The space \((\omega^*)^*\) is not always a continuous image of \((\omega^*)^n\).

by

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Abstract. It is shown that the following statement is relatively consistent with ZFC: "For all \(n \in \omega\), the space \((\omega^*)^n\) is not a continuous image of the space \((\omega^*)^m\).

§ 1. Introduction. By \(\omega^\omega\) we denote the remainder of the Čech–Stone compactification of \(\omega\), the countable discrete space, and by \((\omega^*)^\omega\) the product of \(n\) copies of \(\omega^*\).
It was shown in [12] that the spaces \((\omega^*)^n\) and \((\omega^*)^m\) are not homeomorphic whenever \(n \neq m\). Clearly, if \(n < m\), then \((\omega^*)^n\) is a continuous image of \((\omega^*)^m\). Moreover, if the Continuum Hypothesis holds, then \((\omega^*)^m\) is a continuous image of \(\omega^\omega\) for every \(m\) (see [P]), and hence it is relatively consistent with ZFC that \((\omega^*)^m\) is a continuous image of \((\omega^*)^n\) for arbitrary \(m, n \geq 1\).

Naturally, the question arises whether one can prove in ZFC alone that \((\omega^*)^m\) is a continuous image of \((\omega^*)^n\) for some \(n \geq 1\).

In order to answer the above question we first translate it into the language of Boolean algebras.

Let \(n, k \in \omega\). By \(I_k\) we denote the subset of \(\omega^\omega\) defined as
\[ I_k = \{ \langle x_0, \ldots, x_n \rangle : \exists i < n (x_i < k) \} \]
and let
\[ J_k = \{ x \in \mathcal{P}(\omega^\omega) : \exists k \in \omega (X \in I_k) \} = \bigcup_{i \in \omega} I_i \]
Then \(J_k\) is a proper non-principal ideal in the Boolean algebra \(\mathcal{P}(\omega^\omega)\) of all subsets of the set \(\omega^\omega\).

By \(\mathcal{B}_n\) we denote the subalgebra of \(\mathcal{P}(\omega^\omega)\) generated by the family
\[ \{ X_0 \times X_1 \times \ldots \times X_{n+1} : \forall i < n (X_i \in \omega) \} \]
Obviously, the set \(J_k \cap \mathcal{B}_n\) defined as \(J_k \cap \mathcal{B}_n = I_k \cap \mathcal{B}_n\) is an ideal in \(\mathcal{B}_n\), and it is not hard to see that the Stone space of the Boolean algebra \(\mathcal{B}_n / J_k \cap \mathcal{B}_n\) is homeomorphic to \((\omega^*)^n\).

Therefore the question stated above dualizes as follows: "Is it provable in ZFC that for some \(n \geq 1\) the Boolean algebra \(\mathcal{B}_n / J_k \cap \mathcal{B}_n\) is isomorphic to a subalgebra of \(\mathcal{B}_n / J_{k+1} \cap \mathcal{B}_n\)?"
Identifying subsets of $\omega$ with their characteristic functions, we consider $\mathcal{P}(\omega)$ as a topological space, the topology being induced by the product topology in $2^\omega$. In a similar way, $\mathcal{P}(\omega^n)$ is identified with $2^{\omega^n}$, and $\mathcal{P}(X)$ with $2^X$ for $X \subseteq \omega$. Consequently, we shall speak of dense subsets of $\mathcal{P}(\omega)$, analytic ideals in $\mathcal{P}(\omega)$, continuous functions from $\mathcal{P}(X)$ into $\mathcal{P}(\omega)$, etc.

Let $F, F^*: \mathcal{P}(A) \to \mathcal{P}(\omega)$ be functions, where $A$ is an infinite subset of $\omega$, and let $J \subseteq \mathcal{P}(\omega)$ be an ideal. We say that $F$ preserves intersections mod $J$ if

$$F(X) \cap F(Y) \Delta F(X \cap Y) \subseteq J \text{ for all } X, Y \subseteq A.$$ 

The functions $F$ and $F^*$ are said to be equal mod $J$ if $F^*(X) \Delta F(X) \subseteq J$ for every $X \subseteq A$.

Now, let $J \subseteq \mathcal{P}(\omega)$ be an ideal, $K \subseteq \mathcal{P}(\omega)$ a subfamily. Let $\text{CSP}(J, K)$ abbreviate the following sentence: “For every function $F: \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ preserving intersections mod $J$ there exist an $A \subseteq K$ and a continuous function $F^*: \mathcal{P}(A) \to \mathcal{P}(\omega)$ such that $F^*$ is equal mod $J$ to $F^{1}\mid F(A)$”. Here CSP stands for “continuous selection property”.

It was shown in [11] that the sentence: “For every analytic ideal $J$ and every comeager subset $K \subseteq \mathcal{P}(\omega)$ the statement $\text{CSP}(J, K)$ holds” is relatively consistent with ZFC (see also [11]).

Notice that if $n \geq 1$ and $\omega \to \omega^n$ is a bijection, then the function $G: \mathcal{P}(\omega) \to \mathcal{P}(\omega^n)$ defined as $G(X) = \{\sigma(j): j \in X\}$ is simultaneously an isomorphism of Boolean algebras and a homeomorphism of topological spaces. So we may identify $\mathcal{P}(\omega^n)$ with $\mathcal{P}(\omega)$ and write e.g. CSP$(J_n, K)$, where $J_n$ is the ideal in $\mathcal{P}(\omega^n)$ defined above. Notice that $J_n$ is an $F_n$-subset of $\mathcal{P}(\omega)$, hence an analytic one.

By Fin we denote the ideal of finite subsets of $\omega$ and by $\text{Fin}^* = \mathcal{P}(\omega) \setminus \text{Fin}$ the family of infinite subsets of $\omega$. Notice that Fin* is a comeager subclass of $\mathcal{P}(\omega)$.

By these remarks and by the consistency result mentioned above the negative answer to our initial question follows from

**Theorem 1.** Suppose that $\omega > n \geq 1$ and that $\text{CSP}(J_n, \text{Fin}^*)$ holds. Then the Boolean algebra $\mathcal{B}_n$ does not contain a subalgebra isomorphic to $\mathcal{B}_{n+1}$, and hence the topological space $(\mathcal{B}_n)^{\omega^n}$ is not a continuous image of the space $(\mathcal{B}_n)^{\omega^n}$.

