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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Trivial bundles and near-homeomorphisms

by

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Abstract. Topological characterizations of trivial bundles with fibres N^τ and R^τ are given. These characterizations together with earlier results of E. V. Shchepin and the author are used in the investigation of strong near-homeomorphism of the four standard spaces D^τ, N^τ, I^τ and R^τ.

Introduction. The major problems in zero-dimensional and infinite-dimensional topology concerning topological characterizations of infinite powers of:

(a) the two-point discrete space D;

(b) the countable infinite discrete space N;

(c) the closed unit interval I;

(d) the real line R,

have been solved. The corresponding characterizations for countable powers were obtained respectively by L. E. J. Brouwer [2], P. S. Alexandrov and P. S. Urysohn [1], H. Toruńczyk [13] and [14]. For uncountable powers these results were obtained respectively by E. V. Shchepin [16], A. Ch. Chigogidze [4], E. V. Shchepin [17] and A. Ch. Chigogidze [4].

Let us now consider the parametrical versions of the characterizations mentioned above. For the readers convenience and for the sake of a unified presentation, from a general viewpoint, some of these results are given formulations differing from the original ones. Further comments will be given below.

THEOREM 1D. *Let $\tau \geqslant \omega$. If $f\colon X \to Y$ is a 0-soft map between zero-dimensional* AE(0)*-compacta and if $w(X) = \tau$ then the following conditions are equivalent*:

1. *f is a trivial bundle with fibre D^τ.*

2. *For every zero-dimensional compactum Z of weight $\leqslant \tau$ and every map $g\colon Z \to X$ the set of embeddings $Z \to X$ is dense in the space $C^g_\tau(Z, X)$.*

THEOREM 1N. *Let $\tau \geqslant \omega$. If $f\colon X \to Y$ is a 0-soft map between strongly zero-dimensional* AE(0)*-spaces and if $w(X) = \tau$ then the following conditions are equivalent*:

1. *f is a trivial bundle with fibre N^τ.*

2. *For every strongly zero-dimensional space Z of R-weight $\leqslant \tau$ and every map $g\colon Z \to X$ the set of C-embeddings $Z \to X$ is dense in the space $C^g_\tau(Z, X)$.*

THEOREM 1I. *Let* $\tau \geqslant \omega$. *If* $f\colon X \to Y$ *is a soft map between* ANR-*compacta and if* $w(X) = \tau$ *then the following conditions are equivalent*:

1. f *is a trivial bundle with fibre* I^τ.
2. *For every compactum* Z *of weight* $\leqslant \tau$ *and every map* $g\colon Z \to X$ *the set of embeddings* $Z \to X$ *is dense in the space* $C^g_\tau(Z, X)$.

THEOREM 1R. *Let* $\tau \geqslant \omega$. *If* $f\colon X \to Y$ *is a soft map between* ANR-*spaces and if* $w(X) = \tau$ *then the following conditions are equivalent*:

1. f *is a trivial bundle with fibre* R^τ.
2. *For every space* Z *of* R-*weight* $\leqslant \tau$ *and every map* $g\colon Z \to X$ *the set of* C-*embeddings* $Z \to X$ *is dense in the space* $C^g_\tau(Z, X)$.

Theorems 1D, 1N, 1I and 1R for $\tau = \omega$ were proved respectively by S. Sirota [12], A. Ch. Chigogidze [4], H. Toruńczyk and J. E. West ([15] and unpublished results). The author wishes to thank E. V. Shchepin for information concerning Theorem 1R for $\tau = \omega$. Theorems 1D and 1I for $\tau > \omega$ were proved respectively by E. V. Shchepin [17] and A. Ch. Chigogidze [5]. Theorems 1N and 1R for $\tau > \omega$ are proved in the present paper. Note that taking for Y a one-point space we obtain the above mentioned characterizations of D^τ, N^τ, I^τ and R^τ.

We start with some definitions. Further notions are introduced later in the text, as they are needed.

We consider only Tichonov spaces and continuous maps. Homeomorphisms always are onto. $\operatorname{cov}(X)$ denotes the family of all countable functionally open covers of a space X. If τ is an infinite cardinal and Z, X are given spaces, then $C_\tau(Z, X)$ denotes the set of all maps $Z \to X$ with the topology defined as follows. Given $h \in C_\tau(Z, X)$, the family

$$\{B(h, \{\mathcal{U}_t\colon t \in T\})\colon |T| < \tau \text{ and } \mathcal{U}_t \in \operatorname{cov}(X)\}$$

is a base of open neighbourhoods of h, where

$$B(h, \{\mathcal{U}_t\colon t \in T\}) = \{h' \in C_\tau(Z, X)\colon h' \text{ is } \mathcal{U}_t\text{-close to } h \text{ for each } t \in T\}.$$

Maps in $B(h, \{\mathcal{U}_t\colon t \in T\})$ are said to be $\{\mathcal{U}_t\colon t \in T\}$-*close* to h. For maps $f\colon X \to Y$ and $g\colon Z \to X$ we denote by $C^g_\tau(Z, X)$ the subspace of $C_\tau(Z, X)$ consisting of all maps $h\colon Z \to X$ with $fh = fg$. It is easy to see that if X has a countable base then the space $C_\omega(Z, X)$ coincides with the space $C(Z, X)$ endowed with the limitations topology [14].

