that M_{n+1} is obtained from M_n by placing a Whitehead embedding in each component of M_n. Let $T_1, T_2, \ldots, T_k$ be the components of M_n. Choose disjoint open 3-balls $U_1, U_2, \ldots, U_k$ in 3-space so that T_i is contained in U_i as an unknotted solid torus. The loop γ contracts in 3-space. By general position we may assume that it bounds a singular disk so that for each i, $1 \leqslant i \leqslant k$, the singular disk bounded by γ meets T_i in a finite collection of meridional disks. However, a meridian of $\mathrm{Bd}\,T_i$ bounds a singular disk in $U_i - M_{n+1}$ [Wh]. Hence, γ bounds a singular disk in the complement of M_{n+1}, and our theorem is proved.

References

A] L. Antoine, *Sur l'homeomorphie de deux figures et de leur voisinages*, J. Math. Pures Appl. 86 (1921), 221–325.

[A–S] F. D. Ancel and M. P. Starbird, *The shrinkability of Bing–Whitehead decompositions*, to appear in Topology.

[B] R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. (2) 56 (1952), 354–362.

[C] T. A. Chapman, *Lectures on Hilbert Cube Manifolds*, (Amer. Math. Soc. CBMS regional conference series), No. 28, Amer. Math. Soc., Providence, RI, 1976.

[D] R. J. Daverman, *Embedding Phenomena based upon decomposition theory: wild Cantor sets satisfying strong homogeneity properties*, Proc. Math. Soc. 75 (1979), 177–182.

[D–E] R. J. Daverman and R. D. Edwards, *Wild Cantor sets as approximations to codimension two manifolds*, Topology Appl. 26 (1987), 207–218.

[D–O] D. G. De Gryse and R. P. Osborne, *A wild Cantor set in E^n with simply connected complement*, Fund. Math. 86 (1974), 9–27.

[Wh] J. H. C. Whitehead, *A certain open manifold whose group is unity*, Quart. J. Math. 6 (1935), 268–279.

[Wr] D. G. Wright, *Rigid sets in E^n*, Pacific J. Math. 121 (1986), 245–256.

DEPARTMENT OF MATHEMATICS
BRIGHAM YOUNG UNIVERSITY
Provo, Utah 84602

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Polynomial growth trivial extensions of simply connected algebras

by

Jerzy Nehring and Andrzej Skowroński (Toruń)

Abstract. Let A be a finite-dimensional, basic, connected algebra over an algebraically closed field. Denote by $T(A)$ the trivial extension of A by its minimal injective cogenerator. We show that, if A is simply connected, then the following conditions are equivalent: (i) $T(A)$ is nondomestic of polynomial growth, (ii) $T(A)$ is nondomestic of finite growth, (iii) there exists a tubular algebra B such that $T(A) \simeq T(B)$, (iv) A is tilting-cotilting equivalent to a canonical tubular algebra. Isomorphism classes of such algebras are also determined.

Introduction. Let K denote a fixed algebraically closed field, and A a finite-dimensional K-algebra (associative, with an identity) which we shall assume to be basic and connected. We shall denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules. We recall that A is called *simply connected* (in the sense of [2]) if it is triangular, that is, the ordinary quiver of A has no oriented cycles, and such that, for any presentation $A \simeq KQ/I$ of A as a bound quiver algebra, the fundamental group $\pi(Q, I)$ of (Q, I) [18] is trivial. In the representation-finite case, this notion of simple connectedness coincides with the notion introduced in [6]. Further, A is called *domestic* [20] if there exists a finite number of (parametrising) functors $F_i\colon \operatorname{mod} K[X] \to \operatorname{mod} A$, $1 \leqslant i \leqslant n$, where $K[X]$ is the polynomial algebra in one variable, satisfying the following conditions:

(a) For each i, $F_i = - \underset{K[X]}{\otimes} Q_i$, where Q_i is a $K[X]$-A-bimodule which is finitely generated and free as a $K[X]$-module.

(b) For any dimension d, all but a finite number of isomorphism classes of indecomposable A-modules of K-dimension d are of the form $F_i(M)$, for some i and some indecomposable right $K[X]$-module M.

A is called *n-parametric* if the minimal number of such functors is n. Moreover, for a dimension d, denote by $\mu_A(d)$ the least number of functors $F_i\colon \operatorname{mod} K[X] \to \operatorname{mod} A$, $1 \leqslant i \leqslant \mu_A(d)$, satisfying the above condition (a) and the following condition:

(b′) All but a finite number of isomorphism classes of indecomposable A-modules of K-dimension d are of the form $F_i(S)$ for some i and some simple right $K[X]$-module S.

Then A is *tame* (in the sense of [11]) if $\mu_A(d) < \infty$ for every d. Following [22], A is called of *polynomial growth* if there exists a natural number m such that, for every dimension $d \geqq 2$, $\mu_A(d) \leqq d^m$. Finally, A is of *finite* (*linear*) *growth* if there exists a natural number n such that $\mu_A(d) \leqslant nd$, for every $d \geqslant 1$. It follows from [9] that, if A is domestic, then A is of finite growth.

Recall from [14, 21] that a module T_A is called a *tilting* (resp. *cotilting*) module provided: $\mathrm{Ext}_A^2(T_A, -) = 0$ (resp. $\mathrm{Ext}_A^2(-, T_A) = 0$), $\mathrm{Ext}_A^1(T_A, T_A) = 0$ and the number of nonisomorphic indecomposable direct summands of T_A equal the rank of the Grothendieck group $K_0(A)$ of A. Two algebras A and B are called *tilting-cotilting equivalent* [3] if there exists a sequence of finite-dimensional K-algebras $A = A_0, A_1, \ldots, A_m, A_{m+1} = B$ and a sequence of modules $T^i_{A_i}$, $0 \leqslant i \leqslant m$, such that $A_{i+1} = \mathrm{End}(T^i_{A_i})$ and $T^i_{A_i}$ is either a tilting or cotilting module. It is shown in [2] that if A is tilting-cotilting equivalent to a hereditary algebra of Dynkin type or a hereditary algebra of Euclidean type $\tilde{D}_n$ or $\tilde{E}_p$ or one of Ringel's [21] tame canonical tubular algebra, then A is simply connected (in the above sense).

The *trivial extension* $T(A)$ of A by its minimal injective cogenerator bimodule $D(A) = \mathrm{Hom}_K(A, K)$ is the algebra whose additive structure is that of the group $A \oplus DA$, and whose multiplication is defined by:

$$(a, f)(b, g) = (ab, ag + fb)$$

for $a, b \in A$ and $f, g \in {}_A D(A)_A$. It is known that $T(A)$ is selfinjective, and in fact even symmetric. Trivial extensions have been extensively investigated in representation theory (see [4] for the corresponding references). It is known [17], [1] that for an algebra A the following conditions are equivalent: (i) $T(A)$ is representation-finite, (ii) there exists a tilted algebra B of Dynkin type such that $T(A) \simeq T(B)$, (iii) A is tilting-cotilting equivalent to a hereditary algebra of Dynkin type In particular, if $T(A)$ is representation-finite, then A is (representation-finite) simply connected (see also [26]). Recently, the authors have proved with I. Assem [4] that, for a simply connected algebra A, the following conditions are equivalent: (i) $T(A)$ is representation-infinite domestic, (ii) $T(A)$ is 2-parametric, (iii) there exists a representation-infinite tilted algebra B of Euclidean type $\tilde{D}_n$ or $\tilde{E}_p$ such that $T(A) \simeq T(B)$, (iv) A is tilting-cotilting equivalent to a hereditary algebra of Euclidean type $\tilde{D}_n$ or $\tilde{E}_p$.

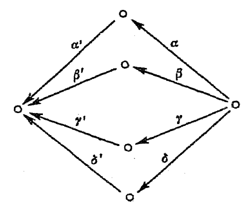
The main objective in this article is to present the following characterisation of all nondomestic polynomial growth trivial extensions of simply connected algebras.