Theorem 1 will be proved in §2 of this paper.

We conclude this introductory section with the following open question: Is it relatively consistent with ZFC that there are $\omega > n \geq m \geq 1$ such that $(\mathcal{B}_n)^{\omega^m}$ is a continuous image of $(\mathcal{B}_n)^{\omega^n}$, but $(\mathcal{B}_n)^{\omega^{n+1}}$ is not a continuous image of $(\mathcal{B}_n)^{\omega^n}$? Obviously, if $(\mathcal{B}_n)^{\omega^n}$ is a continuous image of $(\mathcal{B}_n)^{\omega^n}$ and $n \leq m$, then $(\mathcal{B}_n)^{\omega^{n+1}}$ is a continuous image of $(\mathcal{B}_n)^{\omega^n}$.

**§ 2. Proof of Theorem 1.** Throughout this section we fix $n \in \omega \setminus \{0\}$ and assume that $\text{CSP}(J_n, \text{Fin}^*)$ holds. Moreover, contradicting Theorem 1, we assume that there exists a function $H: \mathcal{B}_{n+1} \to \mathcal{B}_n$ which is an isomorphic embedding of Boolean algebras, and fix such a function $H$ throughout the proof.

Let $H$ be any lifting of $H$, i.e. a function $H: \mathcal{A}_{n+1} \to \mathcal{A}_n$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{A}_{n+1} & \to & \mathcal{A}_n \\
\downarrow H & & \downarrow H \\
\mathcal{B}_{n+1} & \to & \mathcal{B}_n
\end{array}$$

where $\pi_{\omega^n}$ and $\pi_{\omega^{n+1}}$ denote the canonical projections.

We define a function $F: \mathcal{P}(\omega) \to \mathcal{P}(\omega^n)$ by the formula $F(X) = H(X^{\omega+1})$ for every $X \subseteq \omega$. Clearly, the function $F$ preserves intersection mod $J_n$, hence by CSP$(J_n, \text{Fin}^*)$ there exists an infinite subset $A \subseteq \omega$ and a continuous function $F^*: \mathcal{P}(A) \to \mathcal{P}(\omega^n)$ such that $F^*$ and $F$ are equal mod $J_n$.

We fix such a set $A$ and such a function $F^*$ for the remainder of this proof. In order to derive the desired contradiction we show that the function $F^*$ has too nice properties.

**Proposition 2.** There exist an increasing sequence of non-negative integers $(k_n: k \in \omega)$ and a sequence of functions $(G_k: k \in \omega)$ such that $G_k: \mathcal{P}(k_n \setminus A) \to \mathcal{P}(k)$ and $F^*(k_n \setminus A) = G_k(k_n \setminus A)$ for all $k \in \omega$ and $X \subseteq A$.

Proof. Observe that $\mathcal{P}(\omega)$ and $\mathcal{P}(\omega^n)$ are compact metrizable spaces; it will be convenient for our purposes to consider $\mathcal{P}(\omega)$ equipped with the metric $d_0$ defined by the formula

$$d_0(Y_1, Y_2) = 2^{-\min(\max(\|Y_1\|, \|Y_2\|), 1)} \cdot \|Y_1 \setminus Y_2\|.$$ 

It is easily seen that Proposition 2 just asserts that $F^*$ is a uniformly continuous function from $(\mathcal{P}(A), d_0)$ into $(\mathcal{P}(\omega^n), d_0)$.

For the remainder of this section we fixed sequences $(k_n: k \in \omega)$ and $(G_k: k \in \omega)$ satisfying the statement of Proposition 2. Moreover, we assume without loss of generality that $k_0 = 0$. We recall that $I_{k_n}$ was defined as

$$I_{k_n} = \{\langle x_0, \ldots, x_{k_n - 1}\rangle: \exists i < n (x_i < k)\}.$$ 

Since $n$ is fixed, we shall write $I_k$ instead of $I_{k_n}$ in the sequel.

Now we define a concept which will be useful in several places of the proof.

**Definition 3.** Let $k' > k$. We let $\{k', k\} = k^{\omega} \setminus k = \{j \in \omega: k \leq j < k'\}$. A subset $c \subseteq A \setminus I_k \setminus I_k$ is called a $(k, k')$-stabiliser if

$$F^*(a \cup c \cup d) \Delta F^*(b \cup c \cup d) \subseteq I_k \text{ for all } d \subseteq A \setminus I_k \text{ and } a, b \subseteq I_k \setminus A.$$ 

**Proposition 4.** For every $k \in \omega$ there exist $k' > k$ and a $(k, k')$-stabiliser $c \subseteq A \setminus I_k \setminus I_k$. 
Proof. Assume the contrary and let $k$ be such that for every $j > k$ there is no $(k,j)$-stabilizer. Then we may construct inductively:

- an increasing sequence $\langle k(p) : p \in \omega \rangle$ of natural numbers,
- a sequence of pairs $\langle (a_p, b_p) : p \in \omega \rangle$,
- a sequence of finite sets $\langle c_p : p \in \omega \rangle$

such that for all $p \in \omega$ the following conditions hold:

1. $k(0) = k$,
2. $a_p, b_p \subseteq I_k \cap A$,
3. $c_p \subseteq A \cap (I_k, I_{k+1})$,
4. $c_{p+1} \cap I_{k+1} = c_p$,
5. $G_{k+1}(a_p \cup c_p) \Delta G_{k+1}(b_p \cup c_p) \not\subseteq I_{k+1}$.

