Bing-Whitehead Cantor sets

by

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Abstract. We exhibit a collection of Cantor sets in 3-space so that the embedding of a given Cantor set exhibits a high degree of homogeneity. Let X be a Cantor set from the collection and $f\colon F_1 \to F_2$ a one-to-one correspondence between arbitrary finite subsets of X. Then f extends to a homeomorphism of 3-space to itself that sends X onto X. However, these same Cantor sets are not strongly homogeneously embedded, i.e. there are homeomorphisms of X to itself that do not extend to a homeomorphism of 3-space. Daverman [D] has constructed wild Cantor sets in S^n $(n \geqslant 5)$ that are strongly homogeneously embedded. However, it is unknown if a strongly homogeneously embedded Cantor set in S^3 must be tame.

We also give an independent proof of a recent result by Ancel and Starbird that gives necessary and sufficient conditions for Bing–Whitehead decompositions to be shrinkable. These decomposition give rise to the Cantor sets described in this paper.

1. Introduction. R. J. Daverman has constructed examples of wild Cantor sets in $\boldsymbol{R}^n$ and S^n $(n \geqslant 5)$ [D] with the property that every self homeomorphism of these Cantor sets extends to a homeomorphism of $\boldsymbol{R}^n$ or S^n. Such Cantor sets are said to be *strongly homogeneously embedded.* The constructions of Daverman definitely rely on techniques that are only valid in high dimensional spaces. It seems doubtful that such Cantor sets exist in 3-space.

Antoine's necklace [A], the classical wild Cantor set in 3-space, is homogeneously embedded, meaning that for each two points in the Cantor set, there is a homeomorphism of 3-space that takes one point to the other and takes the Cantor set to itself. However, Antoine's necklace is easily seen to not be strongly homogeneously embedded. In fact, if the tori T_1, T_2, T_3, T_4 are the cyclically ordered components of the first stage of a defining sequence for an Antoine Cantor set A, then any homeomorphism of A to itself that fixes a point of $T_1 \cap A$ and sends a point of $T_2 \cap A$ to a point of $T_3 \cap A$ cannot be extended to a homeomorphism of 3-space [Wr].

Suppose that X is a Cantor set in 3-space that is strongly homogeneously embedded, then the linking that occurs in the defining sequences of Antoine's necklace must be avoided. An obvious candidate is the *Bing Cantor set* described in the following paragraph.

R. H. Bing proved that the union of two Alexander crumpled cubes attached by the identity map along the 2-sphere boundaries yields a 3-sphere [B]. His proof showed that a Cantor set in 3-space could be constructed as the intersection of manifolds M_i $i = 0, 1, 2, \ldots$, where the manifold M_0 is an unknotted solid torus and each component of M_i is a solid torus which contains two components of M_{i+1} that are embedded as shown in Figure 1. We call the intersection of

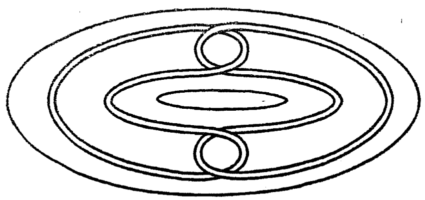

Fig. 1

the M_i a *Bing Cantor set*. If X is a Bing Cantor set, then X exhibits a higher degree of homogeneity than Antoine's necklace for if $f\colon F_1 \to F_2$ is a one-to-one correspondence between finite subsets of X, then there is a homeomorphism of 3-space extending f which takes X to X. This is because, unlike Antoine's necklace, the components of the defining sequence of the Bing Cantor set fall apart. Hence, by going deep enough into the defining sequence M_i for the Bing Cantor set we can find a j so that the points of F_1 are in different components of M_j and the same is true for F_2. Now using the fact that the components of M_j fall apart, it is easy to get a homeomorphism of 3-space to itself that takes X to itself and takes the component of M_j containing a point p of F_1 to the component of M_j containing $f(p)$. This homeomorphism can then be followed by a homeomorphism of 3-space which fixes X and the components of M_j set-wise and takes p to $f(p)$ for p in F_1.

There is a generalization of the Bing Cantor set obtained by interlacing the Whitehead link with the construction of Bing. We let G be an upper semi-continuous decomposition of 3-space consisting of points and components $\bigcap M_i$ where M_1 is a solid torus and M_{i+1} is obtained from M_i by the *Bing Construction* (placing two solid tori in each component as in the construction of the Bing Cantor set — see Figure 1) or by the *Whitehead Construction* (placing a Whitehead link in each component — see Figure 2). We assume that the sequence M_i has an infinite number of Whitehead constructions. Let n_1 be the number of consecutive Bing constructions

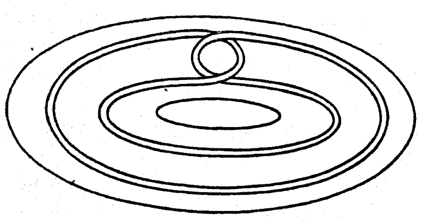

Fig. 2

placed in M_0 before the first Whitehead construction. In general let n_i be the number of consecutive Bing constructions between the $(i-1)$st and ith Whitehead constructions in the sequence M_i. Of course n_i could equal zero if there are consecutive Whitehead constructions. It should be clear [B] that if there are enough Bing constructions, then the decomposition G is shrinkable. In fact a recent result of Ancel and Starbird [A–S] states that the decomposition is shrinkable if and only if the series $\sum_{i=1}^{\infty} \frac{n_i}{2^i}$ diverges. We outline an independent proof of this fact in Appendix A.