Let us now recall some definitions from the general theory of AE(n)-spaces and n-soft maps [4], [6]. As usual, $C(X)$ denotes the set of all continuous real-valued functions on a space X and $C(f)\colon C(Y) \to C(X)$ denotes the operator induced by a map $f\colon X \to Y$. If $Z_0 \subseteq Z$, then $C(Z)/Z_0$ is the set of all elements of $C(Z_0)$ extendible to the whole of Z. Clearly, if $Z_0 \subseteq Z$, then the equality $C(Z_0) = C(Z)/Z_0$ characterizes the C-embedded subspaces. It is well-known that closed subspaces of normal spaces are C-embedded and that compacta are C-embedded in arbitrary ambient (Tichonov) spaces. A C-embedding is an embedding with C-embedded range. It follows from the general definition (Definition 1.1 and Proposition 1.7 from [4]) that each AE(n)-space is realcompact. Consequently, without loss of generality we may restrict ourselves to give the corresponding definition only in the class of realcompact spaces. By *dimension* we mean the dimension dim which is defined by means of finite functionally open covers.

DEFINITION 1. A realcompact space X is called an *absolute extensor in dimensional* n ($n = 0, 1, \ldots, \infty$) if, for any real-compact space Z of dimension $\dim Z \leqslant n$ and any closed subspace Z_0 of Z, each map $f\colon Z_0 \to X$ such that $(C(f))(C(X)) \subseteq C(Z)/Z_0$ can be extended to the whole of Z. The spaces in AE(∞) are called *absolute extensors* (AE).

DEFINITION 2. A *Tichonov space* X is defined to be an absolute (neighbourhood) retract if for any C-embedding of X in an arbitrary Tichonov space Y there exist a retraction of Y (respectively, of some functionally open neighbourhood of X in Y) onto X.

It is easy to conclude that the class AE coincides with the class AR. Under this definition, retracts of all possible powers of the real line (respectively, of functionally open subspaces of powers of the real line) turn out to be precisely absolute (neighbourhood) retracts.

DEFINITION 3. A surjection $f\colon X \to Y$ between AE(0)-spaces is said to be 0-*soft* if, for any strongly zero-dimensional realcompact space Z, any closed subspace Z_0 of Z and any two maps $g\colon Z_0 \to X$ and $h\colon Z \to Y$ such that $C(g)(C(X)) \subseteq C(Z)/Z_0$ and $fg = h/Z_0$, there exists a map $k\colon Z \to X$ such that $fk = h$ and $k/Z_0 = g$.

DEFINITION 4. A surjection $f\colon X \to Y$ between ANR-spaces is said to be *soft* if, for any realcompact space Z, any closed subspace Z_0 of Z, and any two maps $g\colon Z_0 \to X$ and $h\colon Z \to Y$ such that $C(g)(C(X)) \subseteq C(Z)/Z_0$ and $fg = h/Z_0$, there exists a map $k\colon Z \to X$ such that $fk = h$ and $k/Z_0 = g$.

These definitions show that, in the class of compacta, the notions of AE(0) and ANR-spaces, as well as 0-soft and soft maps, agree with the commonly used definitions of these concepts [16], [18]. Consequently, every compact metric space is an AE(0)-space. Moreover, the class of AE(0)-spaces with a countable base coincides with the class of Polish spaces [4]. Similarly, the class of A(N)R-spaces with a countable base coincides with the class of Polish A(N)R-spaces. 0-soft maps between Polish spaces are precisely open surjections [4] and soft maps between metric ANR-compacta are precisely Hurewicz's fibrations with AR-fibers. Let also recall us that any Tychonoff space can be C-embedded in R^τ for a suitable cardinal τ. The smallest infinite cardinal with this property is called the R-*weight* of a space X (notation: R-$w(X)$). The R-weight is countable precisely for Polish spaces. The Weight coincides with the R-weight for non-discrete compacta, as well as for non-discrete AE(0)-spaces [4]. A map $f\colon X \to Y$ is called a *map with Polish kernel* if there exists a Polish space P such that X is C-embedded in the product $Y \times P$ and f coincides with the restriction of the projection $Y \times P \to Y$ onto X. Any undefined notions concerning inverse spectra are used in the sense of [4] and [18].