Since $\mathcal{D}(A \cap A)$ is finite, there exist $a, b \subseteq I_k \cap A$ such that $\langle a, b \rangle = \langle a_p, b_p \rangle$ for infinitely many $p$. Let $C = \bigcup \{c_p : p \in \omega \}$. It follows from (4) that

$$C = \bigcup \{c_p : p \in \omega \} \cup \langle (a, b) \rangle.$$  

From (5) and the choice of the functions $G_k$ we infer that

$$F^*(\omega \cup C) \Delta F^*(\omega \cup C) \not\subseteq J_k.$$  

On the other hand, we have

$$F^*(\omega \cup C) \Delta F^*(\omega \cup C) \subseteq J_k$$  

by the definition of $F^*$, hence

$$H(\omega \cup C)^{\prec \kappa} \Delta H(\omega \cup C)^{\prec \kappa} \subseteq J_k.$$  

But $H$ was chosen to be a lifting of $H_0$, so we must have

$$H((\omega \cup C)^{\prec \kappa}) \Delta H((\omega \cup C)^{\prec \kappa}) \subseteq J_k,$$

since obviously

$$(\omega \cup C)^{\prec \kappa} \subseteq (\omega \cup C)^{\prec \kappa} \subseteq I_{k+1} \subseteq I_{k+1}.$$  

This contradiction concludes the proof of Proposition 4. □

Definition 5. Let $\langle k(p) : p \in \omega \rangle$ be an increasing sequence of natural numbers, and let $b = \langle b_p : p \in \omega \rangle$ and $c = \langle c_p : p \in \omega \rangle$ be sequences of finite subsets of $A$. We say that the triple $(b, c)$ is n-productive if the following conditions hold:

1. $b_p \subseteq A \cap (I_{k(p)}, I_{k(p)+1})$ for $p > 0$ and $b_0 \subseteq A \cap I_{k(0)}$,
2. Assume $X_1, X_2$ are such that for arbitrary $p \in \omega$ and $j \in \{1, 2\}$ either $X_j \cap I_{k(p)} \subseteq I_{k(p)+1}$ or $X_j \cap I_{k(p)} \subseteq I_{k(p)+1}$, where $I_{k(p)} = \langle I_{k(p)} \rangle$ else,

and let $q_0, \ldots, q_{n-1} \in \omega$.

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In this situation, if

$$X_i \Delta X_i \cap (I_{k(q_i)}, I_{k(q_i)+1}) = \emptyset$$  

for all $i < n$,

then

$$F^*(X_1) \Delta F^*(X_2) \cap (\bigcup_{i < \omega} (I_{k(q_i)}, k(q_i)+1))^\ast = \emptyset.$$  

The following lemma will be crucial in the proof.

Lemma 6. There exist sequences $k, b$ and $c$ such that the triple $(k, b, c)$ is n-productive, and moreover

$$\emptyset \neq J_p \not\subseteq \emptyset \text{ for every } p \in \omega.$$  

In order to make it more digestible, the remainder of this paper is organized as follows: First we prove Lemma 6 for the special case $n = 1$, which is considerably easier than the general one, next we show how Theorem 1 follows from Lemma 6, and finally we prove the lemma for all $n$.

Definition 7. Let $\langle k(p) : p \in \omega \rangle$ be a triple of sequences as in Definition 5 such that (1) is satisfied. A subset $X \subseteq A$ will be called $(k(p) : p \in \omega)$-amenable (or amenable, if it is clear from the context which sequences we have in mind) if for arbitrary $p \in \omega$ either $X \cap (I_{k(p)}, I_{k(p)+1}) = b_p$ or $X \cap (I_{k(p)}, I_{k(p)+1}) = c_p$.

Proof of the lemma for $n = 1$. Since $A$ is infinite, it follows from Proposition 4 that there is an increasing sequence $\langle k(p) : p \in \omega \rangle$ and a sequence $\langle a_p : p \in \omega \rangle$ such that $m(0) = 0$ and for every $p \in \omega$ the set $a_p$ is contained in $I_{k(p)} \cap I_{k(p)+1} \cap A$, where $a_{2p}$ is nonempty and $a_{2p+1}$ is an $(m(2p)+1, m(2p+2))$-stabilizer.

Now we define for $p \in \omega$:  

$$k(p) = m(2p), \quad b_p = a_{2p+1}, \quad c_p = a_{2p} \cup a_{2p+1}.$$  

The triple of sequences thus defined obviously satisfies (1) and (3). In order to see that it satisfies (2), let $X_1, X_2 \subseteq A$ be amenable set; and fix $q \in q_0 \in \omega$.

If $q = 0$ and $X_1 \Delta X_2 \cap (I_{k(q)} \cap I_{k(q)+1}) = \emptyset$, then

$$G_{k(q)}(X_1 \cap I_{k(q)}) = G_{k(q)}(X_2 \cap I_{k(q)}),$$  

since by our assumption $I_{k(0)} = I_0 = 0$.

If $q > 0$, then

$$X_i \cap (I_{k(2q-1)}, I_{k(2q)}) = X_i \cap (I_{k(2q-1)}, I_{k(2q)}) = a_{2q-1}.$$  

Since $a_{2q-1}$ is an $(m(2q-1), m(2q))$-stabilizer, it follows that whenever

$$X_i \cap (I_{k(q)}, I_{k(q)+1}) = X_i \cap (I_{k(q)}, I_{k(q)+1}),$$  

and let $q_0, \ldots, q_{n-1} \in \omega$.

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then
\[ G_{i+1}(X_1 \cap b_{i+1}) \cap [k(q), k(q+1)] = G_{i+1}(X_{i+1} \cap [k(q), k(q+1)]), \]
which implies (2) by the choice of the functions \( G_{i+1} \).

This concludes the proof of Lemma 6 for the special case \( n = 1 \).

Derivation of Theorem 1 from Lemma 6. Suppose the triple of sequences \( \langle k, e, f \rangle \) is n-productive and satisfies (3). We define \( X_0, X_1, \ldots, X_n \subseteq A \) by putting for \( i \leq n \) and \( p \in \omega \)
\[ X_i \cap [b_{i+1}, b_{i+1} + 1) = \begin{cases} \emptyset & \text{if } p = \equiv (\mod n + 1), \\ c_p & \text{else}. \end{cases} \]
We denote \( \bigcup_{i \leq n} X_i \) by \( X \).

