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The space $(\omega^*)^{n+1}$ is not always a continuous image of $(\omega^*)^n$

by

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Abstract. It is shown that the following statement is relatively consistent with ZFC: "For all $n \in \omega$, the space $(\omega^*)^{n+1}$ is not a continuous image of the space $(\omega^*)^n$".

§ 1. Introduction. By ω^* we denote the remainder of the Čech–Stone compactification of ω, the countable discrete space, and by $(\omega^*)^n$ the product of n copies of ω^*. It was shown in [vD] that the spaces $(\omega^*)^n$ and $(\omega^*)^m$ are not homeomorphic whenever $n \neq m$. Clearly, if $n < m$, then $(\omega^*)^n$ is a continuous image of $(\omega^*)^m$. Moreover, if the Continuum Hypothesis holds, then $(\omega^*)^n$ is a continuous image of ω^* for every n (see [P]), and hence it is relatively consistent with ZFC that $(\omega^*)^n$ is a continuous image of $(\omega^*)^m$ for arbitrary $m, n \geqslant 1$.

Naturally, the question arises whether one can prove in ZFC alone that $(\omega^*)^{n+1}$ is a continuous image of $(\omega^*)^n$ for some $n \geqslant 1$.

In order to answer the above question we first translate it into the language of Boolean algebras.

Let $n, k \in \omega$. By $I_{k,n}$ we denote the subset of ω^n defined as

$$I_{k,n} = \{\langle x_0, \ldots, x_n\rangle : \exists i < n\ (x_i < k)\}$$

and let

$$J_n = \{X \in \mathscr{P}(\omega^n) : \exists k \in \omega\ (X \subset I_{k,n})\} = \bigcup_{k \in \omega} \mathscr{P}(I_{k,n}).$$

Then J_n is a proper non-principal ideal in the Boolean algebra $\mathscr{P}(\omega^n)$ of all subsets of the set ω^n.

By $\mathscr{B}_n$ we denote the subalgebra of $\mathscr{P}(\omega^n)$ generated by the family

$$\{X_0 \times X_1 \times \ldots \times X_{n-1} : \forall i < n\ (X_i \subseteq \omega)\}.$$

Obviously, the set J_n^- defined as $J_n^- = J_n \cap \mathscr{B}_n$ is an ideal in $\mathscr{B}_n$ and it is not hard to see that the Stone space of the Boolean algebra $\mathscr{B}_n/J_n^-$ is homeomorphic to $(\omega^*)^n$.

Therefore the question stated above dualizes as follows: "Is it provable in ZFC that for some $n \geqslant 1$ the Boolean algebra $\mathscr{B}_{n+1}/J_{n+1}^-$ is isomorphic to a subalgebra of $\mathscr{B}_n/J_n^-$?"

Identifying subsets of ω with their characteristic functions, we consider $\mathscr{P}(\omega)$ as a topological space, the topology being induced by the product topology in 2^{ω}. In a similar way, $\mathscr{P}(\omega^n)$ is identified with 2^{ω^n}, and $\mathscr{P}(X)$ with 2^X for $X \subseteq \omega$. Consequently, we shall speak of meager subsets of $\mathscr{P}(\omega)$, analytic ideals in $\mathscr{P}(\omega)$, continuous functions from $\mathscr{P}(X)$ into $\mathscr{P}(\omega)$, etc.

Let $F, F^*\colon \mathscr{P}(A) \to \mathscr{P}(\omega)$ be functions, where A is an infinite subset of ω, and let $J \subset \mathscr{P}(\omega)$ be an ideal. We say that F *preserves intersections* mod J if

$$F(X) \cap F(Y) \triangle F(X \cap Y) \in J \quad \text{for all } X, Y \subset A.$$

The functions F and F^* are said to be *equal* mod J if $F^*(X) \triangle F(X) \in J$ for every $X \subset A$.

Now, let $J \subset \mathscr{P}(\omega)$ be an ideal, $K \subset \mathscr{P}(\omega)$ a subfamily. Let $\mathrm{CSP}(J, K)$ abbreviate the following sentence: "For every function $F\colon \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ preserving intersections mod J there exist an $A \in K$ and a continuous function $F^*\colon \mathscr{P}(A) \to \mathscr{P}(\omega)$ such that F^* is equal mod J to $F \restriction \mathscr{P}(A)$". Here CSP stands for "continuous selection property".

It was shown in [J] that the sentence: "For every analytic ideal J and every comeager subset $K \subset \mathscr{P}(\omega)$ the statement $\mathrm{CSP}(J, K)$ holds" is relatively consistent with ZFC (see also [J1]).

Notice that if $n \geqslant 1$ and $\sigma\colon \omega \to \omega^n$ is a bijection, then the function $G_\sigma\colon \mathscr{P}(\omega) \to \mathscr{P}(\omega^n)$ defined as $G_\sigma(X) = \{\sigma(j)\colon j \in X\}$ is simultaneously an isomorphism of Boolean algebras and a homeomorphism of topological spaces. So we may identify $\mathscr{P}(\omega^n)$ with $\mathscr{P}(\omega)$ and write e.g. $\mathrm{CSP}(J_n, K)$, where J_n is the ideal in $\mathscr{P}(\omega)^n$ defined above. Notice that J_n is an F_σ-subset of $\mathscr{P}(\omega^n)$, hence an analytic one.

By Fin we denote the ideal of finite subsets of ω and by $\mathrm{Fin}^+ = \mathscr{P}(\omega) \setminus \mathrm{Fin}$ the family of infinite subsets of ω. Notice that Fin^+ is a comeager subfamily of $\mathscr{P}(\omega)$.

By these remarks and by the consistency result mentioned above the negative answer to our initial question follows from.

THEOREM 1. *Suppose that* $\omega > n \geqslant 1$ *and that* $\mathrm{CSP}(J_n, \mathrm{Fin}^+)$ *holds. Then the Boolean algebra* $\mathscr{B}_n/J_n^-$ *does not contain a subalgebra isomorphic to* $\mathscr{B}_{n+1}/J_{n+1}^-$, *and hence the topological space* $(\omega^*)^{n+1}$ *is not a continuous image of the space* $(\omega^*)^n$.

Theorem 1 will be proved in § 2 of this paper.

