On $\frac{1}{2}$-homogeneous ANR-spaces

by

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Abstract. A space X is called $\frac{1}{2}$-*homogeneous* if the action on X of the group $H(X)$ of autohomeomorphisms of X has exactly two orbits. In the paper it is proved that any compact $\frac{1}{2}$-homogeneous $X \in \mathrm{ANR}$ of dimension $\leqslant 2$ either is a polyhedron, or has one orbit which is the union of at most countably many of composants, each being a continuous biunique image of the line E^1.

1. Introduction. This paper is devoted to the study of spaces whose construction from the point of view of homogeneity is similar to manifolds with boundary (whose boundary components have the same position), and which we shall call $\frac{1}{2}$-homogeneous.

In general, a space X will be called $\frac{1}{n}$-*homogeneous* if the action on X of the group $H(X)$ of autohomeomorphisms of X has exactly n orbits, i.e. if there are n subsets $A_1, \ldots, A_n$ of X such that $X = \bigcup_{i=1}^{n} A_i$ and, for any $x \in A_i$, $y \in A_j$, there is an $h \in H(X)$ mapping x to y iff $i = j$. In Krasinkiewicz's paper [11], it is proved that the Sierpiński curve is $\frac{1}{2}$-homogeneous. Using this fact together with Whyburn's characterization of the Sierpiński curve, given in [20], and a reasoning similar to that in Mazurkiewicz's paper [14], it can be proved that the only $\frac{1}{2}$-homogeneous Peano (locally connected) plane continua are the following: plane manifolds with boundary, the Sierpiński curve, locally connected bouquets of α circles, where α is either a natural number $n > 1$ or $\alpha = \aleph_0$, and all connected plane finite (geometrical) multigraphs (where two vertices can be joined by more than one edge) which are homogeneous in the graph-theoretical sense (except S^1). Using Anderson's results [1], [2] concerning the universal Menger curve and homogeneous Peano curves, all the $\frac{1}{2}$-homogeneous Peano curves can also be classified.

The goal of this paper is to prove the following theorem:

THEOREM 1. *Each space $X \in \mathrm{ANR}$ of dimension $\leqslant 2$ which is $\frac{1}{2}$-homogeneous either is a polyhedron, or has one orbit which is the union of at most countably many of composants, each being a continuous biunique image of the line E^1. Moreover, we find a full classification of $\frac{1}{2}$-homogeneous polyhedra by means of homogeneous multigraphs.*

The author does not know whether there exists a $\frac{1}{2}$-homogeneous 2-dimensional connected ANR with one orbit as described above.

Here, ANR-spaces are assumed to be compact. Notice that because of the result of Bing–Borsuk [5] that each at most 2-dimensional homogeneous ANR-space is a closed manifold, we can assume that the space X is connected. Moreover, it can be assumed to be 2-dimensional, because each $\frac{1}{2}$-homogeneous local dendron is either a bouquet of n circles, where $n > 1$, or a homogeneous multigraph (different from S^1) in the sense described above.

The structure of this paper is the following: In Sections 2 and 3 we prepare the main tools needed in the proof. Namely, in Section 2 we describe the homological tools, based on the notion of membrane, defined and applied in [5]. We prove some useful propositions concerning membranes, in particular Theorem 2, which asserts the local connectedness of some membranes in ANR's. In this paper we shall use the Čech homology theory with coefficients in the field Q of rational numbers, which — for compact metric spaces — is equivalent to the Vietoris theory (based on true cycles), used in [5].

In Section 3, we describe the methods from the theory of topological transformation groups. In particular, we prove Theorem 3, which gives a generalization of the well-known Effros–Hagopian theorem (cf. [6] and [9]) concerning homogeneous spaces. Let us notice here that, by using the Effros–Hagopian theorem, the proof by Bing and Borsuk [5] of the theorem concerning homogeneous 2-dimensional ANR's mentioned above can be shortened and simplified. Namely, we can simplify the proof of Theorem 8.1 of that paper if the local homogeneity of X in the assumptions of that theorem is understood as the existence of an open covering of X by homogeneous sets. Indeed, by Arens' results [3] (cf. also [19]), a locally compact and locally connected (metric) space X with the transitive action of $H(X)$ satisfies the assumptions of the Effros–Hagopian theorem. To see how this last theorem can be applied, see the beginning of Section 5 of the present paper and the fact (5) established there.