Claim 8. \( X^{	ext{\upshape \#}} \cup \bigcup_{i \leq n} X_i^{	ext{\upshape \#}} \neq J_n + 1. \)

Proof of the claim. For every \( p \in \omega \) we choose \( z_p \in c_p \setminus b_p \) and let
\[ Z = \{ (z_{i+2}, z_{i+2+1}, \ldots, z_{i+2+p}), p \in \omega \}. \]
Obviously, \( Z \subseteq X^{	ext{\upshape \#}} \cup \bigcup_{i \leq n} X_i^{	ext{\upshape \#}}, \) and \( Z \neq J_n + 1. \)

Since \( H \) is an isomorphic embedding of \( \mathcal{B}_{\cdot} \uplus J_n^\omega \) into \( \mathcal{B}_{\cdot} \uplus J_n^\omega \) and \( H \) is a lifting of \( H \), it follows from claim 8 that \( H(X^{	ext{\upshape \#}} \cup \bigcup_{i \leq n} X_i^{	ext{\upshape \#}}) \neq J_n. \)

On the other hand, \( H \) preserves finite sums \( \mathcal{B}_{\cdot} \) hence
\[ H(X^{	ext{\upshape \#}} \cup \bigcup_{i \leq n} X_i^{	ext{\upshape \#}}) \neq J_n, \]
and consequently
\[ F(X) \setminus \bigcup_{i \leq n} F(X_i) \neq J_n. \]
But this contradicts the following:

Claim 9. \( F(X) \setminus \bigcup_{i \leq n} F(X_i) \in J_n. \)

Proof of the claim. Let \( \bar{z} = \langle z_0, \ldots, z_n \rangle \in \omega^\omega \) and suppose \( \bar{z} \notin J_{n+1} \). Then there exist \( q_0, \ldots, q_n \in \omega \) such that
\[ \bar{z} \in [k(q_0), k(q_0 + 1)] \times [k(q_1), k(q_1 + 1)] \times \cdots \times [k(q_n), k(q_n + 1)]. \]
Moreover, there exists \( s \leq n \) such that \( q(t) = s \) mod \( n + 1 \) for all \( i < n \). We fix such an \( s \) and notice that
\[ X_i \cap [b_{i+1}, b_{i+1} + 1) = c_{i+1} = X \cap [b_{i+1}, b_{i+1} + 1) \]
for all \( i < n \). Notice that both \( X \) and \( X_i \) are \( \langle k, e, f \rangle \)-amenable. Hence it follows from (2) that
\[ F(X) \setminus F(X_i) \cap \bigcup_{i \leq n} [k(q_i), k(q_i + 1)) = \emptyset, \]
and therefore \( \bar{z} \in F(X) \) if and only if \( \bar{z} \in F(X_i) \).

What we have shown is that for every \( z \in F(X) \setminus J_{n+1} \) there exists an \( s \leq n \) such that \( \bar{z} \in F(X_i) \); hence
\[ F(X) \setminus \bigcup_{i \leq n} F(X_i) \in J_{n+1}. \]

This concludes the proof of the claim by the definition of \( J_n. \)

So our task reduces to the
Proof of Lemma 6. For any \( k, t \in \omega \) we define
\[ \mathcal{S}(k, t) = \{ X \subseteq A : \exists \{ Y_{ij} : i < n \land j < t \} (F(X) \setminus J_n = \bigcup_{i \leq n} \bigcap_{j < t} Y_{ij}) \}. \]

Since \( F \) is equal to \( H \) mod \( J_n \) and the range of \( H \) is contained in \( \mathcal{B}_{\cdot} \), it follows that
\[ \bigcup_{k \in \omega} \mathcal{S}(k, t) = \mathcal{B}_{\cdot}. \]

For given \( k, t \in \omega \) and \( \emptyset = \{ Y_{ij} : i < n \land j < t \} \) the set
\[ \{ X \subseteq A : F(X) \setminus J_n = \bigcup_{i \leq n} \bigcap_{j < t} Y_{ij} \} \]
is a closed subset of \( \mathcal{B}_{\cdot} \), hence \( \mathcal{S}(k, t) \) is an analytic subset of \( \mathcal{B}_{\cdot} \), and therefore \( \mathcal{S}(k, t) \) has the Baire property. Obviously, the sets \( \mathcal{S}(k, t) \) increase with increasing parameters \( k \) and \( t \). Hence by Baire's theorem, \( \mathcal{S}(k, t) \) is of second Baire category for \( k, t \) sufficiently large.

For the remainder of this proof we fix numbers \( k' \) and \( t \) and a set \( u \subseteq I_{k'} \cap A \) such that if we put \( [u] = \{ X \subseteq A : X \cap I_{k'} = \emptyset \} \), then \( [u] \subseteq \mathcal{S}(k', t) \) is of first Baire category. We let \( L_1 = [u] \cap \mathcal{S}(k', t) \).

If we could define a continuous function
\[ E : L \rightarrow (\mathcal{B}_{\cdot})^\omega \]
such that
\[ E(X) = \langle h_{ij}(X) : i < n \land j < t \rangle \]
and
\[ F(X) \setminus J_n = \bigcup_{i \leq n} \bigcap_{j < t} Y_{ij}(X) \]
for every \( X \in L \),
when we could prove the lemma by a straightforward generalization of the proof for the case \( n = 1 \). The problem is that the sets \( Y_{ij}(X) \) may not be uniquely determined by \( F(X) \setminus J_n \). Consider the following trivial example for \( n = t = 2 \):

The idea of what follows is, roughly speaking, that we shall find sequences \( E, F \) and \( \bar{c} \) such that, for \( X \in \mathcal{S}(k, t) \)-amenable, although we may not be able to reconstruct the sets \( Y_{ij} \) witnessing that \( F(X) \setminus J_n \) is an element of \( \mathcal{B}_{\cdot} \), we do however
\[ 3 \in \text{Fundamenta Mathematicae} 120.1 \]
possess enough information to reconstruct some finite parts of \( F^k(X) \) from certain information about \( X \) as required in (2).

**Definition 10.** Let \( W \subseteq \omega^\omega \) and \( i < n \). We define a relation \( =_{W,i} \) on \( \omega \times \omega \) as follows: \( z =_{W,i} z' \) if
\[
\forall (x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{n-1}) \in \omega^i \exists (y_0, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_{n-1}) \in \omega^i \langle (x_0, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_{n-1}) \in W \rangle \wedge \langle (x_0, \ldots, x_{i-1}, z', x_{i+1}, \ldots, x_{n-1}) \in W \rangle.
\]

We write \( z =_{W,i} z' \) if \( z =_{W,i} z' \) for all \( i < n \).

**Claim 11.** (a) The relations \( =_{W,i} \) and \( =_{W} \) are equivalence relations for arbitrary \( W \subseteq \omega^\omega \) and \( i < n \).

(b) If \( W \in \mathcal{A}_n \), then the relation \( =_{W} \) splits \( \omega \) into finitely many equivalence classes.