We conclude this introductory section with the following open question: Is it relatively consistent with ZFC that there are $\omega > n > m \geqslant 1$ such that $(\omega^*)^{n+1}$ is a continuous image of $(\omega^*)^n$, but $(\omega^*)^{m+1}$ is not a continuous image of $(\omega^*)^m$?

Obviously, if $(\omega^*)^{n+1}$ is a continuous image of $(\omega^*)^n$ and $n \leqslant m$, then $(\omega^*)^{m+1}$ is a continuous image of $(\omega^*)^m$.

§ 2. Proof of Theorem 1. Throughout this section we fix $n \in \omega \setminus \{0\}$ and assume that $\mathrm{CSP}(J_n, \mathrm{Fin}^+)$ holds. Moreover, contradicting Theorem 1, we assume that there exists a function $\underline{H}\colon \mathscr{B}_{n+1}/J_{n+1}^- \to \mathscr{B}_n/J_n^-$ which is an isomorphic embedding of Boolean algebras, and fix such a function $\underline{H}$ throughout the proof.

Let H be any lifting of $\underline{H}$, i.e. a function $H\colon \mathscr{B}_{n+1} \to \mathscr{B}_n$ such that the following diagram commutes:

$$\begin{array}{ccc} H\colon \mathscr{B}_{n+1} & \longrightarrow & \mathscr{B}_n \\ \downarrow{\scriptstyle \pi_{J_{n+1}^-}} & & \downarrow{\scriptstyle \pi_{J_n^-}} \\ \underline{H}\colon \mathscr{B}_{n+1}/J_{n+1}^- & \longrightarrow & \mathscr{B}_n/J_n^- , \end{array}$$

where $\pi_{J_n^-}$ and $\pi_{J_{n+1}^-}$ denote the canonical projections.

We define a function $F\colon \mathscr{P}(\omega) \to \mathscr{P}(\omega^n)$ by the formula $F(X) = H(X^{n+1})$ for every $X \subset \omega$. Clearly, the function F preserves intersection $\mathrm{mod} J_n$, hence by $\mathrm{CSP}(J_n, \mathrm{Fin}^+)$ there exist an infinite subset $A \subset \omega$ and a continuous function $F^*\colon \mathscr{P}(A) \to \mathscr{P}(\omega^n)$ such that F^* and $F \restriction \mathscr{P}(A)$ are equal $\mathrm{mod} J_n$.

We fix such a set A and such a function F^* for the remainder of this proof. In order to derive the desired contradiction we show that the function F^* has too nice properties.

PROPOSITION 2. *There exist an increasing sequence of non-negative integers* $\langle l_k\colon k \in \omega \rangle$ *and a sequence of functions* $\langle G_k\colon k \in \omega \rangle$ *such that* $G_k\colon \mathscr{P}(l_k \cap A \to \mathscr{P}(k^n)$ *and* $F^*(X) \cap k^n = G_k(l_k \cap X)$ *for all* $k \in \omega$ *and* $X \subset A$.

Proof. Observe that $\mathscr{P}(\omega)$ and $\mathscr{P}(\omega^n)$ are compact metrizable spaces: it will be convenient for our purposes to consider $\mathscr{P}(\omega^n)$ equipped with the metric ϱ_n defined by the formula

$$\varrho_n(Y_1, Y_2) = 2^{-\min\{\max\{y_1, \ldots, y_n\}\colon \langle y_1, \ldots, y_n \rangle \in Y_1 \triangle Y_2\}}.$$

It is easily seen that Proposition 2 just asserts that F^* is a uniformly continuous function from $(\mathscr{P}(A), \varrho_1)$ into $(\mathscr{P}(\omega^n), \varrho_n)$. ■

For the remainder of this section we fixed sequences $\langle l_k\colon k \in \omega \rangle$ and $\langle G_k\colon k \in \omega \rangle$ satisfying the statement of Proposition 2. Moreover, we assume without loss of generality that $l_0 = 0$. We recall that $I_{k,n}$ was defined as

$$I_{k,n} = \{\langle x_0, \ldots, x_{n-1} \rangle\colon \exists i < n \ (x_i < k)\}.$$

Since n is fixed, we shall write I_k instead of $I_{k,n}$ in the sequel.

Now we define a concept which will be useful in several places of the proof.

DEFINITION 3. Let $k^+ > k$. We let $[k, k^+) = k^+ \setminus k = \{j \in \omega\colon k \leqslant j < k^+\}$. A subset $c \subset A \cap l_{k^+} \setminus l_k$ is called a $[k, k^+)$-*stabilizer* if

$$F^*(a \cup c \cup d) \triangle F^*(b \cup c \cup d) \in I_{k^+} \quad \text{for all } d \subset A \setminus l_{k^+} \text{ and } a, b \subset l_k \cap A.$$

PROPOSITION 4. *For every* $k \in \omega$ *there exist* $k^+ > k$ *and a* $[k, k^+)$-*stabilizer* $c \subset A \cap l_{k^+} \setminus l_k$.

Proof. Assume the contrary and let k be such that for every $j>k$ there is no $[k,j)$-stabilizer. Then we may construct inductively:

— an increasing sequence $\langle k(p)\colon p\in\omega\rangle$ of natural numbers,
— a sequence of pairs $\langle\langle a_p, b_p\rangle\colon p\in\omega\rangle$,
— a sequence of finite sets $\langle c_p\colon p\in\omega\rangle$

such that for all $p\in\omega$ the following conditions hold:

(1) $k(0)=k$,
(2) $a_p, b_p\subset l_k\cap A$,
(3) $c_p\subset A\cap[l_k, l_{k(p+1)})$,
(4) $c_{p+1}\cap l_{k(p+1)}=c_p$,
(5) $G_{k(p+1)}(a_p\cup c_p)\,\triangle\, G_{k(p+1)}(b_p\cup c_p)\notin I_{k(p)}$.