In any case, if G is shrinkable, then the image of $\bigcap M_i$ in the decomposition space is a Cantor set which we call a *Bing–Whitehead Cantor set*. Such Cantor sets have been studied by De Gryse and Osborne [D–O] who showed that the complements of such Cantor sets are simply connected. We outline a simple proof of this fact in Appendix 2.

The components of a defining sequence for a Bing–Whitehead Cantor set fall apart in the same manner as the components of a defining sequence for a Bing Cantor set. Hence, Bing–Whitehead Cantor sets exhibit the same degree of homogeneity as Bing Cantor sets. If anything, the components of a defining sequence for a Bing–Whitehead Cantor set seem to be even more loosely held together because the complement of a Bing–Whitehead Cantor set is simply connected.

The goal of this paper is to show that even though the components of a defining sequence for a Bing Cantor set or a Bing–Whitehead Cantor set fit together loosely, such Cantor sets still fail to be strongly homogeneously embedded in 3-space.

2. Definitions and notations. We use $\boldsymbol{R}^n$ and S^n to denote Euclidean n-space and its one point compactification, respectively. By 3-*space* we will mean either $\boldsymbol{R}^3$ or S^3. If M is a manifold, we let $\mathrm{Bd}(M)$ and $\mathrm{Int}(M)$ denote the boundary and interior of M, respectively. We let I denote the unit interval $[0, 1]$ and B^2 denote a 2-dimensional disk. By a solid torus we will mean a space homeomorphic with $B^2 \times S^1$. If X is a subset of a set Y, we let X^c denote the complement of X in Y. A *map* will denote a continuous function. A *Bing link* is an embedding of two solid tori in a solid torus as shown in Figure 1. A *Whitehead link* is the embedding of a solid torus into another solid torus as shown in Figure 2. All embeddings throughout this paper are piecewise linear embeddings.

A *disk with holes* is a compact, connected, planar 2-manifold. Let $f\colon H \to M$ be a map from a disk with holes into a manifold M so that $f(\mathrm{Bd}\,H) \subset \mathrm{Bd}\,M$. Following [D–E], we say that f is *interior-inessential* if there exists a map $f'\colon H \to \mathrm{Bd}\,M$ such that

$$f'|\mathrm{Bd}\,H = f|\mathrm{Bd}\,H.$$

Otherwise, f is said to be *interior-essential*. In addition, a closed subset S of M is said to be *geometrically central* in M if whenever $h\colon H \to M$ is an interior-essential map, then $h(H) \cap S = \emptyset$. Let $\tilde{T}$ be the universal covering space for a solid torus.

Let $\pi\colon \tilde{T} \to T$ be the projection map. If E is either a Bing link or a Whitehead link in T, it is well known [D–E] that $\pi^{-1}(E)$ is geometrically central in $\tilde{T}$.

3. Bing-Whitehead compacta and Cantor sets. Let M_0 be an unknotted solid torus in 3-space. We inductively define M_i, $i > 0$ by letting M_{i+1} be obtained from M_i by placing a Whitehead link in each component of M_i or by placing a Bing link in each component of M_i. We inductively define M_i $i < 0$ so that M_i is a Whitehead link in M_{i-1}. Let $X = \bigcap M_i$. If all but finitely many of the M_i, $i > 0$, are obtained through a Bing construction, we call X a *Bing compactum* (or *Bing Cantor set* if it happens to be a Cantor set). If all but finitely many of the M_i, $i > 0$, are obtained through a Whitehead construction, we call X a *Whitehead compactum.* If infinitely many of the M_i, $i > 0$, are obtained through the Bing construction and infinitely many of the M_i, $i > 0$, are also obtained through the Whitehead construction, then we call X a *Bing–Whitehead compactum* (or *Bing-Whitehead Cantor set* if it happens to be a Cantor set). Let $X^\infty = \bigcap M_i^c$. We call X^∞ a continuum at infinity for the compactum X.

If X is a Bing compactum or Bing–Whitehead compactum, then there must be an infinite number of Bing constructions in the defining sequence. In this case, we let N_k ($k = 1, 2, 3, \ldots$) be the M_i with smallest subscript that has 2^k components. The components of N_1 will be denoted by $N_1(1)$ and $N_1(2)$. The components of N_2 that lie in $N_1(i)$, $i = 1, 2$, will be denoted by $N_2(i, j)$, $j = 1, 2$. In general, the components of N_{k+1} that lie in $N_k(i_1, i_2, i_3, \ldots, i_k)$, $i_j \in \{1, 2\}$, will be denoted by $N_{k+1}(i_1, i_2, i_3, \ldots, i_k, i_{k+1})$, $i_{k+1} \in \{1, 2\}$. A component of X may now be written as a sequence of 1's and 2's. The sequence $i_1, i_2, i_3, \ldots$ denotes the component that is contained in $N_k(i_1, i_2, i_3, \ldots, i_k)$ for each k. In case the component of X consists of a single point, we also let this sequence denote the point.