**Proof of the claim.** Part (a) is obvious.

For the proof of (b) notice that if \( W \in \mathcal{A}_n \), then there exist \( x \in \omega \) and a family \( \{ Y_{i,j}; i < n, j < x \} \) such that \( W = \bigcup_{i < n} \bigcap_{j < x} Y_{i,j} \). It suffices to show that for every \( i < n \), the relation \( =_{W,i} \) splits \( \omega \) into finitely many equivalence classes. So we fix \( i < n \) and set \( P_i(z) = \{ j < x : z \in Y_{i,j} \} \) for \( z \in \omega \). Observe that \( \langle x_0, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_{n-1} \rangle \in W \) iff \( \langle x_0, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_{n-1} \rangle \in \bigcup_{j \in P_i(z)} \bigcap_{i < n} Y_{i,j} \). It follows that \( z =_{W,i} z' \) whenever \( P_i(z) = P_i(z') \), hence the relation \( =_{W,i} \) splits \( \omega \) into at most \( 2^x \) equivalence classes.

**Definition 12.** (a) Let \( r \in \omega \). A subset \( W \subseteq \omega^\omega \) is called \( r \)-semisimple if the relation \( =_{W} \) partitions \( \omega \) into exactly \( n \) nonempty equivalence classes. It is called \( r \)-simple if it is \( r \)-semisimple and every equivalence class of the relation \( =_{W} \) is infinite.

(b) Let \( W \in \mathcal{A}_n \) and \( E \subseteq \omega \). We say that \( E \) is a witness for \( W \) if the relation \( =_{W,\omega} \) splits \( E \) into \( n \) nonempty equivalence classes.

**Claim 13.** Let \( r \in \omega \) and suppose \( W \subseteq \omega^\omega \) is \( r \)-semisimple.

(a) If \( E \subseteq \omega \) is a witness for \( W \) and \( z, z' \in E \), then \( z =_{W,\omega} z' \) iff \( z =_{W,\omega} z' \).

(b) If \( E \subseteq \omega \) is a witness for \( W \) and \( D \subseteq \omega \) is also a witness for \( W \), and moreover \( E \) is a witness for \( W \cap D^* \).

(c) Suppose \( W \subseteq \omega^\omega \) is \( r \)-simple and \( k \in \omega \) then there exists a \( k^* > k \) such that the interval \([k, k^*)\) is a witness for \( W \).

**Proof of the claim.** Parts (a) and (b) follow immediately from Definition 12. We prove (c).

Let \( k, r, W \) be as in the hypothesis. Since all equivalence classes of the relation \( =_{W} \) are infinite, we find numbers \( x_0, \ldots, x_{n-1} > k \) which are representatives of all the equivalence classes of the relation \( =_{W} \). Now let \( k^* = \sup \{ x_j : j < r \} + 1 \). In order to show that \( k^* \) is as required, it suffices to show that for all \( i < n \) and \( y, y' \in [k, k^*) \) the relation \( y =_{W,i} y' \) holds iff the relation \( y =_{W,\omega} y' \) holds.

The "only if" direction follows immediately from Definition 10.

Now suppose that for some \( i < n \) and \( y, y' \in [k, k^*) \) we have \( y \neq_{W,i} y' \), i.e. there exist numbers \( y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{n-1} \) such that \( W \in \mathcal{A}_n \) and
\[
\langle y_0, \ldots, y_{i-1}, y, y_{i+1}, \ldots, y_{n-1} \rangle \in W
\]
and
\[
\langle y_0, \ldots, y_{i-1}, y', y_{i+1}, \ldots, y_{n-1} \rangle \notin W.
\]

By our choice of \( k^* \) there exist numbers \( j_0, \ldots, j_{i-1}, j_{i+1}, \ldots, j_{n-1} \in [k, k^*) \) such that \( j_{0} < j_{1} < \cdots < j_{n-1} \), nonidentically if \( j_{0} < j_{1} < \cdots < j_{n-1} \). By induction over \( j \) one shows that
\[
\langle y_0, \ldots, y_{i-1}, y_j, y_{i+1}, \ldots, y_{n-1} \rangle \in W
\]
and analogously if we replace \( y \) by \( y' \), it follows that
\[
\langle y_0, \ldots, y_{i-1}, y, y_{i+1}, \ldots, y_{n-1} \rangle \in W
\]
and
\[
\langle y_0, \ldots, y_{i-1}, y', y_{i+1}, \ldots, y_{n-1} \rangle \notin W,
\]
witnessing that \( y \neq_{W,\omega} y' \), which concludes the proof of Claim 13.

The following claim says that witnesses allow us to reconstruct \( W \) from the relation \( =_{W} \).

**Claim 14.** Suppose \( W, W' \subseteq \omega^\omega \) are both \( r \)-semisimple, the set \( E \subseteq \omega \) is a witness for \( W \) and \( W \cap E^* = W' \cap E^* \). Assume furthermore that for all \( z, z' \in \omega \) the relation \( z =_{W,\omega} z' \) holds iff the relation \( z =_{W',\omega} z' \) holds. Then \( W = W' \).

**Proof of the claim.** Suppose \( W, W' \subseteq \omega^\omega \) and \( E \subseteq \omega \) satisfy the hypothesis of the claim and let \( z = \langle x_0, \ldots, x_{n-1} \rangle \in \omega^\omega \). We put \( f(E) = \{ |i < n : z_i \notin E \} \). By induction over \( j \) we show that
\[
f(E) \in W \iff f(E) \in W'.
\]

For \( f(E) = 0 \) this is obvious. Suppose \( f(E) \) holds for all \( z \in \omega^\omega \) such that \( f(E) \leq j \), and assume that \( f(E) = j + 1 \). Fix an \( i < n \) such that \( z_i \notin E \). By our assumption there exists a \( z_i \in E \) such that \( z_i =_{W} z_i =_{W'} z_i \). Let \( z' = \langle x_0, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_{n-1} \rangle \). Then \( f(z') = j \), and by our induction hypothesis \( z' \in W \) iff \( z' \in W' \). But the relation \( z_i =_{W} z_i \) implies that \( z \in W \) iff \( z' \in W \), and since \( z_{i} =_{W} z_{i} \), we have \( z \in W \) iff \( z' \in W' \). Hence \( (E) \) holds, concluding the proof of Claim 14.