THEOREM 1. *Let A be a finite-dimensional, basic and connected algebra over an algebraically closed field K. If A is simply connected, then the following conditions are equivalent*:

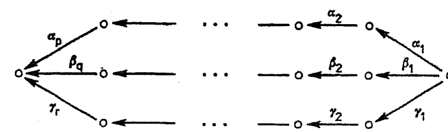
(i) *$T(A)$ is nondomestic of polynomial growth.*
(ii) *$T(A)$ is nondomestic of finite growth.*
(iii) *There exists a tubular algebra B such that $T(A) \simeq T(B)$.*
(iv) *A is tilting-cotilting equivalent to a canonical tubular algebra.*

The trivial extensions of canonical tubular algebras have been described in [15]. In a forthcoming paper we shall show that the polynomial growth trivial extensions $T(A)$ of nonsimply connected algebras A are domestic. Thus our theorem gives a complete characterisation of all nondomestic trivial extensions of polynomial growth.

Recall that following Ringel [21], a canonical algebra of type $(2, 2, 2, 2)$ is given by the quiver



bound by $\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0$, $\alpha\alpha' + \lambda \cdot \beta\beta' + \delta\delta' = 0$, where $\lambda \in K \setminus \{0, 1\}$ and a canonical algebra of type (p, q, r), $p \leqslant q \leqslant r$, is given by the quiver

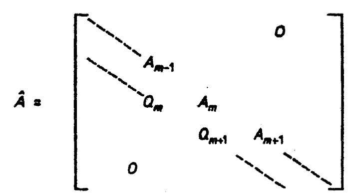


bound by $\alpha_1 \ldots \alpha_r + \beta_1 \ldots \beta_q + \gamma_1 \ldots \gamma_r = 0$.

If (p, q, r) equals $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$, that is $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, or is of type $(2, 2, 2, 2)$ the algebra is called *tubular canonical.*

In the proof we shall use freely results from [2], [3], [4], tilting theory [14], [21], Ringel's theory of tubular algebras [20], [21], and covering techniques developed recently in [8], [10].

§ 1. Preliminaries. Let A be a finite-dimensional algebra. Its *repetitive algebra* $\hat{A}$ is the selfinjective, locally finite-dimensional algebra [17]



in which matrices have finitely many non-zero entries, $A_m = A$, $Q_m = {}_AD(A)_A$ for all $m \in Z$, all the remaining entries are zero, and multiplication is induced from the bimodule structure of $D(A)$ and the zero map $D(A) \underset{A}{\otimes} D(A) \to 0$. The identity maps $A_m \to A_{m+1}$, $Q_m \to Q_{m+1}$ induce an automorphism ν of $\hat{A}$, called the *Nakayama automorphism*, and thus $\hat{A}$ is a Galois covering [6, 13] of $T(A)$ with the infinite cyclic group generated by ν. We say that the algebra $\hat{A}$ is of *polynomial growth* if any full finite subcategory of $\hat{A}$, considered as an algebra, is of polynomial growth. Moreover, following [8], $\hat{A}$ is called *locally support-finite* if, for each object x of $\hat{A}$, the full subcategory of $\hat{A}$ formed by all objects of the support $\operatorname{Supp} M$, where M ranges through all indecomposable finite-dimensional A-modules such that $M(x) \neq 0$, is finite. Then we have the following consequences of [8] (see also [22]).

PROPOSITION 1. *If $T(A)$ is of polynomial growth (resp. domestic) so is $\hat{A}$.*

PROPOSITION 2. *Assume that $\hat{A}$ is locally support-finite and of polynomial (resp. finite) growth. Then $T(A)$ is of polynomial (resp. finite) growth.*

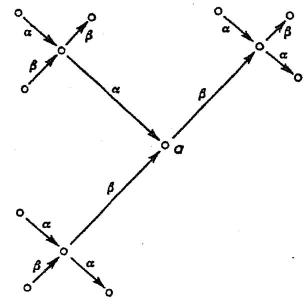
For a locally finite-dimensional K-algebra we shall denote by Q_A its ordinary quiver and by $(Q_A)_0$ the set of vertices of Q_A. For $i \in (Q_A)_0$ we denote by e_i the corresponding idempotent of A, and by $S_A(i)$ the corresponding simple A-module. We shall denote by $P_A(i)$ (resp. $I_A(i)$) the projective cover (resp. injective envelope) of $S_A(i)$.

The one-point extension (resp. coextension) of an algebra A by an A-module M will be denoted by $A[M]$ (resp. $[M]A$). In order to handle module over one-point extensions, we shall use vector space category methods, for which we refer to [20], [21]. Let A be a triangular algebra, and i be a sink of Q_A. The *reflection* $S_i^+ A$ of A at i is the quotient of the one-point extension $T_i^+ A = A[I(i)]$ by the two-sided ideal generated by e_i [17]. Dually, starting with a source j, we define the reflection $S_j^- A$. Clearly, the repetitive algebras of A and $S_i^+ A$ are isomorphic. Also it is shown in [24] that A and $S_i^+ A$ are tilting-cotilting equivalent. Moreover, by [25] $T(A) \simeq T(S_i^+ A)$. The quiver of $S_i^+ A$ is denoted by $\sigma_i^+ Q_A$ and is called a *ν-reflection* of Q_A. The sink i of Q_A is replaced in $\sigma_i^+ Q_A$ by a source i'. A *ν-reflection sequence* of sinks $i_1, \dots, i_t$ is a sequence of vertices of Q_A such that i_s is a sink of $\sigma_{i_{s-1}}^+ \dots \sigma_{i_1}^+ Q_A$ for $1 \leqslant s \leqslant t$.

Finally, we shall denote by τ_A the *Auslander–Reiten translate* [12] in $\operatorname{mod} A$, by τ_A^{-1} its inverse and by Γ_A the Auslander–Reitan quiver of A.

§ 2. Branch enlargements. We first recall from [3] the notion of branch enlargements. An *extension branch* K in a vertex a, called its *root*, is a finite connected full bound subquiver of the following infinite tree, consisting of two types of arrows: the α-arrows and the β-arrows, and bound by all possible relations of the forms

$\alpha\beta = 0$, $\beta\alpha = 0$:



A *coextension branch* K in a is defined dually (reversing all arrows in the figure). The number of vertices in a branch K is called its length and is denoted by $|K|$. We shall agree to consider the empty quiver as a branch of length zero.

Let $A = KQ/I$ be a bound quiver algebra, and (Q', I') be a full bound subquiver of (Q, I) with a source a. Then A is said to be obtained from KQ'/I' by rooting an extension branch (Q'', I'') in a provided that (Q'', I'') is a full bound subquiver of (Q, I) such that:

(1) $Q'_0 \cap Q''_0 = \{a\}$, $Q'_0 \cup Q''_0 = Q_0$.

(2) I is generated by I', I'' and all paths $\beta\gamma$ where $\beta \in Q''_1$ has target a, and $\gamma \in Q'_1$ has source a.

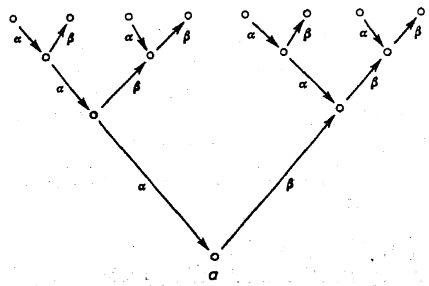
Let C be a tame concealed algebra [21] with a tubular family $(T_\lambda)_{\lambda \in P_1(K)}$, and let $E_1, \dots, E_t$ be pairwise non-isomorphic simple regular C-modules. For each $1 \leqslant i \leqslant t$, we let K_i be an extension branch in a_i, and K'_i be a coextension branch in a'_i, where either K_i or K'_i may be empty. We shall define inductively the *branch enlargement* A of C by the extension branches K_i and the coextension branches K'_i. The algebra $C[E_1, K_1]$ is obtained from the one-point extension $C[E_1]$ with extension vertex a_1 by rooting the branch K_1 in a_1, and, for $1 < j \leqslant t$, $C[E_i, K_i]_{i=1}^{j}$ is obtained from the one-point extension $C[E_i, K_i]_{i=1}^{j-1}[E_j]$ with extension vertex a_j by rooting the branch K_j in a_j. Then $B = C[E_i, K_i]_{i=1}^{t}$ is called the *branch extension* of C at the modules E_i by the extension branches K_i $(1 \leqslant i \leqslant t)$. We now let E'_i be the unique indecomposable B-module whose restriction to C is E_i and whose restriction to K_i is the unique indecomposable module with support consisting of all x in K_i such that there is a non-zero path from x to the root of K_i. Then $[E'_1, K'_1]B$ is obtained from the one-point coextension $[E'_1]B$ with coextension vertex a'_i by rooting K'_1 in a'_1, and, for $1 < j \leqslant t$, ${}_{i=1}^{j}[E'_i, K'_i]B$ is obtained from $[E'_j]_{i=1}^{j-1}[E'_i, K'_i]B$ with coextension vertex a'_j by rooting K'_j in a_j. Then $A = {}_{i=1}^{t}[E'_i, K'_i]B$ is the required branch enlargement of C.