Since $\mathscr{P}(l_k\cap A)$ is finite, there exist $a, b\subset l_k\cap A$ such that $\langle a,b\rangle=\langle a_p,b_p\rangle$ for infinitely many p. Let $C=\bigcup\{c_p\colon p\in\omega\}$. It follows from (4) that

$$C=\bigcup\{c_p\colon p\in\omega\ \&\ \langle a,b\rangle=\langle a_p,b_p\rangle\}.$$

From (5) and the choice of the functions G_k we infer that

$$F^*(a\cup C)\,\triangle\, F^*(b\cup C)\notin J_n.$$

On the other hand, we have

$$F^*(a\cup C)\,\triangle\, H\big((a\cup C)^{n+1}\big)\in J_n \quad\text{and}\quad F^*(b\cup C)\,\triangle\, H\big((b\cup C)^{n+1}\big)\in J_n$$

by the definition of F^*, hence

$$H\big((a\cup C)^{n+1}\big)\,\triangle\, H\big((b\cup C)^{n+1}\big)\notin J_n.$$

But H was chosen to be a lifting of $\underline{H}$, so we must have

$$H\big((a\cup C)^{n+1}\big)\,\triangle\, H\big((b\cup C)^{n+1}\big)\in J_n,$$

since obviously

$$(a\cup C)^{n+1}\,\triangle\,(b\cup C)^{n+1}\in I_{l_k,n+1}\subset J_{n+1}.$$

This contradiction concludes the proof of Proposition 4. ■

Definition 5. Let $\bar{k}=\langle k(p)\colon p\in\omega\rangle$ be an increasing sequence of natural numbers, and let $\bar{b}=\langle b_p\colon p\in\omega\rangle$ and $\bar{c}=\langle c_p\colon p\in\omega\rangle$ be sequences of finite subsets of ω. We say that the triple $\langle\bar{k},\bar{b},\bar{c}\rangle$ is *n-productive* if the following conditions hold:

(1) $b_p\subseteq c_p\subseteq A\cap[l_{k(p)}, l_{k(p+1)})$ for $p>0$ and $b_0\subseteq c_0\subseteq A\cap l_{k(1)}$.

(2) Assume X_1, X_2 are such that for arbitrary $p\in\omega$ and $j\in\{1,2\}$ either $X_j\cap[l'_{k(p)}, l_{k(p+1)})=b_p$ or $X_j\cap[l'_{k(p)}, l_{k(p+1)})=c_p$, where

$$l'_{k(p)}=\begin{cases} l_{k(p)} & \text{else},\\ 0 & \text{if } p=0,\end{cases}$$

and let $q_0,\ldots,q_{n-1}\in\omega$.

In this situation, if

$$X_1\,\triangle\, X_2\cap[l_{k(q_i)}, l_{k(q_i+1)})=\emptyset\quad\text{for all } i<n,$$

then

$$F^*(X_1)\,\triangle\, F^*(X_2)\cap\big(\bigcup_{i<n}[k(q_i), k(q_i+1))\big)^n=\emptyset.$$

The following lemma will be crucial in the proof.

Lemma 6. *There exist sequences $\bar{k}$, $\bar{b}$ and $\bar{c}$ such that the triple $\langle\bar{k},\bar{b},\bar{c}\rangle$ is n-productive, and moreover*

(3) *$c_p\setminus b_p\neq\emptyset$ for every $p\in\omega$.*

In order to make it more digestible, the remainder of this paper is organized as follows: First we prove Lemma 6 for the special case $n=1$, which is considerably easier than the general one, next we show how Theorem 1 follows from Lemma 6, and finally we prove the lemma for all n.

Definition 7. Let $\langle\bar{k},\bar{b},\bar{c}\rangle$ be a triple of sequences as in Definition 5 such that (1) is satisfied. A subset $X\subset A$ will be called $\langle\bar{k},\bar{b},\bar{c}\rangle$*-amenable* (or *amenable*, if it is clear from the context which sequences we have in mind) if for arbitrary $p\in\omega$ either $X\cap[l'_{k(p)}, l_{k(p+1)})=b_p$ or $X\cap[l'_{k(p)}, l_{k(p+1)})=c_p$.

Proof of the lemma for $n=1$. Since A is infinite, it follows from Proposition 4 that there is an increasing sequence $\langle m(p)\colon p\in\omega\rangle$ and a sequence $\langle a_p\colon p\in\omega\rangle$ such that $m(0)=0$ and for every $p\in\omega$ the set a_p is contained in $[l_{m(p)}, l_{m(p+1)})\cap A$, where a_{2p} is nonempty and a_{2p+1} is an $[m(2p+1), m(2p+2))$-stabilizer.

Now we define for $p\in\omega$:

$$k(p)=m(2p),\qquad b_p=a_{2p+1},\qquad c_p=a_{2p}\cup a_{2p+1}.$$

The triple of sequences thus defined obviously satisfies (1) and (3). In order to see that it satisfies (2), let $X_1, X_2\subset A$ be amenable sets and fix $q=q_0\in\omega$.

If $q=0$ and $X_1\,\triangle\, X_2\cap[l_{k(0)}, l_{k(1)})=\emptyset$, then

$$G_{k(1)}(X_1\cap l_{k(1)})=G_{k(1)}(X_2\cap l_{k(1)}),$$

since by our assumption $l_{k(0)}=l_0=0$.

If $q>0$, then

$$X_1\cap[l_{m(2q-1)}, l_{m(2q)})=X_2\cap[l_{m(2q-1)}, l_{m(2q)})=a_{2q-1}.$$

Since a_{2q-1} is an $[m(2q-1), m(2q))$-stabilizer, it follows that whenever

$$X_1\cap[l_{k(q)}, l_{k(q+1)})=X_2\cap[l_{k(q)}, l_{k(q+1)}),$$

then

$$G_{k(q+1)}(X_1 \cap l_{k(q+1)}) \cap [k(q), k(q+1)) = G_{k(q+1)}(X_2 \cap l_{k(q+1)}) \cap [k(q), k(q+1)),$$

which implies (2) by the choice of the functions $G_{k(q+1)}$.

This concludes the proof of Lemma 6 for the special case $n = 1$. ∎

Derivation of Theorem 1 from Lemma 6. Suppose the triple of sequences $\langle \bar{k}, \bar{b}, \bar{c} \rangle$ is n-productive and satisfies (3). We define $X_0, X_1, \ldots, X_n \subset A$ by putting for $i \leqslant n$ and $p \in \omega$

$$X_i \cap [l_{k(p)}, l_{k(p+1)}) = \begin{cases} b_p & \text{if } p \equiv i (\bmod n+1), \\ c_p & \text{else.} \end{cases}$$

We denote $\bigcup_{i \leqslant n} X_i$ by X.

Claim 8. $X^{n+1} \setminus \bigcup_{i \leqslant n} X_i^{n+1} \notin J_{n+1}$.