1. Topological characterizations of trivial bundles with fibres N^τ and R^τ.

In this section we prove Theorems 1N and 1R for $\tau > \omega$.

Proof of Theorem 1R, $\tau > \omega$. (1) $\Rightarrow$ (2). In this case $X \approx Y \times R^\tau$ and f is the projection of $Y \times R^\tau$ onto Y. Let A be any set of cardinality τ. Consider a space Z of R-weight $\leqslant \tau$ and a map $g\colon Z \to Y \times R^A$. Let $\{\mathscr{U}_t\colon t \in T\} \subseteq \operatorname{cov}(Y \times R^A)$ where $\omega \leqslant |T| < \tau$. For each $t \in T$ there exist $\mathscr{V}_t \in \operatorname{cov}(Y)$ and $\mathscr{W}_t \in \operatorname{cov}(R^A)$ with $\mathscr{V}_t \times \mathscr{W}_t \prec \mathscr{U}_t$. Since $|T| < \tau$, there exist a subset B of A of cardinality $|T|$ and $\mathscr{W}'_t \in \operatorname{cov}(R^B)$ such that $\mathscr{W}_t = \pi_B^{-1}(\mathscr{W}'_t)$ for each $t \in T$, where $\pi_B\colon R^A \to R^B$ denotes the natural projection. Since $|A - B| = \tau$, there exists a C-embedding $h'\colon Z \to R^{A-B}$. It easy to see that a map $h = fg\Delta\pi_B qg\Delta h'\colon Z \to Y \times R^A$ (where $q\colon Y \times R^A \to R^A$ denotes the natural projection) is a C-embedding, $\{\mathscr{U}_t\colon t \in T\}$-close to g, and satisfies the desired equality $fh = fg$.

The proof of the implication (2) $\Rightarrow$ (1) involves the main idea of the author [5] and is based on the following proposition (Lemma 7.11 from [4]).

LEMMA 1. *Let $S = \{X_n, p_n^{n+1}, \omega\}$ be an inverse sequence consisting of* ANR-*spaces and soft adjacent projections having Polish kernels. Assume that, for any $n \in \omega$, the space X_{n+1} contains a C-embedded copy of the product $X_n \times R^\omega$ such that $p_n^{n+1}/(X_n \times R^\omega) = \pi_{X_n}$, where $\pi_{X_n}\colon X_n \times R^\omega \to X_n$ is the natural projection. Then the limit projection $p_0\colon \lim S \to X_0$ is a trivial bundle with fibre R^ω.*

Let us now consider a soft map $f\colon X \to Y$ between ANR-spaces with $w(X) = \tau > \omega$. By Theorem 3.2 from [4] (see also [6]) there exist well-ordered continuous spectra $S_X = \{X_\alpha, p_\alpha^\beta, \tau\}$, $S_Y = \{Y_\alpha, q_\alpha^\beta, \tau\}$ and a morphism $\{f_\alpha\colon X_\alpha \to Y_\alpha\}$ between them such that the following conditions are satisfied:

(i) $X = \lim S_X$, $Y = \lim S_Y$ and $f = \lim f_\alpha$.

(ii) X_0 and Y_0 are Polish ANR-spaces and f_0 is a soft map between them.

(iii) For any $\alpha \in \tau$, X_α and Y_α are ANR-spaces and f_α is a soft map between them.

(iv) All adjacent projections in the spectra S_X and S_Y are soft and have Polish kernels.

(v) All adjacent square diagrams (formed by adjacent projection of spectra and the corresponding adjacent elements of the morphism) are soft (see [4], [6]) and, moreover, their characteristic maps (see [4], [8], [18]) have Polish kernels.

We introduce the following notation:

(a) $Z_\alpha^{\alpha+1}$ denotes the fibered product of spaces X_α and $Y_{\alpha+1}$ with respect to maps f_α and $q_\alpha^{\alpha+1}$; $\varphi_\alpha^{\alpha+1}\colon Z_\alpha^{\alpha+1} \to X_\alpha$ and $\psi_\alpha^{\alpha+1}\colon Z_\alpha^{\alpha+1} \to Y_{\alpha+1}$ are the corresponding canonical projections, $\alpha \in \tau$.

(b) $h_\alpha^{\alpha+1}\colon X_{\alpha+1} \to Z_\alpha^{\alpha+1}$ denotes the diagonal product of maps $p_\alpha^{\alpha+1}$ and $f_{\alpha+1}$, $\alpha \in \tau$.

(c) Z_α denotes the fibered product of spaces X_α and Y with respect to maps f_α and q_α; $\varphi_\alpha\colon Z_\alpha \to X_\alpha$ and $\psi_\alpha\colon Z_\alpha \to Y$ are the corresponding canonical projections.

(d) $h_\alpha\colon X \to Z_\alpha$ denotes the diagonal product of p_α and f, $\alpha \in \tau$.