Let A be a branch enlargement of C, and let r_λ denote the rank of the tube T_λ, $\lambda \in P_1(K)$, of Γ_C. The *tubular type* $n_A = (n_\lambda)_{\lambda \in P_1(K)}$ of A is defined by

$$n_\lambda = r_\lambda + \sum_{E_i \in T_\lambda} (|K_i| + |K_i'|)$$

We shall write, instead of $(n_\lambda)_{\lambda \in P_1(K)}$, the finite sequence consisting of at least two n_λ, keeping all those which are larger than 1, and arranging them in non-decreasing order. A branch enlargement A of C is called *domestic* (resp. *tubular*) if n_A is one of the following: (p, q), $1 \leqslant p \leqslant q$, $(2, 2, r)$, $2 \leqslant r$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$ (resp. $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$ or $(2, 2, 2, 2)$).

A *truncated branch* in a (branch in the sense of [21]) is a finite connected full bound subquiver, containing a, of the following infinite tree bound by all possible relations $\alpha\beta = 0$:



If $K_1, \ldots, K_t$ are truncated branches, then the branch extension $B = C[E_i, K_i]_{i=1}^t$ is a tubular extension in the sense of [21]. If n_B is tubular, then B is called a *tubular algebra* [21]. Moreover, if n_B is domestic, then B is a tilted algebra of Euclidean type having a complete slice in its preinjective component, and conversely, every representation-infinite tilted algebra of Euclidean type is either a domestic truncated branch extension or a domestic truncated branch coextension of a tame concealed algebra [21, 4.9].

In our proof we shall use the following lemma.

LEMMA 2.1. *Let A be a truncated branch extension of the tame concealed algebra C. Then the following conditions are equivalent*:

(i) *A is a tubular algebra.*
(ii) *A is nondomestic of finite growth.*
(iii) *A is nondomestic and tame.*

Proof. It follows from [21, 5.2, Theorem 6] that, if A is a tubular algebra, then A is nondomestic of linear growth. Thus (i) implies (ii), and obviously (ii) implies (iii). Moreover, from [21, 4.9] and [4, 2.3], we know that A is domestic if and only if n_A is domestic. Suppose that n_A is neither domestic nor tubular. We shall show that then A is wild. Let B be a full bound subquiver of A containing C, minimal for the property that n_B is neither domestic nor tubular. We shall consider two cases. First assume that there is a simple (has only one neighbour in Q_B) source or sink x of a branch K of B such that the tubular type of the full subcategory $D = B(x)$ of B formed by all objects of B except x is domestic. We claim that x may be assumed to be a source. Indeed, if it is not the case, let a be root of the branch K of B and d denote the maximal distance from a to a vertex in K. If K contains a source i such that the distance from a to i equals d, then we replace x by i. If K contains no such source, let j be an arbitrary vertex of K (thus, a sink) whose distance to a equals d. Since K is truncated branch, j is not the terminal point of a zero-relation in K. Let D^* be the full subcategory of B formed by all objects of B except j. Then B is a one-point coextension of D^* with the coextension vertex j. Applying the APR-tilting module $U_B = \tau_B^{-1}(S_B(j)) \oplus \bigoplus_{i \neq j} P_B(i)$ [5] at j, we obtain an algebra $B^* = \operatorname{End}(U_B)$, which is a one-point extension of D^* and a truncated branch extension of C. Moreover, $n_{D^*} = n_D$ is domestic, and, by [16], B is tame if and only if B^* is. This proves our claim. Thus $B = D[M]$ with extension vertex x. Then D is a tilted algebra of Euclidean type having a complete slice S in its preinjective component. Let T_D be the slice module of S and $H = \operatorname{End}(T_D)$. We want to show that the full subcategory $\mathscr{U}$ of the vector space category $\operatorname{Hom}_D(M, \operatorname{mod} D)$ formed by all objects of the form $\operatorname{Hom}_D(M, X)$ where X is an indecomposable preinjective which is a proper predecessor of S, is wild. Let $N_H = \operatorname{Ext}_B^1(T, M)$. Since M is a regular D-module [21], N_H is a regular H-module. It follows directly by the Brenner–Butler theorem [14] that $\mathscr{U}$ is equivalent to the full subcategory $\mathscr{V}$ of the vector space category $\operatorname{Hom}_H(N_H, \operatorname{mod} H)$ formed by all objects of the form $\operatorname{Hom}_H(N, Y)$ where Y_H is indecomposable preinjective. By [20, 3.5], $\mathscr{V}$ is wild and consequently B is wild.

Now assume that for any simple sink or source of a branch of B the full subcategory $B(x)$ is a tubular algebra. As above we can assume that there is a simple source, say y, in a branch K of B such that the distance from y to the root a of K is the maximal distance from a to a vertex of K. Let $D = B(y)$ and $y \xrightarrow{\alpha} b$ be the unique arrow in Q_B starting at y. We have two cases: y is not a starting point of a zero-relation or there is an arrow $b \xrightarrow{\beta} c$ such that $\alpha\beta = 0$, and obviously c is a simple sink of B. In the former case $M = P_D(b)$ and the latter case $M \simeq P_D(b)/S_D(c) \simeq \tau_D^{-1}(S_D(c))$. We claim that the vector space category $\operatorname{Hom}_D(M, \operatorname{mod} D)$ contains an object $\operatorname{Hom}_D(M, X)$ such that $\operatorname{End}_D(X) \simeq K$ and $\dim_K \operatorname{Hom}_D(M, X) \geqslant 3$. This implies by [20, 2.4] that $B = D[M]$ is wild. By our assumption D is a tubular extension of $C = C_0$, and hence from [21, 5.2, Theorem 3] D is also a tubular coextension of a tame concealed algebra C_1. Moreover, the preinjective component Q_1 of C_1 is also a preinjective component of D. Since all vertices of K are not objects of C, they are, by [21, 5.1] objects of C_1. Thus every object of K belongs to the support of all but a finite number of indecomposable modules from Q_1. Hence, if $M = P_D(b)$ then there is an indecomposable module $X \in Q_1$ such that $\dim_K \operatorname{Hom}_D(M, X) \geqslant 3$ and obviously $\operatorname{End}_D(X) \simeq K$, since Q_1 is a preinjective component of D. Similarly, if $M = \tau_D^{-1} S_D(c))$ then there is an indecomposable module

$Y \in Q_1$ such that $\tau_D^{-1} Y \neq 0$ and $\dim_K \mathrm{Hom}_D(S_D(c), Y) \geqslant 3$. Then, for $X = \tau_D^{-1} Y$, $\mathrm{Hom}_D(M, X) \simeq \mathrm{Hom}_D(\tau_D(M), \tau_D(X)) \simeq \mathrm{Hom}_D(S_D(c), Y)$ and we are done.

§ 3. **Proof of Theorem 1.** Obviously, (ii) implies (i). We shall show that (iv) implies (iii). Assume that A is tilting-cotilting equivalent to a canonical tubular algebra Λ.

Since A is simply connected and not tilting-cotilting equivalent to a hereditary algebra of Dynkin type, there exists [4, Corollary 3.4] a ν-reflection sequence of sinks $i_1, \ldots, i_t$ such that $D = S_{i_t}^+ \ldots S_{i_1}^+ A$ is representation-infinite and clearly $T(A) \simeq T(D)$ (take $D = A$ if A is representation-infinite). By [24], D is tilting-cotilting equivalent to Λ and consequently by [3, Theorem 2.5] D is a branch enlargement of tame concealed algebra C with $n_D = n_\Lambda$. Then by [4, Proposition 2.6] there exists a truncated branch extension B of C such that $T(D) \simeq T(B)$ and $n_D = n_B$. In fact, $B = S_{j_r}^+ \ldots S_{j_1}^+ D$ for a ν-reflection sequence of sinks $j_1, \ldots, j_r$ in Q_D. Therefore B is a tubular algebra of type n_Λ and $T(A) \simeq T(B)$.