Proof of the claim. For every $p \in \omega$ we choose $z_p \in c_p \setminus b_p$ and let

$$Z = \{\langle z_{(n+1)p}, z_{(n+1)p+1}, \ldots, z_{(n+1)p+n} \rangle : p \in \omega\}.$$

Obviously, $Z \subset X^{n+1} \setminus \bigcup_{i \leqslant n} X_i^{n+1}$, and $Z \notin J_{n+1}$. ∎

Since $\underline{H}$ is an isomorphic embedding of $\mathscr{B}_{n+1}/J_{n+1}^-$ into $\mathscr{B}_n/J_n^-$ and H is a lifting of $\underline{H}$, it follows from claim 8 that $H(X^{n+1}) \setminus H(\bigcup_{i \leqslant n} X_i^{n+1}) \notin J_n$.

On the other hand, H preserves finite sums modJ_n, hence

$$H(X^{n+1}) \setminus \Big(\bigcup_{i \leqslant n} H(X_i^{n+1})\Big) \notin J_n,$$

and consequently

$$F(X) \setminus \bigcup_{i \leqslant n} F(X_i) \notin J_n.$$

But this contradicts the following:

Claim 9. $F^*(X) \setminus \bigcup_{i \leqslant n} F^*(X_i) \in J_n$.

Proof of the claim. Let $\bar{z} = \langle z_0, \ldots, z_{n-1} \rangle \in \omega^n$ and suppose $\bar{z} \notin I_{k(0)}$. Then there exist $q_0, \ldots, q_{n-1} \in \omega$ such that

$$\bar{z} \in [k(q_0), k(q_0+1)) \times [k(q_1), k(q_1+1)) \times \ldots \times [k(q_{n-1}), k(q_{n-1}+1)).$$

Moreover, there exists an $s \leqslant n$ such that $q(i) \not\equiv s$ (mod$n+1$) for all $i < n$. We fix such an s and notice that

$$X_s \cap [l_{k(q_i)}, l_{k(q_i+1)}) = c_{q_i} = X \cap [l_{k(q_i)}, l_{k(q_i+1)})$$

for all $i < n$. Notice that both X and X_s are $\langle \bar{k}, \bar{b}, \bar{c} \rangle$-amenable. Hence it follows from (2) that

$$F^*(X) \triangle F^*(X_s) \cap \Big(\bigcup_{i<n} [k(q_i), k(q_i+1))\Big)^n = \emptyset,$$

and therefore $\bar{z} \in F^*(X)$ if and only if $\bar{z} \in F^*(X_s)$.

What we have shown is that for every $z \in F^*(X) \setminus I_{k(0)}$ there exists an $s \leqslant n$ such that $\bar{z} \in F^*(X_s)$; hence

$$F^*(X) \setminus \bigcup_{i \leqslant n} F^*(X_i) \subset I_{k(0)}.$$

This concludes the proof of the claim by the definition of J_n. ∎

So our task reduces to the

Proof of Lemma 6. For any $k, t \in \omega$ we define

$$\mathscr{X}(k, t) = \{X \subset A : \exists \{Y_{i,j} \subset \omega : i < n \,\&\, j < t\} (F^*(X) \setminus I_k = \bigcup_{j<t} \prod_{i<n} Y_{i,j})\}.$$

Since F^* is equal to H modJ_n and the range of H is contained in $\mathscr{B}_n$, it follows that $\bigcup_{k \in \omega} \bigcup_{t \in \omega} \mathscr{X}(k, t) = \mathscr{P}(A)$.

For given $k, t \in \omega$ and $\mathscr{Y} = \{Y_{i,j} : i < n \,\&\, j < t\}$ the set

$$\{X \subset A : F^*(X) \setminus I_k = \bigcup_{j<t} \prod_{i<n} Y_{i,j}\}$$

is a closed subset of $\mathscr{P}(A)$, hence $\mathscr{X}(k, t)$ is an analytic subset of $\mathscr{P}(A)$, and therefore $\mathscr{X}(k, t)$ has the Baire property. Obviously, the sets $\mathscr{X}(k, t)$ increase with increasing parameters k and t. Hence by Baire's theorem, $\mathscr{X}(k, t)$ is of second Baire category for k, t sufficiently large.

For the remainder of this proof we fix numbers k' and t and a set $u \subset l_{k'} \cap A$ such that if we put $[u] = \{X \subset A : X \cap l_{k'} = u\}$, then $[u] \setminus \mathscr{X}(k', t)$ is of first Baire category. We let $L = [u] \cap \mathscr{X}(k', t)$.

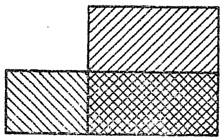
If we could define a continuous function

$E: L \to (\mathscr{P}(\omega))^{t \cdot n}$ such that

$$E(X) = \langle h_{i,j}(X) : i < n \,\&\, j < t \rangle \quad \text{and}$$

$$F^*(X) \setminus I_{k'} = \bigcup_{j<t} \prod_{i<n} Y_{i,j}(X) \quad \text{for every } X \in L,$$

when we could prove the lemma by a straightforward generalization of the proof for the case $n = 1$. The problem is that the sets $Y_{i,j}(X)$ may not be uniquely determined by $F^*(X) \setminus I_{k'}$. Consider the following trivial example for $n = t = 2$:



The idea of what follows is, roughly speaking, that we shall find sequences $\bar{k}$, $\bar{b}$ and $\bar{c}$ such that, for X $\langle \bar{k}, \bar{b}, \bar{c} \rangle$-amenable, although we may not be able to reconstruct the sets $Y_{i,j}$ witnessing that $F^*(X) \setminus I_{k'}$ is an element of $\mathscr{B}_n$, we do however possess enough information to reconstruct some finite parts of $F^*(X)$ from certain information about X as required in (2).

DEFINITION 10. Let $W \subset \omega^n$ and $i < n$. We define a relation $=_{W,i} \subset \omega \times \omega$ as follows: $z =_{W,i} z'$ iff

$$\forall \langle z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n-1}\rangle \in \omega^{n-1}$$
$$(\langle z_0, \ldots, z_{i-1}, z, z_{i+1}, \ldots, z_{n-1}\rangle \in W \Leftrightarrow$$
$$\langle z_0, \ldots, z_{i-1}, z', z_{i+1}, \ldots, z_{n-1}\rangle \in W).$$

We write $z =_W z'$ iff $z =_{W,i} z'$ for all $i < n$.