(e) $Z_{-1} = Y$, $t_{-1}^0 = \psi_0\colon Z_0 \to Z_{-1}$.

(f) $t_\alpha^{\alpha+1}\colon Z_{\alpha+1} \to Z_\alpha$ denotes the diagonal product of maps $p_\alpha^{\alpha+1}\varphi_{\alpha+1}$ and $\psi_{\alpha+1}$, $\alpha \in \tau$.

(g) $r_\alpha\colon Z_\alpha \to Z_\alpha^{\alpha+1}$ denotes the diagonal product of maps φ_α and $q_{\alpha+1}\psi_\alpha$, $\alpha \in \tau$.

It is easy to see that the limit space of the well-ordered continuous inverse spectrum $S = \{Z_\alpha, t_\alpha^{\alpha+1}\}$ is homeomorphic to X and the limit projection $t\colon \lim S \to Y$ coincides with the given map f. It should be observed that, for each $\alpha \geqslant 0$, the square diagram consisting of space $Z_{\alpha+1}$, Z_α, $Z_\alpha^{\alpha+1}$, $X_{\alpha+1}$ and maps $t_\alpha^{\alpha+1}$, r_α, $h_\alpha^{\alpha+1}$ and $\varphi_{\alpha+1}$ is a pull-back square, and hence, by (v), $t_\alpha^{\alpha+1}$ is a soft map and has a Polish kernel.

(*) We now show that for each $\alpha \in \tau$ there exist an index $\beta(\alpha)$ such that the projection $t_\alpha^{\beta(\alpha)}\colon Z_{\beta(\alpha)} \to Z_\alpha$ has a Polish kernel and $Z_{\beta(\alpha)}$ contains a C-embedded copy of the product $Z_\alpha \times R^\omega$ in such a manner that $t_\alpha^{\beta(\alpha)}/(Z_\alpha \times R^\omega) = \pi_{Z_\alpha}$ where $\pi_{Z_\alpha}\colon Z_\alpha \times R^\omega \to Z_\alpha$ is the natural projection. To prove this fact it suffices to show that X contain a C-embedded copy of $Z_\alpha \times R^\omega$ in such a manner that

$$h_\alpha/(Z_\alpha \times R^\omega) = \pi_{Z_\alpha}.$$

It follows from our construction that h_α is a soft map. Hence, there is a map $i\colon Z_\alpha \times R^\omega \to X$ such that $h_\alpha i = \pi_{Z_\alpha}$. Now, fix a subcollection $\{\mathscr{U}'_t\colon t \in T\} \subseteq \operatorname{cov}(X_\alpha)$ with $|T| = w(X_\alpha) < \tau$ satisfying the following condition: for any space Z, any two $\{\mathscr{U}_t\colon t \in T\}$-close maps $Z \to X_\alpha$ coincide. Put $\mathscr{U}_t = p_\alpha^{-1}(\mathscr{U}'_t)$, $t \in T$. Then by (2) there exists a C-embedding $g\colon Z_\alpha \times R^\omega \to X$, $\{\mathscr{U}_t\colon t \in T\}$-close to i, such that $fg = fi$. By (d), $h_\alpha g = (p_\alpha \Delta f)i = h_\alpha i = \pi_{Z_\alpha}$. This completes the proof of (*).

Now, by (*) and Lemma 1, we may assume without loss of generality that all adjacent projections of the spectrum S are trivial bundles with fibre R^ω. By assumption, $w(X) = \tau$ and, by (ii), $w(X_0) = \omega$. Consequently, by (iv), the length of the spectrum S is equal to τ. These facts show that f is a trivial bundle with fibre R^τ. This completes the proof of the theorem.

The proof of Theorem 1N, $\tau > \omega$, is omitted, since it is completely analogous to the proof of Theorem 1R, $\tau > \omega$. It should be only remarked that in this case we must use (instead of Lemma 1) the following proposition (Lemma 7.8 from [4]):

LEMMA 2. *Let $S = \{X_n, p_n^{n+1}, \omega\}$ be an inverse sequence consisting of strongly zero-dimensional* AE(0)-*spaces and* 0-*soft adjacent projections having Polish kernels. Assume that, for any $n \in \omega$, the space X_{n+1} contains a C-embedded copy the product $X_n \times N^\omega$ such that $p_n^{n+1}/(X_n \times N^\omega) = \pi_{X_n}$, where $\pi_{X_n}\colon X_n \times N^\omega \to X_n$ is the natural projection. Then the limit projection $p_0\colon \lim S \to X_0$ is a bundle with fibre N^ω.*

The following corollaries are immediate consequence of Theorems 1N and 1R.