(iii) → (ii). Assume that B is a tubular algebra of tubular type m. We shall show that $R = \hat{B}$ is locally support-finite, nondomestic of finite growth. Then, by Propositions 1 and 2, $T(B)$ is nondomestic of finite growth. Let $B_0 = B$ be a tubular extension of a uniquely determined concealed algebra C_0 and let $\operatorname{ind} C_0 = P_0 \vee T_0 \vee Q_0$ where P_0 is a preprojective component of C_0, Q_0 is a preinjective component of C_0, T_0 is a stable tubular $P_1(K)$-family of Euclidean type, and the ordering from left to right indicates that there are non-zero maps only from any of the classes to itself and to the module classes to its right. From [21, 5.2 Theorem 3] B_0 is also a cotubular algebra, that is, a truncated branch coextension of type m of a uniquely determined tame concealed algebra C_1, say with $\operatorname{ind} C_1 = P_1 \vee T_1 \vee Q_1$. Then by [21, 5.2, Theorem 4]

$$\operatorname{ind} B_0 = P_0 \vee T_0^r \vee M_{0,1} \vee T_1^c \vee Q_1$$

where T_0^r is obtained from T_0 by a finite number of ray insertions [7], [21, 4.5], T_1^c is obtained from T_1 by a finite number of coray insertions [7], [21, 4.6], and $M_{0,1} = \bigvee_{\gamma \in Q_1^0} T_\gamma^0$, for stable tubular $P_1(K)$-families T_γ^0, $\gamma \in Q_1^0$, of tubular type m. Here, for any integer i, Q_{i+1}^i denotes the set of all rational numbers q with $i < q < i+1$. Moreover, all indecomposable projective (resp. injective) B_0-modules are contained in $P_0 \vee T_0^r$ (resp. $T_1^c \vee Q_1$). Observe that R is obtained from B_0 by successive one-point extensions using modules whose restrictions to B_0 belong to $T_1^c \vee Q_1$ or are zero, and successive one-point coextensions using modules whose restrictions to B_0 belong to $P_0 \vee T_0^r$ or are zero. In this process, all stable tubular families T_γ^0, $\gamma \in Q_1^0$, remain unchanged, and consequently they form components of the Auslander–Reiten quiver Γ_R of R. In particular, all shifts $\nu(T_\gamma^0)$, $\gamma \in Q_1^0$, of T_γ^0 by the Nakayama automorphism ν, considered here as the functor $\operatorname{mod} R \to \operatorname{mod} R$, are also components of Γ_R. By [4, Lemma 2.5] there exists a ν-reflection sequence of sinks $i_1, \ldots, i_{t(1)}$ of Q_B such that: $I_B(i_1), \ldots, I_B(i_{t(1)})$ belong to T_1^c and $D_1 = S_{i_{t(1)}}^+ \ldots S_{i_1}^+ B_0$ is a tubular extension of C_1 of type m. Applying [21, 5.2 Theorem 3] we conclude

again that D_1 is truncated branch coextension of type m of a tame concealed algebra C_2. Moreover, if $\operatorname{ind} C_2 = P_2 \vee T_2 \vee Q_2$, then

$$\operatorname{ind} D_1 = P_1 \vee T_1^r \vee M_{1,2} \vee T_2^c \vee Q_2$$

where T_1^r is obtained from T_1 by a finite number of ray insertions, T_2^c is obtained from T_2 by a finite number of coray insertions and $M_{1,2} = \bigvee_{\gamma \in Q_2^1} T_\gamma^1$ for stable tubular $P_1(K)$-families of type m. Then, using [21, 4.9], we conclude that, for $E_1 = T_{i_{t(1)}}^+ \ldots T_{i_1}^+ B_0$,

$$\operatorname{ind} E_1 = P_0 \vee T_0^r \vee M_{0,1} \vee \hat{T}_1 \vee M_{1,2} \vee T_2^c \vee Q_2$$

where $\hat{T}_1 = T_1^{cr} = T_1^{rc}$ is a (nonstable) tubular $P_1(K)$-family of type m obtained from T_1^c (resp. T_1^r) by a finite number of ray (resp. coray) insertions. Moreover, projective-injective indecomposable E_1-modules are in $\hat{T}_1$ and indecomposable injective non-projective E_1-modules are in $T_2^c \vee Q_2$. Repeating the same arguments we deduce that there are (uniquely determined by B) concealed algebras $C_0, C_1, \ldots$, tubular algebras $D_0 = B_0, D_1, D_2, \ldots$ of the same tubular type m such that:

(1) Each D_j is a tubular extension of C_j and a tubular coextension of C_{j+1}.

(2) $D_{j+1} = S_{i_{t(j+1)}}^+ \ldots S_{i_{t(j)+1}}^+ D_j$ for a ν-reflection sequence of sinks $i_{t(j)+1}, \ldots, i_{t(j+1)}$ in Q_{D_j}, $j \geqslant 0$.

(3) If $\operatorname{ind} C_j = P_j \vee T_j \vee Q_j$, then

$$\operatorname{ind} D_j = P_j \vee T_j^r \vee M_{j,j+1} \vee T_{j+1}^c \vee Q_{j+1}$$

where T_j^r is obtained from the tubular family T_j by a finite number of ray insertions, T_{j+1}^c is obtained from T_{j+1} by a finite number of coray insertions, and $M_{j,j+1} = \bigvee_{\gamma \in Q_{j+1}^j} T_\gamma^j$ for stable tubular $P_1(K)$-families T_γ^j, $\gamma \in Q_{j+1}^j$, of type m.

(4) If $E_{j+1} = T_{i_{t(j+1)}}^+ \ldots T_{i_{t(j)+1}}^+ E_j$, then

$$\operatorname{ind} E_{j+1} = P_0 \vee T_0^r \vee \bigvee_{0 \leqslant s \leqslant j} (M_{s,s+1} \vee \hat{T}_{s+1}) \vee M_{j+1,j+2} \vee T_{j+2}^c \vee Q_{j+2}$$

where $\hat{T}_{s+1} = T_{s+1}^{cr} = T_{s+1}^{rc}$ is a (nonstable) tubular $P_1(K)$-family of type m, obtained from T_{s+1}^c (resp. T_{s+1}^r) by a finite number of ray (resp. coray) insertions.

(5) Indecomposable projective-injective (resp. injective nonprojective) E_{j+1}-modules are contained in $\hat{T}_{s+1}$, $0 \leqslant s \leqslant j$ (resp. in $T_{j+2}^c \vee Q_{j+2}$).

Thus, if D_∞ denotes the full subcategory of R formed by all objects of D_j, $j = 0, 1, 2, \ldots$, then

$$\operatorname{ind} D_\infty = P_0 \vee T_0^r \vee \bigvee_{s \geqslant 0} (M_{s,s+1} \vee \hat{T}_{s+1}).$$

For $j \geqslant 0$, we set $J_{j+1} = \{i_{t(j)+1}, \ldots, i_{t(j+1)}\}$. Let n be the least number such that J_{n+1} contains $\nu(x)$ for some vertex x of Q_{B_0}. We shall show that $D_n \simeq \nu(B_0)$

and consequently E_n is isomorphic to the algebra

$$\begin{pmatrix} B_0 & 0 \\ D(B) & B_1 \end{pmatrix}$$

It is known [21, 5.1] that any tube from T_γ^0, $\gamma \in Q_1^0$, consists of sincere B-modules. Therefore, for any two vertices y and z of Q_B, there are non-zero maps

$$P_B(y) \to X \to I_B(z)$$

which are non-isomorphisms and X belongs to $T_{1/2}^0$. This implies that there exists in $\operatorname{ind} R$ a chain of non-zero maps

$$(*) \qquad P_R(\nu(y)) \to \nu(P_B(y)) \to \nu(X) \to \nu(I_B(z)) \to P_R(\nu^2(z))$$

all of them are non-isomorphisms, and $\nu(X)$ belongs to the stable tubular family $\nu(T_{1/2}^0)$ of $\operatorname{ind} R$. Moreover, since by [21, 5.2], $\operatorname{ind} R = \mathscr{A} \vee \nu(T_{1/2}^0) \vee \mathscr{B}$ and $\nu(T_{1/2}^0)$ is stable, there is no a finite chain of non-zero maps