By \( ||z||_W \) we shall denote the equivalence class of the relation \( =_{W} \) containing \( z \).
For a given \( W \in \mathcal{S} \), some of the equivalence classes of the relation \( \equiv_W \) may be finite. Therefore we set \( \overline{W} = \varnothing \text{ if } W \in \mathcal{S}_0 \) or \( \overline{W} = \sup \{ |z_0|_W : z_0 \in \text{Fin} \} \). By Claim 11(b), if \( W \in \mathcal{S}_0 \), then \( k(W) < +\infty \). We write \( \overline{W} = W \setminus I_{\mathcal{S}_0} \), for \( W \in \mathcal{S}_0 \). One easily verifies that \( |z_0|_W k(W) \leq |z_0|_W \) for all \( W \in \mathcal{S}_0 \) and \( z_0 \equiv_W k(W) \), hence every equivalence class of the relation \( \equiv_W \) is infinite.

Now, let us recall that the definitions of \( (k, r) \), for given \( (k, r) \), are given at the beginning of the proof of Lemma 6.

Since the relation \( \equiv_W \) is \( r \)-semi-simillic \( & k(W) < \infty \) defines a Borel subset of \( \mathcal{S}(\omega^n) \), it follows that there exist numbers \( k \geq k \) and \( r \) such that the set

\[
\{ X \in [\omega] \cap (k, r) : F^+(X) \setminus I_r \text{ is } r \text{-semi-simillic } & k(F^+(X) \setminus I_r) \leq k \}
\]

is of second Baire category in \( \mathcal{S}(A) \).

Until the end of this paper we fix \( k, r \geq k \), a set \( v \in [\omega] \cap \mathcal{S}(l_0) \) and a number \( r \) such that the set

\[
S = \{ X \in [\omega] \cap (k, r) : k(F^+(X) \setminus I_r) \geq k \text{ or } F^+(X) \setminus I_r \text{ is not } r \text{-semi-simillic} \}
\]

is of first Baire category in \( \mathcal{S}(A) \). Here \( [\omega] \) denotes the set \( \{ X \in A : X \cap I_r = \varnothing \} \).

In the sequel we write

\[
M = \{ v \} \cap (\mathcal{S}(k, r), r) \setminus S, \quad -\alpha \text{ instead of } -\alpha_0 \setminus \mathcal{S}(k, r) \quad \text{for } X \in M.
\]

Moreover, a subset \( E \subseteq \omega \) will be called a \( \alpha \)-witness for \( X \in M \) if it is a witness for the set \( F^+(X) \setminus I_r \subseteq \alpha^n \).

Sublemma 15. There exists a set \( E \subseteq \omega \) and sequences

\[
E = \langle E(p) : p \in \omega \rangle, \quad b = \langle b_p : p \in \omega \rangle, \quad c = \langle c_p : p \in \omega \rangle
\]

satisfying for all \( p \in \omega \) and \( (k, b, c) \)-amenable sets \( X_0 \) and \( X_1 \):

(0) \( k(0) \geq k \),
(1) \( b_p \leq c_p \leq \langle k(0) \rangle, k_{n+1} \rangle \) for \( p > 0 \) and \( b_0 \leq c_0 \leq \langle k(0) \rangle, k_1 \rangle \),
(2) \( c_p b_p \neq \varnothing \),
(3) \( X_0 \cap E \), \( X_1 \in M \),
(4) \( k \),
(5) \( F^+(X_0 \setminus I_r), k_{n+1} \rangle = \varnothing \),
(6) \( E \cap \{ k(p), k(p+1) \} = \varnothing \),
(7) \( E \cap F^+(X_0) = E \cap F^+(X_1) \),
(8) \( F^+(X_0) \cap F^+(X_1) \subseteq \{ k(p), k(p+1) \} = \varnothing \).

For all \( w \), \( w' \in E \) the relation \( w \leq_\alpha w' \) holds if \( \{ \text{the relation } w \leq_w w' \} \) holds.

Before proving the sublemma we show how to deduce Lemma 6 from it.

We fix a triple \( (E, b, c) \) of sequences satisfying the statement of the sublemma and show that it also satisfies (2) of Definition 5.

Suppose \( X_0 \), \( X_1 \) are amenable and \( q_0, \ldots, q_{n-1} \in \omega \) are such that

\[
X_0 \cap X_1 \cap \bigcup_{i<n} [l_{i+1}, l_{i+1}+1] = \varnothing.
\]

Let

\[
Z_0 = F^+(X_0 \setminus I_{\mathcal{S}_0}) \text{ and } Z_1 = F^+(X_1 \setminus I_{\mathcal{S}_0}).
\]

Furthermore, we set

\[
U = \bigcup \{ k(q), k(q+1) \} \quad \text{and} \quad V_0 = Z_0 \setminus U^n, \quad V_1 = Z_1 \setminus U^n.
\]

We show that \( V_0 = V_1 \), which obviously implies (2).

It follows from (6) and Claim 13(b) that \( C \cap \{ k(q), k(q+1) \} = \varnothing \) for both \( V_0 \) and \( V_1 \). Consequently, if we show that for arbitrary \( z, z' \in U \) we have

\[
z = v_z z' \quad \text{if} \quad z = v_z z', \quad \text{then the equality} \quad V_0 = V_1 \quad \text{becomes an easy consequence of (7) and Claim 14.}
\]

Hence let \( z, z' \in U \) and let \( i, i' < n \) be such that \( z \subseteq \{ k(q), k(q+1) \} \text{ and } z' \subseteq \{ k(q), k(q+1) \} \text{ and } z \subseteq \{ k(q), k(q+1) \} = \varnothing \). By 6 there are numbers \( w \in \{ k(q), k(q+1) \} \) and \( w' \in \{ k(q), k(q+1) \} \) such that \( z = v_z w \) and \( z' = v_z w \). We fix such \( w \) and \( w' \).

Obviously, \( z = v_z z' \quad \text{if} \quad w = v_z w' \).