4. Some general facts. Let T be a solid torus. Let T' be a Whitehead link and T'' a Bing link in T, respectively. In the next two lemmas we consider the manifolds $W = T - \operatorname{Int} T'$ and $B = T - \operatorname{Int} T''$.

LEMMA 4.1. *The manifolds W and B are boundary incompressible; i.e., a loop n the boundary is essential in the boundary if and only if it is essential in the manifold.*

Proof. We give a proof for W. The proof for B is similar. Let $\pi\colon \tilde{T} \to T$ be the projection map from the universal cover $\tilde{T}$ of T. Let γ be a loop in $\operatorname{Bd} T$. If γ is inessential in W, then there is a lift $\tilde{\gamma}$ of γ to the universal covering space $\tilde{T}$. But $\pi^{-1}(T')$ is geometrically central in $\tilde{T}$. Hence, $\tilde{\gamma}$ is inessential in $\operatorname{Bd} \tilde{T}$, and γ is inessential in $\operatorname{Bd} T$.

Suppose that γ is a loop in $\operatorname{Bd} T'$ that is inessential in W. Let R and S be lifts of T' in $\tilde{T}$ that are *adjacent*. Then there is a lift $\tilde{\gamma}$ of γ to the universal covering space $\tilde{T}$ so that $\tilde{\gamma}$ lies on $\operatorname{Bd} R$ and is inessential in the complement of $\operatorname{Int}(R \cup S)$. The loop $\tilde{\gamma}$ cannot go around $\operatorname{Bd} R$ in the meridional direction because it is inessential in the complement of $\operatorname{Int} R$. The loop $\tilde{\gamma}$ cannot go around R in the longitudinal direction

because it is inessential in the complement of $\operatorname{Int} S$. Hence $\tilde{\gamma}$ is inessential in $\operatorname{Bd} R$ and γ is inessential in $\operatorname{Bd} T'$.

LEMMA 4.2. *Let f be a map of the annulus $S^1 \times I$ into W or B so that the boundary components of the annulus are sent into different boundary components of W or B. Then the map f is inessential; i.e., f is homotopic to a constant map.*

Proof. We give the proof for W. The proof for B is only slightly harder and is not given. We assume that $f(S^1 \times \{0\})$ is sent into $\operatorname{Bd} T$ and that $f(S^1 \times \{1\})$ is sent into $\operatorname{Bd} T'$. Since T' is contractible in T, the map f lifts to a map $\tilde{f}$ to the universal cover $\tilde{T}$ of T. Let R and S be *adjacent* lifts of T' so that $\tilde{f}(S^1 \times \{1\})$ is sent into $\operatorname{Bd} R$. Since $\tilde{f} | S^1 \times \{0\}$ bounds homologically in the complement of $\operatorname{Int}(R \cup S)$, we must conclude that $\tilde{f} | S^1 \times \{1\}$ bounds homologically in the complement of $\operatorname{Int}(R \cup S)$. Since $\tilde{f} | S^1 \times \{1\}$ bounds homologically in the complement of $\operatorname{Int} R$, it must be trivial in the meridional factor of $\operatorname{Bd} R$. Since $\tilde{f} | S^1 \times \{1\}$ bounds homologically in the complement of $\operatorname{Int} S$, it must be trivial in the longitudinal factor of $\operatorname{Bd} R$. Hence, $\tilde{f} | S^1 \times \{1\}$ is inessential in $\operatorname{Bd} R$. This implies that $f | S^1 \times \{1\}$ is inessential in $\operatorname{Bd} T'$, and our lemma is proved.

We now let M_i be a defining sequence for a compactum X as described in Section 3. We also let X^∞ denote the continuum at infinity for X. This notation will be used for the remainder of this section.

THEOREM 4.3. *Suppose that v is a loop on $\operatorname{Bd} M_i$. Then v is essential in $\operatorname{Bd} M_i$ if and only if v is essential in $S^3 - (X \cup X^\infty)$.*

Proof. This follows easily from general position and Lemma 4.1.

THEOREM 4.4. *Suppose γ_0, γ_1 are loops in $\operatorname{Bd} M_i$, $\operatorname{Bd} M_j$ ($i \neq j$), respectively, such that γ_0 and γ_1 are homotopic in $S^3 - (X \cup X^\infty)$. Then γ_0 and γ_1 are inessential in $\operatorname{Bd} M_i$, $\operatorname{Bd} M_j$, respectively.*

Proof. Let $f\colon S^1 \times I \to S^3 - (X \cup X^\infty)$ be a map so that $f | S^1 \times \{r\} = \gamma_r$. We assume that f is in general position with respect to the surfaces $\operatorname{Bd} M_k$. By Theorem 4.3 we may assume that $f^{-1}(\operatorname{Bd} M_k)$ consists only of simple closed curves that are essential in the annulus. Since $i \neq j$, there is an essential annulus A of $S^1 \times I$ so that A is sent by f into $M_k - \operatorname{Int} M_{k+1}$ for some k, the boundary components of A being sent into different boundary components of $M_k - \operatorname{Int} M_{k+1}$. Lemma 4.2 and Theorem 4.3 can now be applied to complete the proof.