$$(**) \qquad P_R(\nu^2(z)) = Y_0 \to Y_1 \to \ldots \to Y_r = P_R(\nu(y))$$

We claim that $J_1 \cup \ldots \cup J_n = (Q_B)_0$. Let z be a vertex of Q_B such that $\nu(z)$ belongs to J_{n+1}. Suppose that y is a vertex of Q_B which does not belong to $J_1 \cup \ldots \cup J_n$. Since every injective D_∞-module is projective, y belongs to some J_s, $s \geqslant n+1$. If $s > n+1$ then there exists a finite chain (of non-zero maps) of the form (**), a contradiction. If $s = n+1$, both $P_R(\nu^2(z))$ and $P_R(\nu(y))$ belong to $\hat{T}_{n+1}$. Moreover, there is no non-zero maps between different tubes of $\hat{T}_{n+1}$. Hence, the existence of a chain (*) implies that $P_R(\nu^2(z))$ and $P_R(\nu(y))$ belong to the same tube of the tubular $P_1(K)$-family $\hat{T}_{n+1}$. Thus there is a chain (**) and we have again a contradiction. Therefore $J_1 \cup \ldots \cup J_n$ is the set of all vertices of Q_B. From the minimality of n, the socle of any indecomposable injective D_n-module is of the form $S_R(\nu(x))$ for some vertex x of Q_B, and hence $D_n \simeq \nu(B_0)$. Let now, for $p < q$ in Z, $B_{p,q}$ denote the full subcategory of $\hat{B}$ consisting of the objects of B_r, $p \leqslant r \leqslant q$. We claim that any indecomposable $B_{p,q}$-module is actually a $B_{r,r+1}$-module for some $p \leqslant r \leqslant q-1$. Indeed, $B_{p,q+1}$ is obtained from $B_{p,q}$ by a sequence of one-point extensions by modules whose restrictions to $B_{p,q}$ are either 0 or an indecomposable injective $B_{p,q}$-module. From the previous considerations, it follows that any indecomposable $B_{p,q+1}$-module is either a $B_{p,q}$-module or a $B_{q,q+1}$-module. Dually, any indecomposable $B_{p-1,q}$-module is either a $B_{p-1,p}$-module or a $B_{p,q}$-module. This show our claim. Hence $\hat{B}$ is locally support-finite and

$$\operatorname{ind} \hat{B} = \bigvee_{s \in Z} (M_{s,s+1} \vee \hat{T}_{s+1})$$

Further, using Ringel's description [21, 5.2] of modules in $M_{s,s+1}$ by radical vectors of the corresponding quadratic forms, we infer that $\hat{B}$ is of finite growth. Moreover, for any $s \in Z$, $\nu(\hat{T}_s) = \hat{T}_{s+n}$, $\nu(M_{s,s+1}) = M_{s+n,s+n+1}$, and by [8, Theorem] $\operatorname{ind} T(B)$ consists of the stable tubular $P_1(K)$-families $F_\lambda(T_\gamma^s)$, $\gamma \in Q_{s+1}^s$, $0 \leqslant s \leqslant n-1$, and nonstable tubular $P_1(K)$-families $F_\lambda(\hat{T}_{s+1})$, $0 \leqslant s \leqslant n-1$, all of tubular type m.

(i) → (iv) We shall proceed as in [4, Sections 5 and 6] using the following (more general) lemmas below.

Lemma 3.1. *Let B be an algebra whose bound quiver consists of a full subcategory C which is hereditary of type $\tilde{A}_m$ and objects of a walk connecting two different objects of C, and assume that B is bound only by zero-relations. Then $T(B)$ is not of polynomial growth.*

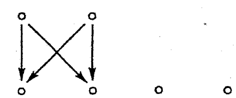
Proof. Use the free closed walks u and v constructed in the proof of [4, Lemma 5.2] and apply arguments as in the proof of [22, Lemma 1].

Lemma 3.2. *Let $B = C[M]$ be a one-point extension of a tame concealed algebra C such that $T(B)$ is tame. Then M is a regular C-module.*

Proof. Since $T(B)$ is tame so is $\hat{B}$. Observe that $C[M]$ and $[M]C$ are full subcategories of $\hat{B}$. Then by [20, 2.5, Lemma 3], M is regular.

Lemma 3.3. *Let $B = C[M]$ be a one-point extension of a tame concealed algebra of type $\tilde{D}_n$ or $\tilde{E}_p$ by a regular module M, and assume that B is of polynomial growth. Then M is a simple regular C-module.*

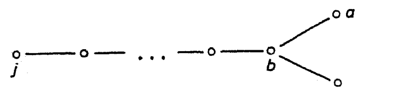
Proof. As in the proof of [4, Lemma 5.4] we can assume that C is hereditary. Suppose that M is not simple regular. Then by [20, 3.5], C is of type $\tilde{D}_n$ and M is regular of regular length two with non-isomorphic simple composition factors. In this case, the vector space category $\operatorname{Hom}_C(M, \operatorname{mod} C)$ is one of two types $(\tilde{D}_n, {}^{n-2}_{n-2})$ or $(\tilde{D}_n, (n-2) \oplus (n-2))$ [20], and hence $\operatorname{mod} B$ contains a full subcategory equivalent to the subspace category [20] of the following poset



Then, combining the arguments from the proof of [22, Lemma 1] with these in the proof of [27, Theorem] (see also [19]), we conclude that B is not of polynomial growth.

Lemma 3.4. *Let $B = C[M]$ be a one-point extension of a hereditary algebra of type $\tilde{A}_m$ by a regular C-module M, and assume that $T(B)$ is of polynomial growth. Then M is a simple regular C-module.*

Proof. Since B is tame, by [20, 3.5], M is of regular length at most two with non-isomorphic simple regular composition factors. Suppose M is not simple regular. Then by Lemma 3.1, M is an indecomposable of regular length 2 lying in a tube of rank at least two. Let i be the extension vertex of $C[M]$ and $P = P_B(i)$. Observe that $\hat{B}$ contains, as a full subcategory, the one-point coextension $D = [P]B$ of B by P. Consider the universal Galois covering $\tilde{D} \to D$ with infinite cyclic group determined by the cycle C. A simple analysis shows that $\tilde{D}$ contains a full subcategory E obtained by identifying the extension vertex of a one-point extension $C'[X]$ of a tame concealed algebra C' of type $\tilde{D}_n$ by a simple regular C'-module X, lying in a tube of rank $n-2$, to the vertex j in a quiver of the form



where $\circ\!-\!\circ$ means $\circ\!\rightarrow\!\circ$ or $\circ\!\leftarrow\!\circ$. Let F be the full subcategory of E formed by all objects of E except a. Then by [21, 4.9] F is a tilted algebra of type $\tilde{D}_m$, $m > n$, with a complete slice in the preinjective component. Let U_F be the slice module [14] of a complete slice S of the preinjective component of F. Then $H = \operatorname{End}(U_F)$ is a hereditary algebra of type $\tilde{D}_m$ and $Y = \operatorname{Ext}^1_F(U_F, P_F(b))$ is an indecomposable regular H-module of regular length 2 lying in the tube of rank $m-2$ (see [4, Lemma 5.6]). Moreover, $E = F[P_F(b)]$ and the vector space category $\operatorname{Hom}_H(Y, \operatorname{mod} H)$ is equivalent to the full subcategory of the vector space category $\operatorname{Hom}_F(P_F(b), \operatorname{mod} F)$ whose indecomposable objects are of the form $\operatorname{Hom}_F(P_F(b), Z)$ where Z belongs to the tube containing M or is an indecomposable preinjective F-module and a proper predecessor of S. From Lemma 3.3, the one-point extension $H[Y]$ is not of polynomial growth. Therefore, by Proposition 1, D, $\tilde{D}$, $\hat{B}$ and hence $T(B)$ are not of polynomial growth.

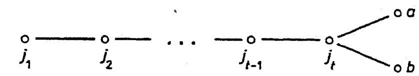
LEMMA 3.5. *Let C be a tame concealed algebra, M a simple regular C-module and $B = C[M][M]$. Then $T(B)$ is not of polynomial growth.*

Proof. Let $D = C[M]$. We can assume that D is tame, that is, by [4, Lemma 2.3] and Lemma 2.1, n_D is either domestic or tubular. Assume first that n_D is domestic. Observe that M is a regular D-module of regular length 2 (see the proof of Lemma 3.3) and hence D is not of polynomial growth for C of type $\tilde{D}_n$ or $\tilde{E}_p$. If D is of type $\tilde{A}_n$, that is, C is hereditary algebra of type $\tilde{A}_{n-1}$, then $\hat{B}$ contains two full subcategories $C[M][M]$ and $[M][M]C$, and, as in the proof of Lemma 3.4, we conclude that $T(B)$ is not of polynomial growth.