Corollary 1N. *Let $\tau \geqslant \omega$ and $f\colon X \to Y$ be a 0-soft map between strongly zero-dimensional* AE(0)*-spaces, $w(X) \leqslant \tau$, and let $\pi_X\colon X \times N^\tau \to X$ be the natural projection. Then the composition $f\pi_X$ is a trivial bundle with fibre N^τ.*

Corollary 1R. *Let $\tau \geqslant \omega$ and $f\colon X \to Y$ be a soft map between* ANR*-spaces, $w(X) \leqslant \tau$, and let $\pi_X\colon X \times R^\tau \to X$ be the natural projection. Then the composition $f\pi_X$ is a trivial bundle with fibre R^τ.*

The corresponding propositions for D and I are also known (see [17] and [5]).

Corollary 1D. *Let $\tau \geqslant \omega$ and $f\colon X \to Y$ be a 0-soft map between zero-dimensional* AE(0)*-compacta, $w(X) \leqslant \tau$, and let $\pi_X\colon X \times D^\tau \to X$ be the natural projection. Then the composition $f\pi_X$ is a trivial bundle with fibre D^τ.*

Corollary 1I. *Let $\tau \geqslant \omega$ and $f\colon X \to Y$ be a soft map between* ANR*-compacta, $w(X) \leqslant \tau$, and let $\pi_X\colon X \times I^\tau \to X$ be the natural projection. Then the composition $f\pi_X$ is a trivial bundle with fibre I^τ.*

Now we formulate a final group of propositions, which will be used in the next section.

Corollary 2R. *Let $\tau > \omega$ and let $f\colon X \to Y$ be a soft map of an R^τ-manifold X onto a space Y with $w(Y) < \tau$. Then f is a trivial bundle with fibre R^τ.*

(An R^τ-manifold is a Tychonov space which admits a countable functionally open cover whose each element is homeomorphic to R^τ[4]; any R^τ-manifold is realcompact; this follows e.g. from [9]).

Proof. By Theorem 2 from [4], $X \approx X' \times R^\tau$ where X' is an R^ω-manifold. Let A be a set of cardinality τ. Then there exist a subset $B \subseteq A$ with $|B| = \max\{\omega, w(Y)\}$ and a map $f'\colon X' \times R^B \to Y$ such that $f = (\mathrm{id}_{X'} \times \pi_B)\cdot f'$, where $\pi_B\colon R^A \to R^B$ denotes the natural projection. It is easy to see that f' is a soft map and consequently, by Corollary 1R, f is a trivial bundle with fibre R^τ.

Similarly we have

Corollary 2N. *Let $\tau > \omega$ and let $f\colon N^\tau_1 \to Y$ be a 0-soft map onto a strongly zero-dimentional space Y with $w(Y) < \tau$. Then f is a trivial bundle with fibre N^τ.*

Corollary 2D (see [17]). *Let $\tau > \omega$ and let $f\colon D^\tau \to Y$ be a 0-soft map with $w(Y) < \tau$. Then f is a trivial bundle with fibre D^τ.*

Corollary 2I (see [17]). *Let $\tau > \omega$ and let $f\colon X \to Y$ be a soft map of a compact I^τ-manifold X onto a space Y with $w(Y) < \tau$. Then f is a trivial bundle with fibre I^τ.*

§ 2. Near-homeomorphisms. In this section Corollaries 2D, 2N, 2I and 2R are used to characterize maps of D^τ, N^τ, I^τ and R^τ into itself which are τ-approximable by homeomorphisms. A map $f\colon X \to Y$ is said to be a τ-near-homeomorphism if, given $\{\mathcal{U}_t\colon t \in T\} \subseteq \mathrm{cov}(Y)$, $|T| < \tau$, there is a homeomorphism $h\colon X \to Y$ $\{\mathcal{U}_t\colon t \in T\}$-close to f. The set of all τ-near-homeomorphisms $X \to Y$ will be denoted by $NH_\tau(X, Y)$. Clearly, if Y is a Polish space then $NH_\omega(X, Y)$ consists of near-homeomorphisms in the usual sense.

Definition 5. Let $\tau \geqslant \omega$. A map $f\colon X \to Y$ between realcompact spaces is said to be *τ-approximatively n-soft* ($n = 0, 1, \ldots, \infty$) if, given a realcompact space Z with $\dim Z \leqslant n$, its closed subspace Z_0, subcollection $\{\mathcal{U}_t\colon t \in T\} \subseteq \mathrm{cov}(Y)$ with $|T| < \tau$, and two maps $g\colon Z_0 \to X$ and $h\colon Z \to Y$ such that $C(g)(C(X)) \subseteq C(Z)/Z_0$ and $fg = h/Z_0$, there exists a map $k\colon Z \to X$ extending g so that fk is $\{\mathcal{U}_t\colon t \in T\}$-close to h. The τ-approximatively ∞-soft maps will be called *τ-approximatively soft maps*.

Evidently every n-soft map is τ-approximatively n-soft for each $\tau \geqslant \omega$.