LEMMA 4.5. *Let J be a meridional simple closed curve on some component T of M_i and V be an open set that contains a component C of $X \cap T$. Then J bounds a disk whose intersection with $X \cup X^\infty$ lies in V.*

Proof. Let D_1 be a disk in T whose boundary is J and whose intersection with M_{i+1} consists of meridional disks of the component of M_{i+1} that contains C. Modify D_1 on $D_1 \cap M_{i+1}$ to obtain a disk D_2 in T whose intersection with M_{i+2} consists of meridional disks of the component of M_{i+2} that contains C. Continuing inductively, we find a disk D_k in T with boundary J whose intersection with M_{i+k} consists of meridional disks of the component of M_{i+k} that contains C. For large enough k, we may assume that the component of M_{i+k} that contains C lies in V. Hence, D_k is our desired disk.

THEOREM 4.6. *A 2-sphere in S^3 cannot separate $X\cup X^\infty$, and hence $S^3-(X\cup X^\infty)$ is irreducible.*

Proof. Suppose S is a 2-sphere in $S^3-(X\cup X^\infty)$. Let i be chosen so that M_i has only one component whose boundary lies in the same complementary domain of S as X^∞. Let J be a meridional simple closed curve of $\mathrm{Bd}\,M_i$. If S separates $X\cup X^\infty$, then there is a component C of X that lies in the complementary domain V of S that does not contain X^∞. By Lemma 4.5, J bounds a disk whose intersection with $X\cup X^\infty$ lies in V. Cutting this disk off on S shows that J is inessential in $X\cup X^\infty$. Hence by Theorem 4.3, J is inessential in $\mathrm{Bd}\,M_i$. However, this contradicts the fact the J is a meridian of $\mathrm{Bd}\,M_i$.

5. The homeomorphism. Now suppose that X is a Bing or Bing–Whitehead Cantor set as described in Section 3. The points of X are denoted by sequences of 1's and 2's. We define a homeomorphism h of the Cantor set by sending the point $i_1, i_2, i_3, i_4, i_5, i_6, \ldots$ to the point $i_2, i_1, i_4, i_3, i_6, i_5, \ldots$

THEOREM 5.1. *The homeomorphism h cannot be extended to S^3 (R^3).*

Proof. Suppose the homeomorphism h extends to a homeomorphism H of S^3 to itself. This gives rise to a second defining sequence $H(N_j)$ for X. For sufficiently large k, $j\geqslant 2k$ implies $H(N_j)$ misses X^∞ and N_j misses $H(X^\infty)$. Let $A=N_{2k}(1,1,1,\ldots,1)$, $A_i=N_{2k+1}(1,1,1,\ldots,1,i)$, and

$$A_{ij}=N_{2k+2}(1,1,1,\ldots,1,i,j)\,.$$

We use a tilde to denote the image of the above sets under the homeomorphism H; e.g., $H(A)=\tilde{A}$. We denote by bars the intersection of the above sets with the Cantor set X; e.g., $\bar{A}=X\cap A$. Notice that $H(\bar{A})=\bar{A}$ and $H(\bar{A}_{ij})=\bar{A}_{ji}$. We may suppose that the surfaces $\mathrm{Bd}\,A$, $\mathrm{Bd}\,A_i$, and $\mathrm{Bd}\,A_{ij}$ are in general position with respect to the image under H of these surfaces. We now show that we may assume that any intersections of the surfaces $\mathrm{Bd}\,A$, $\mathrm{Bd}\,A_i$, and $\mathrm{Bd}\,A_{ij}$ with any of their images under H give rise only to simple closed curves that are essential in each of the containing surfaces. First note by Theorem 4.3 that a simple closed curve in the intersection of any two of the above surfaces is either essential or inessential in both. We use the irreducibility of $S^3-(X\cup X^\infty)$ to remove any inessential simple closed curves.

Now each of $\mathrm{Bd}\,A_1$ and $\mathrm{Bd}\,A_2$ must meet both $\mathrm{Bd}\,\tilde{A}_1$ and $\mathrm{Bd}\,\tilde{A}_2$. This is because $\bar{A}_{11}\cup\bar{A}_{12}\subset A_1$, $\bar{A}_{21}\cup\bar{A}_{22}\subset A_2$, $\bar{A}_{11}\cup\bar{A}_{12}\subset\tilde{A}_1$, and $\bar{A}_{12}\cup\bar{A}_{22}\subset\tilde{A}_2$. Hence by Theorem 4.4 $\mathrm{Bd}\,\tilde{A}_1$ and $\mathrm{Bd}\,\tilde{A}_2$ do not meet any of the sets $\mathrm{Bd}\,A$ or $\mathrm{Bd}\,A_{ij}$. Hence, we may conclude that $A_{11}\cup A_{21} W\tilde{A}_1$ and $A_{12}\cup A_{22}\subset\tilde{A}_2$.