Now assume that n_D is tubular. We shall show that B is wild. Let x be the extension vertex in Q_D corresponding to M. Then, by [7], $P_D(x) = \tau^r_D(M)$ for some positive integer r. From [21, 5.2] D is also a tubular coextension of a tame concealed algebra C', having x in its support, and the preinjective component Q' of C' is the unique preinjective component of D. Then there exists an indecomposable $X \in Q'$ such that $\dim_K \operatorname{Hom}_D(P_D(x), X) \geqslant 3$ and $\tau_D^{-r} X \neq 0$. Put $Y = \tau_D^{-r} X$ and observe that $\operatorname{Hom}_D(M, Y) \simeq \operatorname{Hom}_D(\tau^r_D M, \tau^r_D Y) \simeq \operatorname{Hom}_D(P_D(x), X)$. Thus, $\operatorname{End}_D(Y) = K$, $\dim_K \operatorname{Hom}_D(M, Y) \geqslant 3$ and consequently by [20, 2.4] the vector space category $\operatorname{Hom}_D(M, \operatorname{mod} D)$ is wild. Therefore D, and hence also $T(B)$, is wild.

Lemma 3.6. *Let $B = C[M]$ be a one-point extension of a tame concealed algebra C by a simple regular module M. Let i denote the extension vertex corresponding*

to M, and A is obtained from B by identifying i to the vertex j in a quiver with underlying graph as follows

(*) 

Then $T(A)$ is not of polynomial growth.

Proof. Since $T(eAe) \simeq eT(A)e$ for any idempotent e, without loss of generality, we can assume that the walk $j_1 - \circ - \ldots - \circ - j_t$ has radical square zero. Let D be the full subcategory of A consisting of all vertices except a. Then D is a truncated branch extension of C. Assume that n_D is equal to (p, q), $1 \leqslant p \leqslant q$. In this case A is a bound quiver algebra bound by zero-relations. Let i' be the coextension vertex of the one-point coextension $E = [P_B(i)]A$. Observe that $\hat{A}$ contains a full subcategory F obtained from E by identifying i' to the vertex j_1 in a quiver of the form (*). Consider the universal Galois covering $\tilde{F} \to F$ with infinite cyclic group determined by the cycle C. Then $\tilde{F}$ contains a full subcategory G obtained by identifying the extension vertex of a one-point extension $C'[X]$ of a tame concealed algebra C' of type $\tilde{D}_n$ by a simple regular module X to the vertex j_1 in a quiver of the form (*). Therefore, in order to prove the lemma, it is enough to show by Proposition 1 that if C is of type $\tilde{D}_n$ or $\tilde{E}_p$, then A is not of polynomial growth. We can assume that a is a source. Indeed, if a is a sink, applying the APR-tilting module [5] $T_A = \tau_A^{-1}(S(a)) \oplus \bigoplus_{j \neq a} P_A(j)$, we replace A by an algebra A^* (see the proof of Lemma 2.1) of the same form as A and such that a is a source in A^*, and A is of polynomial growth if and only if A^* is of polynomial growth. Thus $A = D[Y]$ where $Y = P_A(j_t)$. Applying, if necessary, the APR-tilting or the APR-cotilting A-module at the vertex b, we can also assume that j_t is a sink or a source in D. If n_D is neither domestic nor tubular, D is wild, by Lemma 2.1 and [4, Lemma 2.3], and hence A is wild. Assume that n_D is domestic. Then D is a tilted algebra of type $\tilde{D}_n$ or $\tilde{E}_p$ with a complete slice in the preinjective component. Let U_D be the slice module of a complete slice in this component, $H = \operatorname{End}(U_D)$ and $N_H = \operatorname{Ext}^1_D(U, Y)$. Then the vector space category $\operatorname{Hom}_D(Y, \operatorname{mod} D)$ contains a full subcategory $\mathscr{U}$ equivalent to be full subcategory $\mathscr{V}$ of the vector space category $\operatorname{Hom}_H(N, \operatorname{mod} H)$ formed by indecomposable objects of the form $\operatorname{Hom}_H(N, Z)$ for all indecomposable preinjective H-modules Z. On the other hand, we have, by [14], the connecting Auslander–Reiten sequence in $\operatorname{mod} H$

$$0 \to \operatorname{Hom}_D(U, I) \to W \to N \to 0$$

where $I = I_D(i_t)$. The middle term W is determined up to extension by the short exact sequence

$$0 \to \operatorname{Ext}^1_D(U, \operatorname{rad} Y) \to W \to \operatorname{Hom}_D(U, I/\operatorname{soc} I) \to 0$$

[14], [21]. A simple analysis shows that for any orientation of the quiver $j_{t-1} — j_t — b$, the above connecting Auslander–Reiten sequence has two indecomposable middle terms, and hence N is an indecomposable regular module of regular length at least two. Then by Lemma 3.3, $H[N]$ is not of polynomial growth. Hence $\mathscr{V}$, and consequently A, is not of polynomial growth. Assume now that n_D is tubular. Then D is a tubular algebra and, as in the proof of Lemma 3.5, we show that there exists a preinjective D-module X such that $\mathrm{End}_D(X) = K$ and $\dim_K \mathrm{Hom}_D(Y, X) \geqslant 3$. Hence $A = D[Y]$ is, by [20, 2.4], wild.

Now assume that A is simply connected and $T(A)$ is nondomestic of polynomial growth. If A is representation-finite, then since $\hat{A}$ is not locally representation-finite there exists, by [4, Corollary 3.4] a ν-reflection sequence of sinks $i_1, \ldots, i_t$ such that $S_{i_t}^+ \ldots S_{i_1}^+ A = B$ is representation-infinite but $S_{i_{t-1}}^+ \ldots S_{i_1}^+ A$ is representation-finite. From [4, Lemma 3.1] B is simply connected and, by [24], tilting-cotilting equivalent to A. Thus we can assume that A is representation-infinite. Then, using Lemmas 3.1, …, 3.6 and their duals, we prove in the same way as in [4, Section 6] that A contains a tame concealed algebra C as a full convex subcategory, and is a branch enlargement of C. From [4, Proposition 2.6] there exists a truncated branch extension D of C such that $n_A = n_D$ and $T(A) \simeq T(D)$. Since $T(A)$ is nondomestic, by [4, Theorem] and [3, Theorem 2.5], $n_D = n_A$ is nondomestic and hence, by [4, Lemma 2.3], D is nondomestic. On the other hand, D is tame since $T(D)$ is tame. Therefore, D is a tubular algebra and $n_A = n_D$ is tubular. Then, by [3, Theorem 2.5] D, and thus A, is tilting-cotilting equivalent to a canonical tubular algebra. This finishes the proof of Theorem 1.

§ 4. Isomorphisms of trivial extensions. Two algebras A and B are said to be *reflection-equivalent* if there exists a sequence of algebras $A = A_0, A_1, \ldots, A_{m+1} = B$ such that $A_{r+1} \simeq S_{i_r}^+ A_r$ or $S_{j_r}^- A_r$, $0 \leqslant r \leqslant m$, for a sink i_r or a source j_r of Q_{A_r}.

We shall show the following theorem.

THEOREM 2. *Let A and R be two simply connected algebras such that $T(A)$ and $T(R)$ are nondomestic of polynomial growth. Then the following conditions are equivalent*:

(i) $T(A) \simeq T(R)$.

(ii) $\hat{A} \simeq \hat{R}$.

(iii) A and R are *reflection-equivalent*.