Lemma 3. *A map $f\colon X \to Y$ between Polish spaces is ω-approximatively n-soft iff, given a Polish space Z with $\dim Z \leqslant n$, its closed subspace Z_0, open cover $\mathcal{U} \in \mathrm{cov}(Y)$ and two maps $g\colon Z_0 \to X$ and $h\colon Z \to Y$ such that $fg = h/Z_0$, there exists a map $k\colon Z \to X$ extending g so that fk is $\mathcal{U}$-close to h.*

Proof. It is easy to see that for ω-approximatively n-soft maps between Polish spaces the condition of the lemma is satisfied. Let us prove the converse. Let Z be a realcompact space of dimension $\dim Z \leqslant n$, Z_0 be a closed subspace of Z, $\{\mathcal{U}_t\colon t \in T\} \subseteq \mathrm{cov}(Y)$, $|T| < \omega$, and $g\colon Z_0 \to X$ and $h\colon Z \to Y$ be maps such that $C(g)(C(X)) \subseteq C(Z)/Z_0$ and $fg = h/Z_0$. Represent Z as a limit space of some factorizing sigma-spectrum $S_Z = \{Z_\alpha, p_\alpha^\beta, A\}$ consisting of Polish spaces of dimension $\dim Z_\alpha \leqslant n$ (see [7], Theorem 1). Consider also the spectrum

$$S_{Z_0} = \{\mathrm{cl}_{Z_\alpha}(p_\alpha(Z_0)), p_\alpha^\beta, A\}$$

associated with the closed subspace Z_0 of Z. By the spectral theorem of E. V. Shchepin [18], there exist an index $\alpha_1 \in A$ and a map $h_{\alpha_1}\colon Z_{\alpha_1} \to Y$ such that $h = h_{\alpha_1}p_{\alpha_1}$. Let us also note that the spectrum S_{Z_0} is not factorizing, in general. But the condition $C(g)(C(X)) \subseteq C(Z)/Z_0$ is sufficient for the existence of an index $\alpha_2 \in A$ and a map $g_{\alpha_2}\colon \mathrm{cl}_{Z_{\alpha_2}}(p_{\alpha_2}(Z_0)) \to X$ such that $g = g_{\alpha_2}p_{\alpha_2}/Z_0$. Without loss of generality we may assume that $\alpha_1 = \alpha_2 = \alpha$, $h_{\alpha_1} = h_\alpha$ and $g_{\alpha_2} = g_\alpha$. Since $fg_\alpha/p_\alpha(Z_0) = h_\alpha/p_\alpha(Z_0)$ and $p_\alpha(Z_0)$ is dense in $\mathrm{cl}_{Z_\alpha}(p_\alpha(Z_0))$, we conclude that $fg_\alpha = h_\alpha/\mathrm{cl}_{Z_\alpha}(p_\alpha(Z_0))$. By assumption, there exists a map $k_\alpha\colon Z_\alpha \to X$ extending g_α so that fk_α is $\mathcal{U}$-close to h_α, where $\mathcal{U}$ refines each $\mathcal{U}_t$, $t \in T$. Clearly, the map $k = k_\alpha p_\alpha$ has the desired properties. This completes the proof.

It should be noted that if X and Y are compact spaces then a map $f\colon X \to Y$ is τ-approximatively n-soft iff the condition from Definition 5 holds only for compact spaces Z. Clearly, in this case the inclusion $C(g)(C(X)) \subseteq C(Z)/Z_0$ is automatically satisfied. Moreover, if X and Y are metrizable compacta, then, as in Lemma 3, a map $f\colon X \to Y$ is ω-approximatively n-soft iff the condition from that definition holds, only for metrizable compacta.

Now we formulate the main results of this section.

Theorem 2D. *Let $\tau \geqslant \omega$ and $f \in C_\tau(D^\tau, D^\tau)$. Then $f \in NH_\tau(D^\tau, D^\tau)$ iff f is a τ-approximatively 0-soft map*

Theorem 2N. *Let $\tau \geqslant \omega$ and $f \in C_\tau(N^\tau, N^\tau)$. Then $f \in NH_\tau(N^\tau, N^\tau)$ iff f is a τ-approximatively 0-soft map.*

Theorem 2I. *Let $\tau \geqslant \omega$ and $f \in C_\tau(M, M')$ where M and M' are compact I^τ-manifolds. Then $f \in NH_\tau(M, M')$ iff f is a τ-approximatively soft map.*

Theorem. 2R. *Let $\tau \geqslant \omega$ and $f \in C_\tau(M, M')$ where M and M' are R^τ-manifolds. Then $f \in NH_\tau(M, M')$ iff f is a τ-approximatively soft map.*