Let D be a disk which lies in A_1, whose intersection with A_{11} is the boundary of D and a longitude of A_{11}, and which meets A_{12} precisely in two meridional disks of A_{12}. We assume that D is in general position with respect to $\mathrm{Bd}\,\tilde{A}_1$. Now consider the curves in $D\cap\mathrm{Bd}\,\tilde{A}_1$. Because $\mathrm{Bd}\,\tilde{A}_1$ separates A_{11} from A_{12}, $D\cap\mathrm{Bd}\,\tilde{A}_1$ contains a simple closed curve J that is homotopic in $S^3-(X\cup X^\infty)$ to either a longitude of A_{11} or a meridian of A_{12}. So J is homotopic in $S^3-(X\cup X^\infty)$ to an essential curve γ_0 in A_{1i}, $i=1$ or 2. If J were inessential in $\mathrm{Bd}\,\tilde{A}_1$, then γ_0 is inessential in $S^3-(X\cup X^\infty)$, a contradiction to Theorem 4.3. Hence, J is essential in $\mathrm{Bd}\,\tilde{A}_1$. Let γ_1 be a simple closed curve of $\mathrm{Bd}\,A_1\cap\mathrm{Bd}\,\tilde{A}_1$. Then γ_1 is essential in $\mathrm{Bd}\,\tilde{A}_1$ and misses J. So γ_1 and J are homotopic in $\mathrm{Bd}\,\tilde{A}_1$. Now γ_0 is essential in $\mathrm{Bd}\,N_{2k+2}$, γ_1 is essential in $\mathrm{Bd}\,N_{2k+1}$, and γ_0 is homotopic to γ_1 in $S^3-(X\cup X^\infty)$. This, however, contradicts Theorem 4.4 and the theorem is proved for S^3.

We consider S^3 to be the one point compactification of R^3. If the homeomorphism h could be extended to R^3, then h could be extended to S^3 by sending the point at infinity to itself. Hence, the theorem is also true for R^3.

Appendix A

Let $X=\bigcap M_i$ denote a compactum as described in Section 3. We assume that the sequence M_i ($i>0$) has infinitely many Whitehead constructions. Let n_1 be the number of consecutive Bing constructions placed in M_0 before the first Whitehead construction. In general, let n_i be the number of consecutive Bing constructions between the $(i-1)$st and ith Whitehead constructions in the sequence M_i. Let G be the decomposition of 3-space consisting of points and components of X.

THEOREM (Ancel–Starbird). *The decomposition G is shrinkable if and only if the series $\sum_{i=1}^{\infty}\frac{n_i}{2^i}$ diverges.*

Proof of divergence implies shrinkability.

DEFINITION A1. Let $R_1, R_2, \ldots, R_k, B_1, B_2, \ldots, B_k$ be disjoint meridional disks in a solid torus T. Let $R=\bigcup R_i$ and $B=\bigcup B_i$. We say that (R, B) is a *k-interlacing collection of meridional disks* if each component of $T-(R\cup B)$ has exactly one R_i and one B_j in its closure. It may be helpful to think of the disks in R and B as being colored red and blue, respectively.

DEFINITION A2. Let R and B be disjoint sets and T a solid torus. We say that (R, B) is a k-interlacing for T if there are subsets R' and B' of R and B respectively so that (R', B') is a k-interlacing collection of meridional disks, but it is impossible to find such subsets that form a $(k+1)$-interlacing collection of meridional disks.

The following two lemmas are quite simple and the proofs will not be given.

Lemma A3. *If $k>0$ and (R, B) is a k-interlacing for a solid torus T so that each of $R\cap T$ and $B\cap T$ is the union of finitely many disjoint meridional disks of T, then it is possible to put a Whitehead link T' in T so that (R, B) is a $(2k-1)$-interlacing of T' and each of $R\cap T'$ and $B\cap T'$ is the union of finitely many disjoint meridional disks of T'.*

Lemma A4. *Let $k>0$ and (R, B) be a k-interlacing for a solid torus T so that each of $R\cap T$ and $B\cap T$ is the union of finitely many disjoint meridional disks of T. Then it is possible to put two solid tori T_1 and T_2 so that $T_1\cup T_2$ is a Bing link in T, (R, B) is a $(k-1)$ interlacing of each T_i, and each of $R\cap T_i$ and $B\cap T_i$ is the union of finitely many disjoint meridional disks of T_i.*

Note. We agree that (R, B) is a 0-*interlacing* of T in case T misses either R or B. With this agreement, Lemma A4 makes sense for $k = 1$. Also, it is clear that if (R, B) is a 0-interlacing of T, then (R, B) is a 0-interlacing for any solid torus contained in T.