CLAIM 11. (a) *The relations* $=_{W,i}$ *and* $=_W$ *are equivalence relations for arbitrary* $W \subset \omega^n$ *and* $i < n$.

(b) *If* $W \in \mathscr{B}_n$, *then the relation* $=_W$ *splits* ω *into finitely many equivalence classes.*

Proof of the claim. Part (a) is obvious.

For the proof of (b) notice that if $W \in \mathscr{B}_n$, then there exist $s \in \omega$ and a family $\{Y_{i,j}: i<n, j<s\}$ such that $W = \bigcup_{j<s} \prod_{i<n} Y_{i,j}$. It suffices to show that for every $i < n$ the relation $=_{W,i}$ splits ω into finitely many equivalence classes. So we fix $i < n$ and set $P_i(z) = \{j < s: z \in Y_{i,j}\}$ for $z \in \omega$. Observe that $\langle z_0, \ldots, z_{i-1}, z, z_{i+1}, \ldots, z_{n-1}\rangle \in W$ iff $\langle z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n-1}\rangle \in \bigcup_{j \in P_i(z)} \prod_{r \in n \setminus \{i\}} Y_{ij}$. It follows that $z =_{W,i} z'$ whenever $P_i(z) = P_i(z')$, hence the relation $=_{W,i}$ splits ω into at most 2^s equivalence classes. ■

DEFINITION 12. (a) Let $r \in \omega$. A subset $W \subset \omega^n$ is called *r-semisimple* if the relation $=_W$ partitions ω into exactly r nonempty equivalence classes. It is called *r-simple* if it is r-semisimple and every equivalence class of the relation $=_W$ is infinite.

(b) Let W be r-semisimple and let $E \subset \omega$. We say that E is a *witness* for W if the relation $=_{W \cap E^n}$ splits E into r nonempty equivalence classes.

CLAIM 13. *Let* $r \in \omega$ *and suppose* $W \subset \omega^n$ *is* r*-semisimple.*

(a) *If* $E \subset \omega$ *is a witness for* W *and* $z, z' \in E$, *then* $z =_W z'$ *iff* $z =_{W \cap E^n} z'$.

(b) *If* $E \subset D \subset \omega$ *and* E *is a witness for* W, *then* D *is also a witness for* W, *and moreover* E *is a witness for* $W \cap D^n$.

(c) *Suppose* $W \subset \omega^n$ *is* r*-simple and* $k \in \omega$. *Then there exists a* $k^+ > k$ *such that the interval* $[k, k^+)$ *is a witness for* W.

Proof of the claim. Parts (a) and (b) follow immediately from Definition 12. We prove (c).

Let k, r, W be as in the hypothesis. Since all equivalence classes of the relation $=_W$ are infinite, we find numbers $x_0, \ldots, x_{r-1} \geqslant k$ which are representatives

of all the equivalence classes of the relation $=_W$. Now let $k^+ = \sup\{x_j: j < r\} + 1$. In order to show that k^+ is as required, it suffices to show that for all $i < n$ and $y, y' \in [k, k^+)$ the relation $y =_{W,i} y'$ holds iff the relation $y =_{W \cap [k,k^+)^n, i} y'$ holds.

The "only if" direction follows immediately from Definition 10.

Now suppose that for some $i < n$ and $y, y' \in [k, k^+)$ we have $y \neq_{W,i} y'$, i.e. there exist numbers $y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{n-1}$ such that w.l.o.g.

$$\langle y_0, \ldots, y_{i-1}, y, y_{i+1}, \ldots, y_{n-1}\rangle \in W$$

and

$$\langle y_0, \ldots, y_{i-1}, y', y_{i+1}, \ldots, y_{n-1}\rangle \notin W.$$

By our choice of k^+ there exist numbers

$$\hat{y}_0, \ldots, \hat{y}_{i-1}, \hat{y}_{i+1}, \ldots, \hat{y}_{n-1} \in [k, k^+)$$

such that $\hat{y}_j =_{W,j} y_j$ for $j \in n \setminus \{i\}$. By induction over j one shows that

$$\langle y_0, \ldots, y_{i-1}, y, y_{i+1}, \ldots, y_{n-1}\rangle \in W$$

iff $\langle \hat{y}_0, \ldots, \hat{y}_j, y_{j+1}, \ldots, y, \ldots, y_{n-1}\rangle \in W$ and analogously if we replace y by y'. It follows that

$$\langle \hat{y}_0, \ldots, \hat{y}_{i-1}, y, \hat{y}_{i+1}, \ldots, \hat{y}_{n-1}\rangle \in W \quad \text{and}$$
$$\langle \hat{y}_0, \ldots, \hat{y}_{i-1}, y', \hat{y}_{i+1}, \ldots, \hat{y}_{n-1}\rangle \notin W,$$

witnessing that $y \neq_{W \cap [k,k^+)^n, i} y'$, which concludes the proof of Claim 13. ■

The following claim says that witnesses allow us to reconstruct W from the relation $=_W$.

CLAIM 14. *Suppose* $W, W' \subset \omega^n$ *are both* r*-semisimple, the set* $E \subset \omega$ *is a witness for* W *and* $W \cap E^n = W' \cap E^n$. *Assume furthermore that for all* $z, z' \in \omega$ *the relation* $z =_W z'$ *holds iff the relation* $z =_{W'} z'$ *holds. Then* $W = W'$.

Proof of the claim. Suppose W, W' and E satisfy the hypothesis of the claim and let $\bar{z} = \langle z_0, \ldots, z_{n-1}\rangle \in \omega^n$. We put $j(\bar{z}) = |\{i < n: z_i \notin E\}|$. By induction over j we show that

$$(*) \qquad \bar{z} \in W \quad \text{iff} \quad \bar{z} \in W'.$$

For $j(\bar{z}) = 0$ this is obvious. Suppose $(*)$ holds for all $\bar{z}^* \in \omega^n$ such that $j(\bar{z}^*) \leqslant j$, and assume that $j(\bar{z}) = j+1$. Fix an $i < n$ such that $z_i \notin E$. By our assumption there exists a $\hat{z}_i \in E$ such that $\hat{z}_i =_W z_i =_{W'} \hat{z}_i$. Let $\bar{z}^* = \langle z_0, \ldots, z_{i-1}, \hat{z}_i, z_{i+1}, \ldots, z_{n-1}\rangle$. Then $j(\bar{z}^*) = j$, and by our induction hypothesis $\bar{z}^* \in W$ iff $\bar{z}^* \in W'$. But the relation $\hat{z}_i =_{W,i} z_i$ implies that $\bar{z} \in W$ iff $\bar{z}^* \in W$, and since $\hat{z}_i =_{W',i} z_i$ we have $\bar{z} \in W'$ iff $\bar{z}^* \in W'$. Hence $(*)$ holds, concluding the proof of Claim 14. ■

By $\|z\|_W$ we shall denote the equivalence class of the relation $=_W$ containing z.