On the other hand, by (5) we have

\[
V_0 \cap \{ k(q), k(q+1) \} = V_1 \cap \{ k(q), k(q+1) \} = \varnothing.
\]

Since \( E \cap \{ k(q), k(q+1) \} = \varnothing \) for both \( V_0 \) and \( V_1 \), it follows from (5) and from Claim 13 that \( z \neq z' \text{ if } w = w' \text{ if } z \neq z' \text{ if } w = w' \). By an analogous reasoning one can show that \( z = z' \text{ if } w = w' \text{ if } z = z' \text{ if } w = w' \). Hence the relation \( z = z' \text{ if } w = w' \text{ holds if the relation } z = z' \text{ holds.}
\]

We have thus shown that the triple of sequences \( \langle E, b, c \rangle \) satisfying (0), (1) and (3)-(5) satisfies (2) of Definition 5 as well, and hence the proof of Lemma 6 reduces to the Proof of Sublemma 15. Recall that the definitions of \( M \) and \( S \) were given before the statement of the sublemma. Let \( S = \bigcup S_\alpha \), where \( S_\alpha \) is a nowhere dense subset of \( \mathcal{S}(A) \) and \( S_\alpha \subseteq S_{\alpha+1} \) for all \( \alpha \). We construct inductively and increasing sequence of natural numbers

\[
\mathcal{M} = \langle m(p) : p \in \omega \rangle
\]

and a sequence \( \mathcal{E} = \langle e_p : p \in \omega \cup \{ -1 \} \rangle \) of finite subsets of \( A \) such that:

(0) \( e_{-1} = \langle 0, I_{\mathcal{S}_0} \rangle \),
(1) \( m(0) \geq k \),
(2) \( e_{-1} \subseteq \{ e_p : p \in \omega \} \),

and for all \( p \in \omega \) we have:

(3) \( e_p \subseteq \langle k(0), k_{n+1} \rangle \),
(4) \( \text{if} \quad m \leq m + 1, \quad \text{then} \quad \langle m \rangle \subseteq \langle m \rangle \),
(5) \( \langle m(p), m(p+1) \rangle \) into exactly \( r \) nonempty equivalence classes.
(vi) If \( X \subseteq A \) and \( X \cap \bigcup_{n \in \mathbb{N}} b_n = a \), then \( X \neq S_p \).
(vii) \( \{n \mid p \neq q \} \geq 1 \).
(viii) \( \mathcal{B}_{p+1} \) is an \( (m(4p+3), m(4p+4)) \)-stabilizer.

First we should convince ourselves that sequences satisfying (i)–(viii) exist. It is obvious that constructing \( m(p) \) and \( e_n \) inductively we can take care of (i)-(iv) and (vii). Proposition 4 tells us that we can deal with (viii) as well. In order to see how to take care of (vi), notice that \( S_p \) is nowhere dense in the topology of \( \mathcal{P}(A) \) and that there are only finitely many candidates for \( X \cap \mathcal{B}_{p+1} \), so we can deal with all of them successively extending initial fragments of \( \mathcal{B}_{p+1} \).

It remains to show that at stage \( 4p \) of the construction we can make sure that (v) holds. In order to see this, notice that given any \( p, \) by the definition of \( M \) and the Baire category theorem there exists \( X \in M \) such that \( X \cap \mathcal{B}_{p,q} = \emptyset \). Since \( X \in M \), the set \( \mathcal{F}^*(X) \cap \mathcal{B}_q = Y \) is r-simple, and hence by Claim 13(c) there exists an \( m(4p+1) > m(4p) \) such that \( m(4p), m(4p+1) \) is a witness for \( Y \). By the choice of the functions \( G_n \) we have \( D_p = Y \cap \{n \mid m(4p+1) \} \) whenever \( X \cap \mathcal{B}_{p+1} \) = \( \emptyset \).

For the remainder of this proof we fix \( \mathcal{F} \) and \( \mathcal{B} \) satisfying (i)–(viii). Suppose that we are given two sequences \( \mathcal{B} = \{b_n \mid n \in \mathbb{N} \} \) and \( \mathcal{B}' = \{b_n' \mid n \in \mathbb{N} \} \) such that \( b_n \subseteq b_n' \subseteq \mathcal{B}_{p+1} \) for all \( p \). We define:

\[
\begin{align*}
\eta_0 &= \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3 \cup \varepsilon_4, \\
\eta_p &= \varepsilon_4 \cup \mathcal{B}_{p+1} \cup b_n' \cup \mathcal{B}_{p+1}, & \text{for} \ p \geq 1.
\end{align*}
\]

Moreover, we put \( \kappa(p) = m(4p) \) for every \( p \).

Sequences \( \mathcal{B}, \mathcal{B}' \) which are defined by the above method from some sequences \( \mathcal{B} \) and \( \mathcal{B}' \) will be called feasible. Moreover, a subset \( X \subseteq C \) will be called feasible if it is \( (\mathcal{B}, \mathcal{B}', \varepsilon) \)-amenable for some feasible sequences \( \mathcal{B}, \mathcal{B}' \).

Let \( \mathcal{B}, \mathcal{B}' \) be feasible. Conditions (1) and (3) of the sublemma are satisfied by the choice of \( b_n \) and \( b_n' \). Moreover, (4) follows from (iii), (vii), the definition of \( M \) and the choice of \( S_p \). (0) is a consequence of the definition of \( \mathcal{B} \).

In order to see that (5) holds as well, suppose \( X_0 \) and \( X_1 \) are feasible and let \( p \in \omega \) be such that

\[
X_0 \cap \bigcup_{n \in \mathbb{N}} b_n = X_1 \cap \bigcup_{n \in \mathbb{N}} b_n' = a_p.
\]

If \( p = 0 \), then

\[
\mathcal{F}^*(X_0) \cap (m(p+1)) = a_0 = \mathcal{B}_{p+1} \cap (m(p+1)).
\]

If \( p > 0 \), then \( X_0 \cap b_n \cup \mathcal{B}_{p+1} = X_1 \cap (b_n' \cup \mathcal{B}_{p+1}) = a_{p-1} \). But according to (viii), the set \( \mathcal{B}_{p+1} \) was chosen to be an \( (m(4p+1), m(4p+4)) \)-stabilizer, so

\[
\mathcal{F}^*(X_0) \cap \mathcal{F}^*(X_1) \cap (m(p+1)) \subseteq \mathcal{B}_{p+1}
\]

by the definition of a stabilizer; hence (5) holds.

Now let \( E = \bigcup_{p=0}^{\infty} m(4p), m(4p+1) \). We show that \( E \) satisfies (6).

Indeed, arguing as in the proof of (5), we notice that

\[
\mathcal{F}^*(X_0) \cap (m(4p), m(4p+1)) = \mathcal{B}_p
\]

for arbitrary \( p \in \omega \), and now (6) is an immediate consequence of (4), the choice of \( M \) and Claim 13(b).

It remains to show that we can find sequences \( \mathcal{B} \) and \( \mathcal{B}' \) such that (7) and (8) are satisfied for \( \mathcal{B}, \mathcal{B}' \) and \( \mathcal{E} \) defined as above.