We assume now that the series $\sum_{i=1}^{\infty}\frac{n_i}{2^i}$ diverges. We show how to construct a homeomorphism h of 3-space, fixed outside M_0, so that the components of $h(M_r)$ are small for some integer r. We may assume, without loss of generality, that the B^2 factor of M_0 is small. Let (R, B) be a k-interlacing of M_0 so that each of $R\cap M_0$ and $B\cap M_0$ is the union of finitely many disjoint meridional disks of M_0 and so that any connected subset of M_0 that misses R or B is small. Choose n so that the partial sum $\sum_{i=1}^{n}\frac{n_i}{2^i}$ is larger than $k/2$. We now choose the homeomorphism h so that $h(M_{j+1})$ is embedded in $h(M_j)$ as in Lemmas A3 and A4 through the nth Whitehead construction. Let M_r be the set obtained with the nth Whitehead construction. A little arithmetic reveals that (R, B) is an m-interlacing of each component of M_r where m equals the maximum of zero and

$$2^{n+1}\left[\frac{k}{2}-\sum_{i=1}^{n}\frac{n_i}{2^i}-\sum_{i=1}^{n}\frac{1}{2^{i+1}}\right].$$

But n was chosen so that m is equal to zero. Hence, the components of $h(M)$ are all small. The proof that divergence implies shrinkability now follows by applying the Bing shrinking criterion [C] and the above argument to components of M_i.

Proof of shrinkability implies divergence.

Definition A5. Let H be a properly embedded disk with holes in a solid torus T so that the inclusion map is interior-essential: i.e., the inclusion maps on $\mathrm{Bd}\,H$ cannot be extended to a map of H into $\mathrm{Bd}\,T$. We call H a *meridional disk with holes* for the solid torus T.

Theorem A6. *Let H be a properly embedded disk with holes in a solid torus T. Then H is a meridional disk with holes if and only if the inclusion $f\colon H\to T$ lifts to a map $\tilde{f}$ from H to the universal cover $\tilde{T} = B^2\times \mathbf{R}$ and $\tilde{f}(H)$ separates $\tilde{T}$ into two unbounded complementary domains.*

Proof. We assume that H is a meridional disk with holes and choose an orientation of H which induces an orientation on the boundary curves of H. Let $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k$ be the oriented boundary curves of H that are nontrivial in $\mathrm{Bd}\,T$. Since H is embedded in T, the nontrivial boundary curves of H are parallel in $\mathrm{Bd}\,T$; i.e., $\gamma_1 = \pm\gamma_j$ in the first homology of $\mathrm{Bd}\,T$. If any one of these is nontrivial in the longitudinal factor then we must have $\sum_{i=1}^{k}\gamma_i = 0$ in the first homology of $\mathrm{Bd}\,T$. Because the fundamental group of $\mathrm{Bd}\,T$ is abelian, $\sum_{i=1}^{k}\gamma_i = 0$ implies that the inclusion map on $\mathrm{Bd}\,H$ can be extended to a map of H into $\mathrm{Bd}\,T$. This contradicts the fact that the disk with holes H is a meridional disk with holes.

Hence we may assume that all of the boundary curves of H must be either trivial or meridional simple closed curves of $\mathrm{Bd}\,T$. Therefore, the inclusion map $f\colon H\to T$ lifts to a map $\tilde{f}\colon H\to\tilde{T}$ from H to the universal cover $\tilde{T}$ of T. Now $\tilde{T}$ is homeomorphic with $B^2\times\mathbf{R}$. We may choose an identification between $\tilde{T}$ and $B^2\times\mathbf{R}$ so that under this identification $\{0\}\times\mathbf{R}$ is in general position with respect to $\tilde{f}(H)$. The orientable surface $\tilde{f}(H)$ separates $\tilde{T} = B^2\times\mathbf{R}$ into two components. Hence, under an orientation of $\tilde{f}(H)$ and $\{0\}\times\mathbf{R}$ we find that the algebraic intersection number must be 0 or ±1. Linking theory then shows that $\sum_{i=1}^{k}\gamma_i$ is the algebraic intersection number. We have already established that $\sum_{i=1}^{k}\gamma_i$ is not zero. Hence, the algebraic intersection number is ±1 which is the same as showing that $\tilde{f}(H)$ separates the universal covering space into two unbounded complementary domains.

The proof of the other direction of the theorem is much easier and is left to the reader.

Note. We have shown that the nontrivial boundary curves of a meridional disk with holes H in a solid torus T must be meridional simple closed curves of T whose algebraic sum in the first homology of $\mathrm{Bd}\,T$ is ±1.

Lemma A7. *Let H_1 and H_2 be disjoint meridional disks with holes in a solid torus T. Let W_1 and W_2 be the closures of the complementary domains of $H_1\cup H_2$ in T. Let P be a compact planar 2-manifold properly embedded in $T-H_1\cup H_2$. If P separates H_1 from H_2 in W_i, then $W_i\cap P$ contains a meridional disk with holes.*

Proof. By lifting W_i to the universal covering space of T, we see that the lift of $W_i\cap P$ separates the universal covering space into complementary domains at least two of which are unbounded. Hence, some component of the lift of $W_i\cap P$ must separate the universal covering space into two unbounded complementary domains. The projection of this component into $W_i \cap P$ is the desired meridional disk with holes of $W_i \cap P$.