For a given $W\in\mathscr{B}_n$ some of the equivalence classes of the relation $=_W$ may be finite. Therefore we set for $W\subset\omega^n$ $k(W)=\sup\{z\colon \|z\|_W\in\mathrm{Fin}\}$. By Claim 11 (b), if $W\in\mathscr{B}_n$, then $k(W)<+\infty$. We write $W^-=W\setminus I_{k(W)}$ for $W\in\mathscr{B}_n$. One easily verifies that $\|z\|_W\setminus k(W)\subset\|z\|_{W^-}$ for all $W\in\mathscr{B}_n$ and $z>k(W)$, hence every equivalence class of the relation $=_{W^-}$ is infinite.

Now, let us recall that the definitions of $u, k', t, \mathscr{X}(k', t)$ are given at the beginning of the proof of Lemma 6.

Since the relation "W is r-semisimple & $k(W)<m$" defines a Borel subset of $\mathscr{P}(\omega^n)$, it follows that there exist numbers $k\geqslant k'$ and r such that the set

$$\{X\in[u]\cap\mathscr{X}(k', t)\colon F^*(X)\setminus I_{k'} \text{ is } r\text{-semisimple } \& \ k(F^*(X)\setminus I_{k'})\leqslant k\}$$

is of second Baire category in $\mathscr{P}(A)$.

Until the end of this paper we fix $k^*\geqslant k'$, a set $v\in[u]\cap\mathscr{P}(l_{k^*})$ and a number r such that the set

$$S=\{X\in[v]\cap\mathscr{X}(k', t)\colon k(F^*(X)\setminus I_{k'})\geqslant k^* \text{ or } F^*(X)\setminus I_{k^*} \text{ is not } r\text{-semisimple}\}$$

is of first Baire category in $\mathscr{P}(A)$. Here $[v]$ denotes the set $\{X\subset A\colon X\cap l_{k^*}=v\}$.

In the sequel we write

$$M=[v]\cap\mathscr{X}(k^*, t)\setminus S, \text{ and } =^*_X \text{ instead of } =_{F^*(X)\setminus I_{k^*}} \quad \text{for } X\in M.$$

Moreover, a subset $E\subset\omega$ will be called a $*$-*witness* for $X\in M$ if it is a witness for the set $F^*(X)\setminus I_{k^*}\subset\omega^n$.

Now we are ready to formulate the crucial statement.

SUBLEMMA 15. *There exist a set $E\subset\omega$ and sequences*

$$\bar{k}=\langle k(p)\colon p\in\omega\rangle, \quad \bar{b}=\langle b_p\colon p\in\omega\rangle, \quad \bar{c}=\langle c_p\colon p\in\omega\rangle$$

satisfying for all $p\in\omega$ and $\langle\bar{k}, \bar{b}, \bar{c}\rangle$-amenable sets X_0 and X_1:

(0) $k(0)\geqslant k^*$,

(1) $b_p\subseteq c_p\subseteq A\cap[l_{k(p)}, l_{k(p+1)})$ *for* $p>0$ *and* $b_0\subseteq c_0\subseteq A\cap l_{k(1)}$,

(3) $c_p\setminus b_p\neq\emptyset$,

(4) $X_0, X_1\in M$,

(5) *If* $X_0\triangle X_1\cap[l_{k(p)}, l_{k(p+1)})=\emptyset$, *then*

$$F^*(X_0)\triangle F^*(X_1)\cap[k(p), k(p+1))^n=\emptyset,$$

(6) $E\cap[k(p), k(p+1))$ *is a $*$-witness for X_0 and X_1,*

(7) $E^n\cap F^*(X_0)=E^n\cap F^*(X_1)$,

(8) *For all $w, w'\in E$ the relation $w=^*_{X_0}w'$ holds iff the relation $w=^*_{X_1}w'$ holds.*

Before proving the sublemma we show how to deduce Lemma 6 from it.

We fix a triple $\langle\bar{k}, \bar{b}, \bar{c}\rangle$ of sequences satisfying the statement of the sublemma and show that it also satisfies (2) of Definition 5.

Suppose X_0, X_1 are amenable and $q_0, \ldots, q_{n-1}\in\omega$ are such that

$$X_0\triangle X_1\cap\bigcup_{i<n}[l_{k(q_i)}, l_{k(q_i+1)})=\emptyset.$$

Let

$$Z_0=F^*(X_0)\setminus I_{k(0)} \text{ and } Z_1=F^*(X_1)\setminus I_{k(0)}.$$

Furthermore, we set

$$U=\bigcup_{i<n}[k(q_i), k(q_i+1)) \quad\text{and}\quad V_0=Z_0\cap U^n, \quad V_1=Z_1\cap U^n.$$

We show that $V_0=V_1$, which obviously implies (2).

It follows from (6) and Claim 13(b) that $E\cap[k(q_i), k(q_i+1))$ is a witness for both V_0 and V_1. Consequently, if we show that for arbitrary $z, z'\in U$ we have $z=_{V_0}z'$ iff $z=_{V_1}z'$, then the equality $V_0=V_1$ becomes an easy consequence of (7) and Claim 14.

Hence let $z, z'\in U$ and let $i, i'<n$ be such that $z\in[k(q_i), k(q_i+1))$ and $z'\in[k(q_{i'}), k(q_{i'}+1))$. By (6) there are numbers $w\in[k(q_i), k(q_i+1))\cap E$ and $w'\in[k(q_{i'}), k(q_{i'}+1))\cap E$ such that $z=_{V_0}w$ and $z'=_{V_0}w'$. We fix such w and w'. Obviously, $z=_{V_0}z'$ iff $w=_{V_0}w'$.