Proof. The implications (iii) → (ii) and (ii) → (i) are obvious. We shall show that (i) → (iii). From Theorem 1, there are two tubular algebras B and Λ such that A is reflection-equivalent to B, R is reflection-equivalent to Λ, and $T(B) \simeq T(A) \simeq T(R) \simeq T(\Lambda)$. Consider the following diagram of functors

$$\begin{array}{ccc} \mathrm{mod}\,\hat{\Lambda} & & \mathrm{mod}\,\hat{B} \\ \downarrow{\scriptstyle F_\lambda^\Lambda} & & \downarrow{\scriptstyle F_\lambda^B} \\ \mathrm{mod}\,T(\Lambda) & \xrightarrow[\simeq]{\Phi} & \mathrm{mod}\,T(B) \end{array}$$

where F_λ^Λ and F_λ^B are the push-down functors [6, 13] and Φ is induced by an isomorphism $\psi\colon T(\Lambda) \to T(B)$. We know [8] that every module X of $\mathrm{mod}\,T(\Lambda)$ is of the form $F_\lambda^\Lambda(M)$ for some $M \in \mathrm{mod}\,\hat{\Lambda}$, and hence

$$\mathrm{Hom}_{T(\Lambda)}(F_\lambda(M), F_\lambda(N)) \simeq \bigoplus_{r \in Z} \mathrm{Hom}_{\hat{\Lambda}}(M, \nu^r N)$$

where M and N belong to $\mathrm{mod}\,\hat{\Lambda}$. Similarly, every module Y from $\mathrm{mod}\,T(B)$ is of the form $F_\lambda^B(U)$, for some $U \in \mathrm{mod}\,\hat{B}$ and

$$\mathrm{Hom}_{T(B)}(F_\lambda^B(U), F_\lambda^B(W)) \simeq \bigoplus_{r \in Z} \mathrm{Hom}_{\hat{B}}(U, \nu^r W).$$

if U and W belong to $\mathrm{mod}\,\hat{B}$. Observe that, if P and P' are two indecomposable projective $\hat{B}$-modules then $\mathrm{Hom}_{\hat{B}}(P, P') \neq 0$ if and only if $\mathrm{Hom}_{\hat{B}}(\nu^{-1}P', P) \neq 0$. Hence, since $Q_{\hat{B}}$ has no oriented cycles, if P and P' are two non-isomorphic indecomposable projective $\hat{B}$-modules with $\mathrm{Hom}_{\hat{B}}(P, P') \neq 0$ then F_λ^B induces isomorphisms

$$(*)\qquad \begin{aligned} &\mathrm{Hom}_{T(B)}(F_\lambda^B P, F_\lambda^B P') \simeq \mathrm{Hom}_{\hat{B}}(P, P'), \\ &\mathrm{Hom}_{T(B)}(F_\lambda^B P, F_\lambda^B P) \simeq \mathrm{Hom}_{\hat{B}}(P, P) \oplus \mathrm{Hom}_{\hat{B}}(P, \nu(P)). \end{aligned}$$

We have similar relations between indecomposable projective modules over $T(\Lambda)$ and $\hat{\Lambda}$. Moreover,

$$\mathrm{ind}\,\hat{B} = \bigvee_{s \in Z} (\mathcal{M}_{s,s+1} \vee \hat{\mathcal{T}}_{s+1})$$

and

$$\mathrm{ind}\,\hat{\Lambda} = \bigvee_{s \in Z} (\mathcal{M}'_{s,s+1} \vee \hat{\mathcal{T}}'_{s+1})$$

where $\mathcal{M}_{s,s+1}$ (resp. $\mathcal{M}'_{s,s+1}$) is formed by stable tubular $P_1(K)$-families, and, for any $s \in Z$, $\hat{\mathcal{T}}_{s+1}$ (resp. $\hat{\mathcal{T}}'_{s+1}$) is a nonstable tubular $P_1(K)$-family. Denote by $\mathscr{P}_\Lambda$ the full subcategory of $\mathrm{ind}\,\hat{\Lambda}$ formed by the indecomposable projective $\hat{\Lambda}$-modules $P_{\hat{\Lambda}}(i)$, $i \in (Q_\Lambda)_0$, where we identify Λ with Λ_0. Obviously $\Lambda \simeq \mathrm{End}_{\hat{\Lambda}}(P)$, where $P = \bigoplus_{i \in (Q_\Lambda)_0} P_{\hat{\Lambda}}(i)$. Let q be the least integer such that $\hat{\mathcal{T}}'_{q+1}$ contains a module from $\mathscr{P}_\Lambda$. Since $F_\lambda^B(\hat{\mathcal{T}}_{s+1})$, $0 \leqslant s \leqslant n-1$, form all nonstable tubular $P_1(K)$-families in $\mathrm{ind}\,T(B)$, $\Phi F_\lambda^\Lambda(\hat{\mathcal{T}}'_{q+1}) = F_\lambda^B(\hat{\mathcal{T}}_{r+1})$ for some r, $0 \leqslant r \leqslant n-1$. From the proof of Theorem 1 we deduce that $\mathscr{P}_\Lambda$ is the full subcategory of $\mathrm{ind}\,\hat{\Lambda}$ formed by all projective $\hat{\Lambda}$-modules from the tubes $\hat{\mathcal{T}}'_{q+1}, \ldots, \hat{\mathcal{T}}'_{q+n}$. Observe also that $\Phi F_\lambda^\Lambda(\hat{\mathcal{T}}'_{q+j}) = F_\lambda^B(\hat{\mathcal{T}}_{r+j})$ for $1 \leqslant j \leqslant n$. Let $\mathscr{P}_B$ be the full subcategory of $\mathrm{ind}\,\hat{B}$ consisting of all projective B-modules of the tubes $\hat{\mathcal{T}}_{r+1}, \ldots, \hat{\mathcal{T}}_{r+n}$. Observe that the endomorphism algebra of the direct sum of all projective modules from $\mathscr{P}_B$ is isomorphic to the algebra $D_r \simeq D_{r+n}$, defined in the proof of Theorem 1 (implication (iii) → (ii)). Let $\mathscr{P}$ be the category whose objects are indecomposable projective $T(B)$-modules and, for $P(i) = P_{T(B)}(i)$, $P(j) = P_{T(B)}(j)$, $i, j \in (Q_{T(B)})_0$,

$$\mathrm{Hom}_{\mathscr{P}}(P(i), P(j)) = \begin{cases} \mathrm{Hom}_{T(B)}(P(i), P(j)) & \text{if } i \neq j \text{ and } \\ & \mathrm{Hom}_{\hat{B}}(P_{\hat{B}}(i), P_{\hat{B}}(j)) \neq 0 \\ \mathrm{End}_{T(B)}\big(P(i)/J(\mathrm{End}_{T(B)}(P(i))\big) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

where $J(\operatorname{End}_{T(B)}(P(i)))$ is the Jacobson radical of $\operatorname{End}_{T(B)}(P(i))$. Thus $\operatorname{Hom}_P(P(i), P(i)) \simeq K$. Now, using the formulas (*), we conclude that the functors $\Phi F_\lambda^A\colon \operatorname{mod}\hat{A} \to \operatorname{mod}T(B)$ and $F_\lambda^B\colon \operatorname{mod}\hat{B} \to \operatorname{mod}T(B)$ induce isomorphisms of categories $\mathscr{P}_A \simeq \mathscr{P}$ and $\mathscr{P}_B \simeq \mathscr{P}$. Consequently $\Lambda \simeq D_r$. But $D_r = S_{i_{t(r)}}^+ \ldots S_{i_1}^+ B$, for the ν-reflection sequence of sinks $i_{t_1}, \ldots, i_{t(r)}$, constructed in the proof of Theorem 1, and so Λ and B are reflection-equivalent. Hence A and R are reflection-equivalent, and Theorem 2 is proved.

The following corollary is an immediate consequence of the above proof.

COROLLARY 1. *Let A and R be two tubular algebras and the Grothendieck groups $K_0(A)$ and $K_0(R)$ have rank n. Then the following conditions are equivalent*:

(i) $T(A) \simeq T(R)$

(ii) $\hat{A} \simeq \hat{R}$

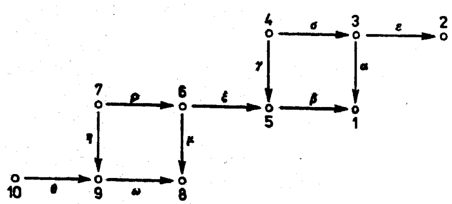
(iii) $R \simeq S_{i_t}^+ \ldots S_{i_1}^+ A$ *for a ν-reflection sequence of sinks* $i_1, \ldots, i_t$, $t \leqslant n$, *of* Q_A.

(iv) $A \simeq S_{i_s}^+ \ldots S_{i_1}^+ R$ *for a ν-reflection sequence of sinks* $i_1, \ldots, i_s$, $s \leqslant n$, *of* Q_R.

§ 5. Remarks. (1) It follows directly from Theorem 1 and [23] that a nondomestic trivial extension of a simply connected algebra is stably equivalent to the trivial extension of a canonical tubular algebra.