DEFINITION A8. We define a *k-interlacing collection of meridional disks with holes* by replacing *meridional disks* in Definition A1 by *meridional disks with holes*.

DEFINITION A9. Let R and B be disjoint sets and T a solid torus. We say that (R, B) is a *k-interlacing* for T if there are subsets R' and B' of R and B, respectively, so that (R', B') is a k-interlacing collection of meridional disks with holes, but it is impossible to find such subsets that form a $(k+1)$-interlacing collection of meridional disks with holes.

Note: Definition A9 generalizes Definition A2.

LEMMA A10. *Suppose R and B are disjoint 2-manifolds properly embedded in a solid torus T so that (R, B) is a k-interlacing for T. If T' is a Whitehead link in T that is in general position with respect to $R \cup B$, then (R, B) is an m-interlacing for T' where $m \geqslant 2k-1$.*

LEMMA A11. *Suppose R and B are disjoint 2-manifolds properly embedded in a solid torus T so that (R, B) is a k-interlacing for T. If T_1 and T_2 are solid Tori that form a Bing link in T and are in general position with respect to $R \cup B$, then (R, B) is an m-interlacing for either T_1 or T_2 where $m \geqslant k-1$.*

We give the proof of A10. The proof of A11 is similar. We first need a definition.

DEFINITION A12. Let D_1 and D_2 be disjoint meridional disks with holes in $B^2 \times \mathbf{R}$ each of which separates $B^2 \times \mathbf{R}$ into two unbounded complementary domains. We say D_1 is *less than* D_2, and write $D_1 < D_2$, if D_2 lies in the complementary domain of $B^2 \times \mathbf{R} - D_1$ whose $\mathbf{R}$ coordinates are bounded below but not above.

Proof of Lemma A10. Since (R, B) is a k-interlacing for T, there are disjoint meridional disks with holes $R_1, R_2, \ldots, R_k, B_1, B_2, \ldots, B_k$ so that $\bigcup R_i = R' \subset R$, $\bigcup B_i = B' \subset B$, and (R', B') is a k-interlacing collection of meridional disks with holes for T. Let $B^2 \times \mathbf{R}$ be the universal cover of T. The lifts of R_i and B_i separate $B^2 \times \mathbf{R}$ into two unbounded complementary domains. Let $R_i(j)$ and $B_i(j)$ $(-\infty < j < \infty)$ be the lifts of R_i and B_i $(1 \leqslant i \leqslant k)$, respectively. Without loss of generality we may suppose that the subscripts have been chosen so that $R_1(0) < B_1(0) < R_2(0) < B_2(0) < \ldots < R_k(0) < B_k(0)$ and for each i $(1 \leqslant i \leqslant k)$ and integers $r < s$ we have both $R_i(r) < R_i(s)$ and $B_i(r) < B_i(s)$. Because the union of all ifts of T' is geometrically central in $B^2 \times \mathbf{R}$, each $R_i(j)$ and $B_i(j)$ must meet a lift of T' in meridional disk with holes. By relabeling, if necessary, we may assume that $R_1(0)$ meets a given lift $\tilde{T}'$ of T' in a meridional disk with holes, but all other lifts of R_i and B_i that are less than $R_1(0)$ do not meet $\tilde{T}'$ in a meridional disk with holes. We claim that $B_k(0)$ must also meet $\tilde{T}'$ in a meridional disk with holes. For some $n \geqslant 0$, $B_k(n)$ must meet $\tilde{T}'$ in a meridional disk with holes. If $n = 0$, our claim is valid. Otherwise, since $B_1(0)$ separates $R_1(0)$ from $B_k(n)$, we can apply Lemma A7 to see that $B_1(0)$ meets $\tilde{T}'$ in a meridional disk with holes.

Let $H_1 \subset R_1(0)$, $H_2 \subset B_k(0)$ be meridional disks with holes in $\tilde{T}'$. Let W_1 and W_2 be the closures of the complementary domains of $H_1 \cup H_2$ in $\tilde{T}'$. By Lemma A7 and the fact that each of $B_1(0), R_2(0), B_2(0), \ldots, R_k(0)$ separates $R_1(0)$ from $B_k(0)$ in $B_2 \times \mathbf{R}$, we see that each of $B_1(0), R_2(0), B_2(0), \ldots, R_k(0)$ meets both W_1 and W_2 in a meridional disk with holes of $\tilde{T}'$. By choosing these disks with holes inductively, we obtain (R'', B''), a $(2k-1)$-interlacing collection of meridional disks with holes where $R'' \subset \bigcup_{i=1}^{k} R_i(0)$ and $B'' \subset \bigcup_{i=1}^{k} B_i(0)$. Projecting this interlacing from $\tilde{T}'$ down to T', we see that the conclusion of Lemma A10 is true.