We fix \( x_0, x_1, \ldots, x_{r-1} \in m(0), m(1) \) which are assumed to be representatives of all equivalence classes of the relation \( = \). Notice that the \( x_0, \ldots, x_{r-1} \) may be chosen independently of the choice of \( \mathcal{B} \) and \( \mathcal{B}' \). Given a feasible \( X \), we define a function \( f_X : E \to r \) as follows:

\[
f_X(w) = j \iff w = x_j.
\]

We show that there are feasible sequences \( \mathcal{B}, \mathcal{B}' \) such that \( f_{X_0} = f_X \), for all \( \mathcal{B}, \mathcal{B}' \)-amenable sets \( X_0 \) and \( X_1 \). It is not hard to see that for such sequences (8) will be satisfied.

For \( p > 0 \), we choose \( x_0, \ldots, x_{r-1} \in m(4p), m(4p+1) \) to be representatives of all equivalence classes of the relation \( = \). For every feasible \( X \) we define functions \( f_{X, p} : r \to r \) as follows:

\[
f_{X, p}(j) = j' \iff x_j(j) = j'_j.
\]

Notice that for \( w \in m(4p), m(4p+1) \) and arbitrary feasible \( X \) we have: \( f_{X, p}(w) = j' \iff x_j(f_{X, p}^{-1} w) = j'_j \). If \( f_{X, p}(w) = j' \), then \( f_{X, p}^{-1} w = x_j(j') \), for all \( e_n \in \omega \).

Claim 16. Let \( p \in \omega \) and \( X_0, X_1 \) be such that

\[
X_0 \Delta X_1 \cap \bigcup_{n \in \mathbb{N}} b_n = \emptyset.
\]

Then \( f_{X_0, p} = f_{X_1, p} \).

Proof of the claim. Repeating the argument used for demonstrating (5) shows that there is a set \( V \) such that

\[
\mathcal{F}^*(X_0) \cap (m(4p), m(4p+5)) = \mathcal{F}^*(X_1) \cap (m(4p), m(4p+5)) = V
\]

whenever \( X_0, X_1 \) are as in the hypothesis of the claim. On the other hand, by (6) and Claim 13(b) we know that \( V \) is a witness for both \( X_0 \) and \( X_1 \). It follows that we have \( w = x_n w' \iff w = w' \iff w = w' \), for \( w, w' \in m(4p), m(4p+5) \). To conclude the proof of the claim it suffices to observe that the function \( f_{X, p} \) is completely determined by the relation \( = \) restricted to \( m(4p), m(4p+5) \).

Claim 16 tells us that for feasible \( X \) the function \( f_{X, p} \) depends only on \( X \cap \mathcal{B}_{p+1} \), as well as on \( e_{p+1} \). In other words, given a subset \( d_{e_{p+1}} \), there is a unique bijection \( f_{X, p}(d_{e_{p+1}}) \) from \( X \cap \mathcal{B}_{p+1} \) to \( r \). But there are only \( r! \) bijections from \( r \) to \( r \), and hence by (vii) we can find
\[ b_0 \subseteq b_1 \subseteq c_0 \subseteq \cdots \text{ such that } f_0(b_1) = f_0(c_1). \]  This shows that there are feasible sequences \( b, \bar{c} \) such that \( f_{x_0} = f_{x_1} \) for all \( \langle x, \bar{b}, \bar{c}, \bar{z} \rangle \)-amenable sets \( X_0, X_1 \), which concludes the proof of (5).

(7) is an easy consequence of (8) and Claim 14, because \( X_0 \cap (m(0), m(1)) \) is a \( \tau \)-witness and \( F^*(X_0) \cap m(1)^* = F^*(X_1) \cap m(1)^* \) for all amenable \( X_0 \) and \( X_1 \).

This concludes the proof of Sublemma 15, Lemma 6 and Theorem 1.

References


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Sur le nombre de côtés d’une sous-variété

by

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Abstract. Let \( A \) be a connected, locally connected and locally closed subset of a metric space \( X \) which is locally two-sided in \( X \) in the sense that every point \( x \) of \( A \) has arbitrary small connected neighbourhoods \( U \) such that \( U \cap A \) has exactly two components whose closure contains \( x \). We use elementary methods from sheaf theory to study when \( A \) is globally two-sided in \( X \) (i.e., \( A \) has a connected neighbourhood \( V \) such that \( V \setminus A \) is not connected). We give some applications to concrete examples.

1. Introduction and notations. Soient \( X \) une variété de dimension \( n+1 \), et \( A \) une variété connexe de dimension \( n \) (pas nécessairement fermée) contenue dans \( X \). On dit que \( A \) a deux côtés dans \( X \) si elle a un voisinage ouvert connexe \( W \) dans \( X \) tel que \( W \setminus A \) ait exactement deux composantes; sinon, on dit que \( A \) n’a qu’un côté dans \( X \). Le problème de reconnaître quand \( A \) a deux côtés se pose naturellement, et divers résultats partiels sont connus, l’un des plus généraux étant celui de Rushing [7] selon lequel une \( n \)-variété simplement connexe localement plane dans \( X \) a deux côtés (et a même un double coller dans \( X \)). Rushing remarque aussi que les techniques de la topologie algébrique ne semblent pas suffire à montrer qu’une \( n \)-variété orientable non fermée dans \( S^{n+1} \) a deux côtés. Nous montrerons dans cet article que l’utilisation des premiers éléments de la théorie des faisceaux permet de caractériser les sous-variétés ayant deux côtés (voir le corollaire 2.2). L’avantage de notre approche abstraite est qu’elle s’applique à des espaces beaucoup plus généraux que les variétés; il suffit que \( A \) soit un sous-espace localement fermé et localement connexe d’un espace métrique \( X \) "séparant localement \( X \) en deux morceaux". A titre d’exemple d’applications de ce raisonnement général, nous prouverons les résultats suivants:

1. (i) Soit \( A \) un sous-ensemble connexe et localement connexe d’un espace métrique \( X \) qui a en tout point un double coller local dans \( X \) (voir section 2 pour la définition). Alors

   (a) Si \( A \) n’admet pas de revêtement non trivial à deux feuilles, \( A \) a un double coller dans \( X \).

   (b) Pour toute distance admissible \( d \) sur \( X \), \( A \) a un double coller dans \( X \) si, et