Remark. The validity of Theorem 2D for $\tau = \omega$ easily follows from the following obvious fact: $f\colon D^\omega \to D^\omega$ is a near-homeomorphism iff f is a surjection. Similarly, the validity of Theorem 2N for $\tau = \omega$ follows from the following fact: $f\colon N^\omega \to N^\omega$ is a near-homeomorphism iff $f(N^\omega)$ is dense in N^ω. Let us also recall that a map between compact I^ω-manifolds, as well as between R^ω-manifolds, is a near-homeomorphism iff it is a fine homotopy equivalence (see [3], [10], [11]). Clearly, ω-approximatively soft maps in both cases are fine homotopy equivalences [11]. On the other hand, it is an easy exercise to show that every near-homeomorphism between Polish ANR's is ω-approximatively soft. These remarks yield the validity of Theorem 2I and 2R for $\tau = \omega$.

The proofs of the above theorems in the case of $\tau > \omega$ are completely similar. So we give, for illustration, only the proof of Theorem 2R.

Proof of Theorem 2R, $\tau > \omega$. Suppose that $f\colon M \to M'$ is a τ-approximatively soft map between R^τ-manifolds and let $\{\mathscr{U}_t\colon t \in T\}$ be any subcollection of $\operatorname{cov}(M')$ with $\omega \leqslant |T| = \lambda < \tau$. Let us represent M and M' as the limit spaces of factorizing λ-spectra (see [18]) $S_M = \{M_\alpha, p_\alpha^\beta, A\}$ and $S_{M'} = \{M'_\alpha, q_\alpha^\beta, A\}$ consisting of R^λ-manifolds and soft limit projections (the possibility of such representation follows from [4], Theorem 2). Without loss of generality we may suppose that a map f is the limit map of some morphism

$$\{f_\alpha\colon M_\alpha \to M'_\alpha, A\}\colon S_M \to S_{M'}.$$

Since $S_{M'}$ is a factorizing λ-spectrum, there exist an index $\alpha \in A$ and covers $\mathscr{U}_t^\alpha \in \operatorname{cov}(M'_\alpha)$ such that $\mathscr{U}_t = q_\alpha^{-1}(\mathscr{U}_t^\alpha)$ for each $t \in T$. Fix a subcollection $\{\mathscr{V}_t^\alpha\colon t \in T\}$ of $\operatorname{cov}(M'_\alpha)$ such that, for any space Z, any two $\{\mathscr{V}_t^\alpha\colon t \in T\}$-close maps of Z into M'_α coincide (recall that $w(M'_\alpha) = \lambda$). Put $\mathscr{V}_t = q_\alpha^{-1}(\mathscr{V}_t^\alpha)$, $t \in T$.

We show that the map $f_\alpha\colon M_\alpha \to M'_\alpha$ is soft. Consider a realcompact space Z, its closed subspace Z_0 and two maps $g_\alpha\colon Z_0 \to M_\alpha$ and $h_\alpha\colon Z \to M'_\alpha$ such that $C(g_\alpha)(C(M_\alpha)) \subseteq C(Z)/Z_0$ and $f_\alpha g_\alpha = h_\alpha/Z_0$. Since the limit projection p_α of the spectrum S_M is a soft map, and hence a retraction, there exists a map $i\colon M_\alpha \to M$ such that $p_\alpha i = \mathrm{id}_{M_\alpha}$. Let $g = ig_\alpha$. It is easy to see that $q_\alpha fg = h_\alpha/Z_0$ and $C(fg)(C(M')) \subseteq C(Z)/Z_0$. Consequently, by the softness of the limit projection q_α of the spectrum $S_{M'}$, there exists a map $h\colon Z \to M'$ such that $fg = h/Z_0$ and $q_\alpha h = h_\alpha$. Evidently $C(g)(C(M)) \subseteq C(Z)/Z_0$. By assumption, f is a τ-approximatively soft map. Hence, there exists a map $k\colon Z \to M$ extending g so fk is $\{\mathscr{V}_t\colon t \in T\}$-close to h. Put $k_\alpha = p_\alpha k$. It is an immediate consequence of our construction that $g_\alpha = k_\alpha/Z_0$ and $f_\alpha k_\alpha = h_\alpha$. Thus f_α is a soft map.

Thus the composition $f_\alpha p_\alpha\colon M \to M'_\alpha$ and the limit projection $q_\alpha\colon M' \to M'_\alpha$ of the spectrum $S_{M'}$ are soft maps. Let us now recall that $w(M'_\alpha) < w(M')$. Hence, by Corollary 2R, the maps $f_\alpha p_\alpha$ and q_α are trivial bundles with fibre R^τ. Since the space M'_α is a common base of both bundles, we conclude that there exists a homeomorphism $s\colon M \to M'$ such that $q_\alpha s = f_\alpha p_\alpha$. Clearly, s is $\{\mathscr{U}_t\colon t \in T\}$-close to f and consequently f is a τ-near-homeomorphism.