In Section 4, we prove Theorem 1 in the case when X has a locally separating point, in part in a somewhat more general form, for spaces which are not necessarily ANR's.

Finally, in Section 5, we prove Theorem 1 in the remaining case, first showing that either one orbit has the described structure, or X is a polyhedron. Our strategy in this proof is the following: First (by a procedure similar to that in [5]) we identify two orbits A, B of the action of $H(X)$ on X where $A = \bigcup_{i=0}^{\infty} A_i$ is an F_σ subset of X and $B = X \setminus A$ is a dense G_δ set in X. Next, after some preparations by means of Baire's theorem, we establish the fact (4) that each set A_i, $i = 0, 1, \ldots$, is locally connected.

In the next part of the proof, we need Theorem 3, and therefore we verify its assumptions, so as to use it for the set A with the action of $H(X)$ on it. Then we can show the fact (7) that no set A_i, $i = 0, 1, \ldots$, contains a ramification point, and therefore each A_i is locally an (open) arc. Further, we establish the fact (10) that such a small arc locally disconnects the space X into at least three components. Then we consider three cases 1°, 2° and 3°, according to the structure of the components of A; we eliminate cases 1° and 2°, proving in (14) that the components of A form a finite sequence, consisting of simple closed curves. In case 1° the orbit A has the structure described in the assumptions of Theorem 1. The fact (14) easily implies that X must be a polyhedron.

After this long proof, using the results of Section 4, it is not difficult to classify all $\frac{1}{2}$-homogeneous polyhedra.

Notice that Theorem 1 can be extended to the case when the space X is locally compact, but non-compact, with the conclusion being that X is a locally finite CW-complex, except the described case. Indeed, as in [5], all membranes can be constructed in compact subsets of X (cf. a comment in Section 2). The proofs of Lemmas 1 and 2 from Section 4 can be modified for this case, so as to assert that the set of points which separate (locally separate) the space X is locally finite (cf. a comment in Section 4). The proof, given in Section 5, that the orbit A is the union of a finite number of disjoint simple closed curves can be adapted to this case, with the finite number of curves replaced by a locally finite sequence. Observe that Theorem 3 can also be applied to the orbit A with the action of $H(X)$ on it, because — by Arens' results mentioned above — the assumptions concerning the group $G = H(X)$ are satisfied. The result concerning the orbit A (resp. the points which separate, or locally separate, the space X) together with the homogeneity of the orbit $B = X \setminus A$ (resp. of the set of remaining points of X) easily imply — as in our proof — that X is a locally finite CW-complex, except the described case.

Evidently, for $k > 2$ there are $\frac{1}{k}$-homogeneous ANR's (even 1-dimensional), which are not polyhedra. Also, for $n > 2$ there are $\frac{1}{2}$-homogeneous n-dimensional ANR-sets which are non-polyhedral generalized manifolds, and have — for instance — one point at which they are not manifolds (obtained by contracting a wild arc in a manifold to a point). However, some kind of classification of such $\frac{1}{2}$-homogeneous n-dimensional generalized manifolds would be interesting.

Also, in connection with the above-mentioned Krasinkiewicz theorem [11], and Bestvina's recent very-interesting results [4] on the homogeneity of the universal Menger continuum M_n^{2n+1}, one can ask whether the space M_n^m is $\frac{1}{k}$-homogeneous for some k, in particular, if the space M_n^{2n} is $\frac{1}{2}$-homogeneous.

2. Membranes in ANR-spaces. As mentioned in the Introduction, we define a membrane as in [5], but using the Čech homology groups with coefficients in Q. Our general reference in homology theory will be Eilenberg and Steenrod's book [7] and sometimes Spanier's book [18].