On the other hand, by (5) we have

(§) $$V_0\cap[k(q_i), k(q_i+1))^n=V_1\cap[k(q_i), k(q_i+1))^n=W.$$

Since $E\cap[k(q_i), k(q_i+1))$ is a witness for both V_0 and V_1, it follows from (§) and from Claim 13 that $z=_{V_0}w$ iff $z=_W w$ iff $z=_{V_1}w$. By an analogous reasoning one can show that $z'=_{V_1}w'$. Therefore, $z'=_{V_1}z$ iff $w'=_{V_1}w$, which is by (8) equivalent to $w'=_{V_0}w$. Hence the relation $z'=_{V_0}z$ holds iff the relation $z'=_{V_1}z$. holds.

We have thus shown that the triple of sequences $\langle\bar{k}, \bar{b}, \bar{c}\rangle$ satisfying (0), (1) and (3)–(8) satisfies (2) of Definition 5 as well, and hence the proof of Lemma 6 reduces to the

Proof of Sublemma 15. Recall that the definitions of M and S were given before the statement of the sublemma. Let $S=\bigcup_{p\in\omega}S_p$, where S_p is a nowhere dense subset of $\mathscr{P}(A)$ and $S_p\subseteq S_{p+1}$ for all $p\in\omega$.

We construct inductively and increasing sequence of natural numbers $\bar{m}=\langle m(p)\colon p\in\omega\rangle$ and a sequence $\bar{e}=\langle e_p\colon p\in\omega\cup\{-1\}\rangle$ of finite subsets of A such that:

(i) $e_{-1}\subset[0, l_{m(0)})$,

(ii) $m(0)\geqslant k^*$,

(iii) $e_{-1}\in[v]$,

and for all $p\in\omega$ we have:

(iv) $e_p\subset[l_{m(p)}, l_{m(p+1)})$.

(v) If we let $D_p=G_{m(p+1)}(v\cup e_{p-1}\cup e_p)\cap[m(p), m(p+1))^n$, then the relation $=_{D_{4p}}$ splits $[m(4p), m(4p+1))$ into exactly r nonempty equivalence classes.

(vi) If $X \subset A$ and $X \cap [l_{m(4p+1)}, l_{m(4p+2)}) = e_{4p+1}$, then $X \notin S_p$.
(vii) $|e_{4p+2}| \geqslant r!$
(viii) e_{4p+3} is an $[m(4p+3), m(4p+4))$-stabilizer.

First we should convince ourselves that sequences satisfying (i)–(viii) exist. It is obvious that constructing $m(p)$ and e_p inductively we can always take care of (i)–(iv) and (vii). Proposition 4 tells us that we can deal with (viii) as well. In order to see how to take care of (vi), notice that S_p is nowhere dense in the topology of $\mathscr{P}(A)$ and that there are only finitely many candidates for $X \cap l_{m(4p+1)}$, so we can deal with all of them successively extending initial fragments of e_{4p+1}.

It remains to show that at stage $4p$ of the construction we can make sure that (v) holds. In order to see this, notice that given any p, by the definition of M and the Baire category theorem there exists $X \in M$ such that $X \cap l_{m(4p)} = v \cup e_{4p-1}$. Since $X \in M$, the set $F^*(X) \setminus I_{k^*} = Y$ is r-simple, and hence by Claim 13(c) there exists an $m(4p+1) > m(4p)$ such that $[m(4p), m(4p+1))$ is a witness for Y. By the choice of the functions G_k we have $D_{4p} = Y \cap [m(4p), m(4p+1))^n$ whenever $X \cap m(4p+1) = v \cup e_{4p-1} \cup e_{4p}$.

For the remainder of this proof we fix sequences $\bar{m}$ and $\bar{e}$ satisfying (i)–(viii).

Suppose that we are given two sequences $\bar{b}' = \langle b'_p \colon p \in \omega \rangle$ and $\bar{c}' = \langle c'_p \colon p \in \omega \rangle$ such that $b'_p \subsetneq c'_p \subseteq e_{4p+2}$ for all p. We define:

$$b_0 = e_{-1} \cup e_0 \cup e_1 \cup b'_0 \cup e_3\,,$$
$$c_0 = e_{-1} \cup e_0 \cup e_1 \cup c'_0 \cup e_3\,,$$
$$b_p = e_{4p} \cup e_{4p+1} \cup b'_p \cup e_{4p+3}\,, \quad \text{and}$$
$$c_p = e_{4p} \cup e_{4p+1} \cup c'_p \cup e_{4p+3} \quad \text{for } p > 0\,.$$

Moreover, we put $k(p) = m(4p)$ for every $p \in \omega$.

Sequences $\bar{k}$, $\bar{b}$ and $\bar{c}$ which are defined by the above method from some sequences $\bar{b}'$ and $\bar{c}'$ will be called *feasible*. Moreover, a subset $X \subset A$ will be called *feasible* if it is $\langle \bar{k}, \bar{b}, \bar{c} \rangle$-amenable for some feasible sequences $\bar{k}$, $\bar{b}$, $\bar{c}$.

Let $\bar{k}$, $\bar{b}$, $\bar{c}$ be feasible. Conditions (1) and (3) of the sublemma are satisfied by the choice of b'_p and c'_p. Moreover, (4) follows from (iii), (vi), the definition of M and the choice of S_p. (0) is a consequence of the definition of $\bar{k}$.

In order to see that (5) holds as well, suppose X_0 and X_1 are feasible and let $p \in \omega$ be such that

$$X_0 \cap [l_{k(p)}, l_{k(p+1)}) = X_1 \cap [l_{k(p)}, l_{k(p+1)}) = a_p\,.$$

If $p = 0$, then

$$F^*(X_0) \cap k(p+1)^n = G_{k(p+1)}(e_{-1} \cup a_p) = F^*(X_1) \cap k(p+1)^n\,.$$

If $p > 0$, then $X_0 \cap [l_{m(4p-1)}, l_{m(4p)}) = X_1 \cap [l_{m(4p-1)}, l_{m(4p)}) = e_{4p-1}$. But according to (viii), the set e_{4p-1} was chosen to be an $[m(4p-1), m(4p))$-stabilizer, so

$$F^*(X_0) \triangle F^*(X_1) \cap k(p+1)^n \subset I_{m(4p)} = I_{k(p)}$$

by the definition of a stabilizer; hence (5) holds.

Now let $E = \bigcup_{p \in \omega} [m(4p), m(4p+1))$. We show that E satisfies (6).

Indeed, arguing as in the proof of (5), we notice that

$$F^*(X_0) \cap [m(4p), m(4p+1))^n = D_{4p}$$

for arbitrary $p \in \omega$, and now (6) is an immediate consequence of (4), the choice of M, and Claim 13(b).