We now show that if the decomposition is shrinkable, then $\sum_{i=1}^{\infty} \frac{n_i}{2^i}$ diverges. Suppose that $\sum_{i=1}^{\infty} \frac{n_i}{2^i}$ converges. Let k be a positive integer so that $k/2$ is greater than $\sum_{i=1}^{\infty} \frac{n_i}{2^i} + 1$. Let (R, B) be a k-interlacing for M_0 so that each of $R \cap M_0$ and $B \cap M_0$ is the union of finitely many disjoint meridional disks of M_0. We further assume that $R \cup B$ is in general position with each M_i for $i \geqslant 1$. Let M_r be the set obtained with the nth Whitehead construction. By Lemmas A10 and A11, (R, B) is an m-interlacing for some component of M_r where m is greater than or equal to

$$2^{n+1}\left[\frac{k}{2} - \sum_{i=1}^{n} \frac{n_i}{2^i} - \sum_{i=1}^{n} \frac{1}{2^{i+1}}\right].$$

But k was chosen so that this number is positive for any choice of n. Hence for all i some component of M_i must be large enough to meet both R and B. But this contradicts the fact that the diameter of the components of M_i tend to zero as i gets large. This contradiction arose from the supposition that $\sum_{i=1}^{\infty} \frac{n_i}{2^i}$ converges. Therefore we conclude that $\sum_{i=1}^{\infty} \frac{n_i}{2^i}$ diverges.

Appendix B

THEOREM. (De Gryse–Osborne). *Let X be a Bing–Whitehead compactum in 3-space. Then the complement of X is simply connected.*

Proof. Let M_i be a defining sequence for X as described in Section 3 so that $X = \bigcap M_i$. Let γ be a loop in the complement of X. Choose n sufficiently large so that γ lies in the complement of M_n. Without loss of generality we may assume that M_{n+1} is obtained from M_n by placing a Whitehead embedding in each component of M_n. Let $T_1, T_2, \ldots, T_k$ be the components of M_n. Choose disjoint open 3-balls $U_1, U_2, \ldots, U_k$ in 3-space so that T_i is contained in U_i as an unknotted solid torus. The loop γ contracts in 3-space. By general position we may assume that it bounds a singular disk so that for each i, $1 \leqslant i \leqslant k$, the singular disk bounded by γ meets T_i in a finite collection of meridional disks. However, a meridian of $\mathrm{Bd}\, T_i$ bounds a singular disk in $U_i - M_{n+1}$ [Wh]. Hence, γ bounds a singular disk in the complement of M_{n+1}, and our theorem is proved.

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Polynomial growth trivial extensions of simply connected algebras

by

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Abstract. Let A be a finite-dimensional, basic, connected algebra over an algebraically closed field. Denote by $T(A)$ the trivial extension of A by its minimal injective cogenerator. We show that, if A is simply connected, then the following conditions are equivalent: (i) $T(A)$ is nondomestic of polynomial growth, (ii) $T(A)$ is nondomestic of finite growth, (iii) there exists a tubular algebra B such that $T(A) \simeq T(B)$, (iv) A is tilting-cotilting equivalent to a canonical tubular algebra. Isomorphism classes of such algebras are also determined.

Introduction. Let K denote a fixed algebraically closed field, and A a finite-dimensional K-algebra (associative, with an identity) which we shall assume to be basic and connected. We shall denote by $\mathrm{mod}\, A$ the category of finite-dimensional right A-modules. We recall that A is called *simply connected* (in the sense of [2]) if it is triangular, that is, the ordinary quiver of A has no oriented cycles, and such that, for any presentation $A \simeq KQ/I$ of A as a bound quiver algebra, the fundamental group $\pi(Q, I)$ of (Q, I) [18] is trivial. In the representation-finite case, this notion of simple connectedness coincides with the notion introduced in [6]. Further, A is called *domestic* [20] if there exists a finite number of (parametrising) functors $F_i\colon \mathrm{mod}\, K[X] \to \mathrm{mod}\, A$, $1 \leqslant i \leqslant n$, where $K[X]$ is the polynomial algebra in one variable, satisfying the following conditions:

(a) For each i, $F_i = - \underset{K[X]}{\otimes} Q_i$, where Q_i is a $K[X]$-A-bimodule which is finitely generated and free as a $K[X]$-module.

(b) For any dimension d, all but a finite number of isomorphism classes of indecomposable A-modules of K-dimension d are of the form $F_i(M)$, for some i and some indecomposable right $K[X]$-module M.

A is called *n-parametric* if the minimal number of such functors is n. Moreover, for a dimension d, denote by $\mu_A(d)$ the least number of functors $F_i\colon \mathrm{mod}\, K[X] \to \mathrm{mod}\, A$, $1 \leqslant i \leqslant \mu_A(d)$, satisfying the above condition (a) and the following condition:

(b′) All but a finite number of isomorphism classes of indecomposable A-modules of K-dimension d are of the form $F_i(S)$ for some i and some simple right $K[X]$-module S.