\psi^{-1}((-\infty, a])$ and $\psi^{-1}([b, \infty))$ are closed. Let $c = y - d$. Then, by (3), $[c, a] \subseteq R \setminus D$ whence $\psi^{-1}([c, a]) = G([c, a] \times R)$. Since

$$G|(R \setminus (-1, 2)) \times R = \mathrm{id}$$

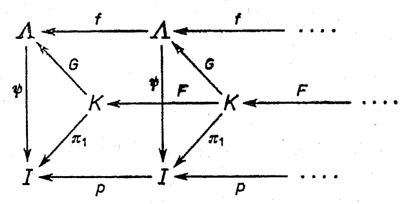
and the set $[c, a] \times [-1, 2]$ is compact, it follows that $\psi^{-1}([c, a])$ is a closed set. Also, $\psi^{-1}((-\infty, c]) = (-\infty, c] \times R$ since

$$\psi^{-1}((-\infty, c]) = G((-\infty, c] \times R \setminus H) \cup \bigcup_{(k,n);\, p^n(c_k) < y} V_k^n$$

and since, by (3), for each rectangle B_k^n we have either $B_k^n \subseteq (-\infty, c] \times R$ or $B_k^n \cap ((-\infty, c] \times R) = \emptyset$. Thus, the preimage $\psi^{-1}((-\infty, a])$ is closed. Similarly, $\psi^{-1}([b, \infty))$ is closed. ■

CLAIM 20. *The dynamical system (f, Λ) is conjugate to the inverse limit of (p, I).*

Proof. Take a continuous map ψ such that $\psi \circ G = \pi_1$ and consider the following infinite diagram:



By Claim 16, Λ is a compact set and f: $\Lambda \to \Lambda$ is a homeomorphism. The map F: $K \to K$ is one-to-one and onto since F is one-to-one. By Lemma 18, the map G: $K \to \Lambda$ is onto. By condition (viii), π_1 does map K into I, therefore ψ takes Λ into I indeed.

We will check that the diagram commutes. We have $\psi \circ G = \pi_1$, also $\pi_1 \circ F = p \circ \pi_1$ by (1) and $G \circ F = f \circ G$ by Lemmata 9 and 7(d). Hence

$$\psi \circ f \circ G = \psi \circ G \circ F = \pi_1 \circ F = p \circ \pi_1 = p \circ \psi \circ G,$$

but G is surjective, so $\psi \circ f = p \circ \psi$.

Since $\psi \circ f = p \circ \psi$, the formula

$$(S(z))_n = \psi \circ f^{-n}(z)$$

defines a function S: $\Lambda \to \varprojlim(p, I)$ satisfying $S \circ f = \hat{p} \circ S$. We claim that S conjugates (f, Λ) to $(\hat{p}, \varprojlim(p, I))$, and it remains to prove that S is a homeomorphism. By compactness of Λ and continuity of S, it suffices to show that S is one-to-one and onto.

In order to show that S is onto take an element $u = (u_0, u_1, u_2, \ldots) \in \varprojlim(p, I)$ and let $x_n = [\chi(u_{n+1}), \chi(u_{n+2}), \ldots]$ for $n \in N$. Observe that by definition $F(u_n, x_n) = (u_{n-1}, x_{n-1})$ for $n = 1, 2, \ldots$, and consequently, $(u_n, x_n) \in K$ for each n. Hence $u = S(G(u_0, x_0))$ by commutativity of the diagram.

To show that S is one-to-one suppose that $a, a' \in \Lambda$ and $S(a) = S(a')$ and let (t, x), $(t', x') \in K$ satisfy $a = G(t, x)$ and $a' = G(t', x')$. For any $n \in N$ we have

$\pi_1 \circ F^{-n}(t, x) = \pi_1 \circ F^{-n}(t', x')$. Hence $t = \pi_1(t, x) = \pi_1(t', x') = t'$ and, by the identity

$$F(u, [v_0, v_1, \ldots]) = (p(u), [\chi(u), v_0, v_1, \ldots]),$$

$$x = [\chi \circ \pi_1 \circ F^{-1}(t, x), \chi \circ \pi_1 \circ F^{-2}(t, x), \ldots] = [\chi \circ \pi_1 \circ F^{-1}(t', x'), \chi \circ \pi_1 \circ F^{-2}(t', x'), \ldots] = x'.$$