It remains to show that we can find sequences $\bar{b}'$ and $\bar{c}'$ such that (7) and (8) are satisfied for $\bar{b}$, $\bar{c}$ and E defined as above.

We fix $x_0(0), \ldots, x_0(r-1) \in [m(0), m(1))$ which are assumed to be representatives of all equivalence classes of the relation $=_{D_0}$. Notice that the $x_0(j)$'s may be chosen independently of the choice of $\bar{b}'$ and $\bar{c}'$. Given a feasible X we define a function $f_X \colon E \to r$ as follows: $f_X(w) = j$ iff $w =^*_X x_0(j)$.

We show that there are feasible sequences $\bar{k}$, $\bar{b}$, $\bar{c}$ such that $f_{X_0} = f_{X_1}$ for all $\langle \bar{k}, \bar{b}, \bar{c} \rangle$-amenable sets X_0 and X_1. It is not hard to see that for such sequences (8) will be satisfied.

For $p > 0$ we choose $x_p(0), \ldots, x_p(r-1) \in [m(4p), m(4p+1))$ to be representatives of all equivalence classes of the relation $=_{D_p}$. For every feasible X we define functions $f_{p,X} \colon r \to r$ as follows:

$$f_{p,X}(j) = j' \quad \text{iff} \quad x_p(j) =^*_X x_{p+1}(j')\,.$$

Notice that for $w \in [m(4p), m(4p+1))$ and arbitrary feasible X we have: $f_X(w) =$ iff $x_p(f_{p-1,X} \circ f_{p-2,X} \circ \ldots \circ f_{0,X}(j)) =^*_X w$ iff $x_p(f_{p-1,X} \circ f_{p-2,X} \circ \ldots \circ f_{0,X}(j)) =_{D_p} wj$. It follows that for feasible X_0, X_1 the functions f_{X_0} and f_{X_1} are equal iff $f_{p,X_0} = f_{p,X_1}$ for all $p \in \omega$.

Claim 16. *Let $p \in \omega$ and X_0, X_1 be such that*

$$X_0 \triangle X_1 \cap [l_{k(p)}, l_{k(p+1)}) = \emptyset\,.$$

Then $f_{p,X_0} = f_{p,X_1}$.

Proof of the claim. Repeating the argument used for demonstrating (5) shows that there is a set V such that

$$F^*(X_0) \cap [m(4p), m(4p+5))^n = F^*(X_1) \cap [m(4p), m(4p+5))^n = V$$

whenever X_0, X_1 are as in the hypothesis of the claim. On the other hand, by (6) and Claim 13(b) we know that V is a $*$-witness for both X_0 and X_1. It follows that we have $w =_{X_0} w'$ iff $w =_{X_1} w'$ iff $w =_V w'$ for $w, w' \in [m(4p), m(4p+5))$. To conclude the proof of the claim it suffices to observe that the function $f_{p,X}$ is completely determined by the relation $=_X$ restricted to $[m(4p), m(4p+5))$. ∎

Claim 16 tells us that for feasible X the function $f_{p,X}$ depends only on $X \cap e_{4p+2}$. In other words, given a subset $d_p \subset e_{4p+2}$ there is a unique bijection $f_p[d_p] \colon r \to r$ such that $f_{p,X} = f_p[d_p]$ whenever X is feasible and $X \cap e_{4p+2} = d_p$. But there are only $r!$ bijections from r to r, and hence by (vii) we can find

$b'_p \subsetneq c'_p \subseteq e_{4p+2}$ such that $f_p[b'_p] = f_p[c'_p]$. This shows that there are feasible sequences $\bar{b}$, $\bar{c}$ such that $f_{X_0} = f_{X_1}$ for all $\langle \bar{k}, \bar{b}, \bar{c} \rangle$-amenable sets X_0, X_1, which concludes the proof of (8).

(7) is an easy consequence of (8) and Claim 14, because $E \cap [m(0), m(1))$ is a $*$-witness and $F^*(X_0) \cap m(1)^n = F^*(X_1) \cap m(1)^n$ for all amenable X_0 and X_1.

This concludes the proof of Sublemma 15, Lemma 6 and Theorem 1. ■ ■ ■

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Sur le nombre de côtés d'une sous-variété

by

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Abstract. Let A be a connected, locally connected and locally closed subset of a metric space X which is locally two-sided in X in the sense that every point x of A has arbitrary small connected neighbourhoods $\mathfrak{U}$ such that $\mathfrak{U} \setminus A$ has exactly two components whose closure contains x. We use elementary methods from sheaf theory to study when A is globally two-sided in X (i.e., A has a connected neighbourhood V such that $V \setminus A$ is not connected). We give some applications to concrete examples.

1. Introduction et notations. Soient X une variété de dimension $n+1$, et A une variété connexe de dimension n (pas nécessairement fermée) contenue dans X. On dit que A a deux côtés dans X si elle a un voisinage ouvert connexe W dans X tel que $W \setminus A$ ait exactement deux composantes; sinon, on dit que A n'a qu'un côté dans X. Le problème de reconnaître quand A a deux côtés se pose naturellement, et divers résultats partiels sont connus, l'un des plus généraux étant celui de Rushing [7] selon lequel une n-variété simplement connexe localement plate dans X a deux côtés (et a même un double collier dans X). Rushing remarque aussi que les techniques de la topologie algébrique ne semblent pas suffire à montrer qu'une n-variété orientable non fermée dans S^{n+1} a deux côtés. Nous montrerons dans cet article que l'utilisation des premiers éléments de la théorie des faisceaux permet de caractériser les sous-variétés ayant deux côtés (voir le corollaire 2.2). L'avantage de notre approche abstraite est qu'elle s'applique à des espaces beaucoup plus généraux que les variétés; il suffit que A soit un sous-espace localement fermé et localement connexe d'un espace métrique X "séparant localement X en deux morceaux". A titre d'exemple d'applications de ce raisonnement général, nous prouverons les résultast suivants:

(1) Soit A un sous-ensemble connexe et localement connexe d'un espace métrique X qui a en tout point un double collier local dans X (voir section 2 pour la définition). Alors

(a) Si A n'admet pas de revêtement non trivial à deux feuillets, A a un double collier dans X.

(b) Pour toute distance admissible d sur X, A a un double collier dans X si, et