Conversely, suppose that $f\colon M \to M'$ is a τ-near-homeomorphism between R^τ-manifolds and that $\tau > \omega$. Let Z be a realcompact space, Z_0 be its closed subspace, $\{\mathscr{U}_t\colon t \in T\} \subseteq \operatorname{cov}(M')$, $\omega \leqslant |T| = \lambda < \tau$, and let $g\colon Z_0 \to M$ and $h\colon Z \to M'$ be maps such that $C(g)(C(M)) \subseteq C(Z)/Z_0$ and $fg = h/Z_0$. Represent M' as the limit space of some factorizing λ-spectrum $S = \{M'_\alpha, q_\alpha^\beta, A\}$ with soft limit projections. Then, as above, there exist an index $\alpha \in A$ and covers $\mathscr{U}_t^\alpha \in \operatorname{cov}(M'_\alpha)$ such that $\mathscr{U}_t = q^{-1}(\mathscr{U}_t^\alpha)$ for each $t \in T$. Since $w(M'_\alpha) = \lambda$, we can fix a subcollection $\{\mathscr{V}_t^\alpha\colon t \in T\} \subseteq \operatorname{cov}(M'_\alpha)$ such that, for any space Z, any two $\{\mathscr{V}_t^\alpha\colon t \in T\}$-close maps of Z into M'_α coincide. Put $\mathscr{V}_t = q_\alpha^{-1}(\mathscr{V}_t^\alpha)$, $t \in T$. By assumption, there exists a homeomorphism $s\colon M \to M'$ which is $\{\mathscr{V}_t\colon t \in T\}$-close to f. Then we have $q_\alpha f = q_\alpha s$ and consequently the map $q_\alpha f$, equal to the composition of soft maps q_α and s, is also a soft map. Hence, there exists a map $k\colon Z \to M$ such that $k/Z_0 = g$ and $q_\alpha fk = q_\alpha h$. By the last equality, the composition fk is $\{\mathscr{U}_t\colon t \in T\}$-close to h. Hence f is a τ-approximatively soft map. This completes the proof.

Corollary 3D. *Let $\tau \geqslant \omega$. Then the projection $\pi_1\colon D^\tau \times D^\tau \to D^\tau$ is a τ-near-homeomorphism.*

Corollary 3N. *Let $\tau \geqslant \omega$. Then the projection $\beta_1\colon N^\tau \times N^\tau \to N^\tau$ is a τ-near-homeomorphism.*

Corollary 3I. *Let $\tau \geqslant \omega$ and X be a compact I^τ-manifold. Then the projection $\pi_X\colon X \times I^\tau \to X$ is a τ-near-homeomorphism.*

Corollary 3R. *Let $\tau \geqslant \omega$ and X be an R^τ-manifold. Then the projection $\pi_X\colon X \times R^\tau \to X$ is a τ-near-homeomorphism.*

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The second Peano derivative as a composite derivative

by

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Abstract. Differentiable functions $f\colon R\to R$ which simultaneously have a second derivative in the Peano sense, f_2, and a second derivative in the composite sense, $(f')'_c$, are investigated. It is shown that $\{x\colon (f')'_c(x)\neq f_2(x)\}$ is a scattered set, i.e. a countable set not dense in any perfect set. As a corollary it follows that f_2 is the derivative of f' in the composite sense.

1. One of the long outstanding problems concerning Peano derivatives is the lack of a precise description of in what sense an $(n+1)$th Peano derivative can be considered as a derivative of the associated (n)th Peano derivative. In this paper we provide an answer to that problem in the case when $n=1$ and the derivative is taken in the composite sense. To make the presentation as readily intelligible as possible requires a little background information.

There is a wealth of information about certain aspects of the class of Peano derivatives. The interested reader should see for example the excellent survey [2]. It is also safe to say that all known properties of these functions are also properties of approximate derivatives, see [4], [7]. However, for approximately differentiable functions $f\colon R\to R$ and its approximate derivative, g, the following property is known to hold, [6],:

For any fixed perfect set P, there is an open interval, (a, b) having nonempty intersection with P, such that for any x in $(a, b)\cap P$,

$$\lim_{\substack{h\to 0\\ x+h\in P}} \frac{f(x+h)-f(x)}{h} = g(x).$$

It is naturally reasonable to hope an analogous situation holds for the class of Peano derivatives. In [6], the above enclosed relationship for a pair of functions f and g was formalized by saying f was compositely differentiable to g and that g was a composite derivative of f.

Using that terminology, we can rephrase the previously mentioned problem as: Does the nth Peano derivative compositely differentiate to the $(n+1)$th Peano derivative?

Historically, Denjoy has provided partial answers to that problem, [1]. He established that if besides the $(n+1)$th P-derivative, the $(n+2)$th Peano is also assumed