Thus, $(t, x) = (t', x')$ and so $a = a'$. ■

Note that if it was assumed in Part 2 that $\mathrm{sign}(Q_k') = \mathrm{sign}(p')|I_k$ then F preserves orientation, and otherwise F reverses orientation. By Lemma 8 and since $F|\mathrm{bd}(B_k^n) = \tilde{F}|\mathrm{bd}(B_k^n)$, the same is true for $\tilde{F}$ and so, by (12), also for f. This concludes the proof of Theorem B.

Proof of Theorem A. Let $g\colon M \to M$ be a diffeomorphism of a two-dimensional manifold M. Take a map $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ satisfying the conclusion of Theorem B. If M is orientable, assume that f and g both preserve, or both reverse, orientation. Denote $\Lambda = \bigcap_{n=0}^{\infty} f^n(\mathbf{R}^2)$, let D be a C^∞ disk such that $\Lambda \subseteq D$ and $f(D) \subseteq \mathrm{int}(D)$, and take a C^∞-embedding $h\colon D \to \mathrm{int}(M)$. By condition (b), $f|D$ can be approximated in the C^1 topology by a C^∞ diffeomorphism $f_1\colon D \to f(D)$ which coincides with f in a neighbourhood of $\mathrm{bd}(D)$. Let f_1 be close to f enough to be isotopic with f via the linear parametrization. Then, the C^1-diffeomorphism $g_1\colon M \to M$ satisfying $g_1 \circ h \circ f_1 = h \circ f|D$ and $g_1|M \setminus h \circ f(D) = \mathrm{id}$, is diffeotopic to the identity. If M is orientable then $g \circ h$ preserves orientation if and only if $h \circ f_1$ does. Hence, by Th. 3.1 of Ch. 8 of [1], there is a C^∞-diffeomorphism $g_2\colon M \to M$, diffeotopic to the identity, such that $g_2 \circ g \circ h = h \circ f_1$. The C^1-diffeomorphism $\varphi = g_1 \circ g_2 \circ g$ is thus diffeotopic to g, of class C^∞ outside $h(Z)$, and such that $\varphi \circ h = h \circ f|D$. ■

4. Appendix. We will show that in our construction the diffeomorphism $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ is not of class C^2 unless $C = \emptyset$, that is, p is monotonic. More precisely, the derivative Df is not Lipschitz continuous in any neighbourhood of Z. To verify this, first observe that, for some neighbourhoods U_n of $\varphi_{1,n}(\mathrm{bd}(B))$ we have $f|U_n = F|U_n$, and hence

$$\partial\pi_1 \circ f/\partial x_1(\varphi_{1,n}(0, 1)) = \partial\pi_1 \circ F/\partial x_1(\varphi_{1,n}(0, 1)) = \varepsilon_{1,n}$$

for every sufficiently great n. On the other hand we have $f(V_1^n) = V_1^{n+1}$, so there exist points $z_n \in B_1^n$ with $\varepsilon_{1,n}(\partial\pi_1 \circ f/\partial x_1)(z_n) = a_{1,n+1} a_{1,n}^{-1}$. Now suppose that

$$\|Df(z_n) - Df(\varphi_{1,n}(0, 1))\| < L \cdot \mathrm{dist}(z_n, \varphi_{1,n}(0, 1))$$

for each n. Since $\mathrm{diam}(B_1^n) < 3a_{1,n}$ this implies $(1 - a_{1,n+1} a_{1,n}^{-1}) \leqslant 3La_{1,n}$, and hence we have $a_{1,n} > Kn^{-1}$ for some $K > 0$. It follows that the series $\sum_{n=1}^{\infty} a_{1,n}$ is divergent, and this is impossible since the rectangles B_1^n have disjoint interiors and each of them intersects the line $\{p^n(c_1)\} \times \mathbf{R}$ along a segment of length $a_{1,n}$. ■

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