(2) Let B be a tubular algebra and $r(B)$ denote the rank of the Grothendieck group $K_0(B)$. It follows from our proof of Theorem 1 that the number $n(B)$ of nonstable tubular $P_1(K)$-families in $\Gamma_{T(B)}$ coincides with the number of tubular algebras of the form $S_{i_t}^+ \ldots S_{i_1}^+ B$ for a ν-reflection sequence of sinks $i_1, \ldots, i_t$ in Q_B, $t \leqslant r(B)$, and hence $n(B) \leqslant r(B) \leqslant 10$. Moreover, using [21, Section 5], one can deduce that $3 \leqslant n(B)$. If B is a canonical tubular algebra then $n(B) = 3$ by [15]. On the other hand, for any positive integer m, $3 \leqslant m \leqslant 9$, it is not difficult to find a tubular algebra B such that $n(B) = m$. We do not know if there exists a tubular algebra B such that $n(B) = 10$. We end the paper with an example of a tubular algebra B such that $n(B) = 9$.

Let B be the bound quiver algebra KQ/I where Q is the quiver and I is the ideal in the quiver algebra KQ generated by the elements $\gamma\beta - \sigma\alpha$, $\eta\omega - \varrho\mu$ and $\varrho\xi\beta$. Then B



is a one-point extension (resp. one-point coextension) of the tame concealed algebra C_0 (resp. C_1) of type $\tilde{E}_8$, formed by all vertices of Q except 7 (resp. except 1), by a simple regular C_0-module (resp. C_1-module) lying in the tube of rank 5. Thus B is a tubular (and cotubular) algebra of tubular type (2, 3, 6). Then, in our notations from the proof of Theorem 1, we have the following nine (nonisomorphic) tubular algebras reflection-equivalent to B: $D_1 = S_1^+ B$, $D_2 = S_2^+ D_1$, $D_3 = S_8^+ D_2$, $D_4 = S_5^+ D_3$, $D_5 = S_9^+ S_3^+ D_4$, $D_6 = S_6^+ D_5$, $D_7 = S_4^+ D_6$, $D_8 = S_{10}^+ D_7$ and $D_9 = S_7^+ D_8 = B$.

References

[1] I. Assem, D. Happel and O. Roldán, *Representation-finite trivial extension algebras*, J. Pure Appl. Algebra 33 (1984), 235–242.
[2] I. Assem and A. Skowroński, *On some classes of simply connected algebras*, Proc. London Math. Soc. 56 (1988), 417–450.
[3] — — *Algebras with cycle-finite derived categories*, Math. Ann. 280 (1988), 441–463.
[4] I. Assem, J. Nehring and A. Skowroński, *Domestic trivial extensions of simply connected algebras*, Tsukuba J. Math. (1988), to appear.
[5] M. Auslander, M. I. Platzeck and I. Reiten, *Coxeter functors without diagrams*, Trans. Amer. Math. Soc. 250 (1979), 1–46.
[6] K. Bongartz and P. Gabriel, *Covering space in representation-theory*, Invent. Math. 65 (1981/82), 331–378.
[7] G. d'Este and C. M. Ringel, *Coherent tubes*, J. Algebra 87 (1984), 150–201.
[8] P. Dowbor and A. Skowroński, *On Galois coverings of tame algebras*, Arch. Math. 44 (1985) 522–529.
[9] — — *On the representation type of locally bounded categories*, Tsukuba J. Math., 10 (1986), 63–72.
[10] — — *Galois coverings of representation-infinite algebras*, Comment. Math. Helvetici 62 (1987), 311–337.
[11] Yu. Drozd, *Tame and wild matrix problems*, in: *Representations and Quadratic forms*, Kiev 1979, 39–73 (in Russian).
[12] P. Gabriel, *Auslander–Reiten sequences and representation-finite algebras*, Lecture Notes in Math. 831 (1980), 1–71.
[13] — *The universal cover of a representation-finite algebra*, Lecture Notes in Math. 903 (1981), 68–105.
[14] D. Happel and C. M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. 274 (1982), 399–443.
[15] — — *The derived category of a tubular algebra*, Lecture Notes in Math. 1177 (1986), 156–180.
[16] M. Hoshino, *Splitting torsion theories induced by tilting modules*, Comm. Algebra 11 (1983), 427–441.
[17] D. Hughes and J. Waschbüsch, *Trivial extensions of tilted algebras*, Proc. London Math. Soc. 46 (1983), 347–364.
[18] R. Martinez and J. A. de la Peña, *The universal cover of a quiver with relations*, J. Pure Appl. Algebra 30 (1983), 277–292.
[19] L. A. Nazarova and A. V. Rojter, *On a problem of I. M. Gelfand*, Funkc. Anal. Appl. 7 (1973), 54–69 (in Russian).
[20] C. M. Ringel, *Tame algebras*, Lecture Notes in Math. 831 (1980), 137–287.
[21] — *Tame algebras and integral quadratic forms*, Lecture Notes in Math. 1099 (1984).
[22] A. Skowroński, *Group algebras of polynomial growth*, Manuscripta Math. 59 (1987), 499–516.
[23] H. Tachikawa and T. Wakamatsu, *Tilting functors and stable equivalences for selfinjective algebras*, J. Algebra, 109 (1987), 138–165.
[24] — — *Applications of reflection functors for selfinjective algebras*, Lecture Notes in Math. 117 (1986), 308–327.

[25] T. Wakamatsu, *Note on trivial extensions of artin algebras*, Comm. Algebra 12 (1984) 33–41.
[26] K. Yamagata, *On algebras whose trivial extensions are of finite type* II, Preprint (1983).
[27] A. G. Zavadskij and L. A. Nazarova, *On the finiteness and boundedness of a number of parameters*, Inst. Math. Acad. Sci. Ukrainian SSR, Preprint 27 (1981), 21–29 (in Russian).

NSTITUTE OF MATHEMATICS
NICHOLAS COPERNICUS UNIVERSITY
Chopina 12/18
87-100 Toruń
Poland

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Non-trivial homeomorphisms of $\beta N \setminus N$ without the continuum hypothesis

by

Saharon Shelah (Jerusalem) and **J. Steprans** (Toronto)

Abstract. The problem of constructing non-trivial homeomorphisms of $\beta N \setminus N$ without assuming the continuum hypothesis is examined.

In [3] Shelah showed that it is consistent that all automorphisms of $\mathscr{P}(\omega)/Finite$, or, equivalently, all autohomeomorphisms of $\beta N \setminus N$, are trivial in the sense that they are induced by almost-permutations of the integers (an almost-permutation of ω is an injective function from ω to ω whose domain and range are both cofinite). In [2] W. Rudin showed that the continuum hypothesis implies thet there is a non-trivial autohomeomorphism by showing that there are in fact $2^{2^{\aleph_0}}$ such homeomorphisms. It is the purpose of this paper to examine the question of how to construct non-trivial autohomeomorphisms in the absence of the continuum hypothesis. The reader should be warned that $\beta N \setminus N$ and $\mathscr{P}(\omega)/Finite$ will be used almost interchangeably. As well, subsets of the integers will routinely be confused with clopen sets in $\beta N \setminus N$.

At this point the reader may be wondering why the argument assuming $2^{\aleph_0} = \aleph_1$ does not generalize to $MA_{\aleph_1}$ and make the rest of this paper pointless. The reason, of course, is that an induction of length greater than ω_1 may run into a Hausdorff gap and stop. In fact it will be shown in [4] that PFA implies that all autohomeomorphisms of $\beta N \setminus N$ are trivial and so this is consistent with $MA_{\aleph_1}$. This raises the following unanswered question:

QUESTION. Is it consistent with $MA_{\aleph_1}$ that there is a non-trivial autohomeomorphism of $\beta N \setminus N$?

The first result towards obtaining non-trivial autohomeomorphisms of $\beta N \setminus N$ without the continuum hypothesis is due to Frolik [1]. He showed that the set of fixed points of any 1-1 continuous function from an extremally disconnected space to itself form a clopen set. To see how this can be used to construct non-trivial autohomeomorphisms of $\beta N \setminus N$ consider the following lemma.

LEMMA 1. *Suppose that $\mathscr{I}$ is an ideal on ω generated by an $\subseteq^*$-ascending sequence $\{A_\alpha : \alpha \in \varkappa\}$. Suppose further that f_α is an almost-permutation of A_α for*