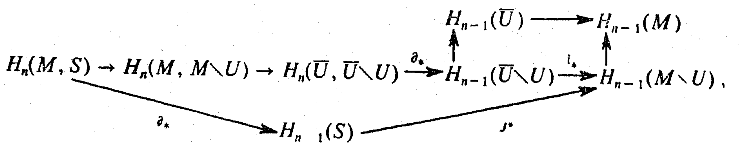
Assume that X is a metric space, S is a compact subset of X and $0 \neq \gamma \in H_n(S)$. Thus S is a carrier of γ. A compact set M containing S will be called a *membrane* of γ

spanned on S if $i_*(\gamma) = 0$, but for any compact proper subset M' of M containing S we have $i'_*(\gamma) \neq 0$, where $i\colon S \to M$, $i'\colon S \to M'$ are the inclusion maps. If $M \supset S$ and $i_*(\gamma) = 0$, then the Brouwer reduction theorem and the continuity axiom for the Čech homology groups imply the existence of a membrane of γ contained in M. If S is a continuous image of the circle S^1, then by a membrane spanned on S we shall always mean a membrane of the element $\gamma \in H_1(S)$ which is the image of a generator of $H_1(S^1)$ under the induced homomorphism.

If M is a membrane of $\gamma \in H_{n-1}(S)$ spanned on S, then it follows from the exactness axiom (which is valid, since the coefficients are in the field Q) that there exists a $\zeta \in H_n(M, S)$ such that $\partial_*(\zeta) = \gamma$, where $\partial_*\colon H_n(M, S) \to H_{n-1}(S)$ denotes the boundary homomorphism. Let U be an open subset of $M \setminus S$ and consider the homomorphisms

$$H_n(M, S) \to H_n(M, M\setminus U) \to H_n(\overline{U}, \overline{U}\setminus U),$$

the first of which is natural, and the second is the excision isomorphism. Let $\zeta' \in H_n(\overline{U}, \overline{U}\setminus U)$ denote the image of ζ and let γ' denote the image of ζ' under the boundary homomorphism $\partial_*\colon H_n(\overline{U}, \overline{U}\setminus U) \to H_{n-1}(\overline{U}\setminus U)$. Considering the commutative diagram



one sees (cf. [5]) that:

(1) $\gamma' \neq 0$ and $\overline{U}$ is a membrane of γ' spanned on $\overline{U}\setminus U$. Moreover, if $i\colon \overline{U}\setminus U \to M\setminus U$, $j\colon S \to M\setminus U$ are the natural maps, then $i_*(\gamma') = j_*(\gamma)$.

If M and M' are two membranes of γ spanned on $S \subset M \cap M'$, then it follows from the Mayer–Vietoris sequence

$$H_n(M \cup M') \to H_{n-1}(M \cap M') \to H_{n-1}(M) \oplus H_{n-1}(M')$$

that $M \neq M'$ implies $H_n(M \cup M') \neq 0$. Consequently, if $M \cup M'$ is a subset of an n-dimensional space X which is contractible in X, then we have $M = M'$. Hence if M is a membrane of $\gamma \in H_{n-1}(S)$ spanned on S, $h_t\colon S \to X$, $t \in I$, is a homotopy such that $h_0 = \mathrm{id}$, M' is a membrane of $h_{1*}(\gamma)$ spanned on $h_1(S)$ and $M \cup \bigcup_{t\in I} h_t(S) \cup M'$ is contractible in X, where $\dim X = n$, then $M \setminus \bigcup_{t\in I} h_t(S) = M' \setminus \bigcup_{t\in I} h_t(S)$. Thus (cf. [5]):

(2) If X is an n-dimensional ANR, the membrane M is sufficiently small, h is an autohomeomorphism of X sufficiently close to the identity (and therefore closely homotopic to it), then the membranes M and $h(M)$ coincide outside a small neighborhood of S.

Now, we shall prove two propositions needed in the proof of Theorem 1.

PROPOSITION 1. *If M_0 is a membrane spanned on a compact set S_0 such that $H_1(S_0) = Q$, then $M_0 \setminus S_0$ is connected.*

Proof. Let γ be a generator of $H_1(S_0)$. By the exactness axiom, there is a $\zeta \in H_2(M_0, S_0)$ such that $\partial_*(\zeta) = \gamma$, where $\partial_*\colon H_2(M_0, S_0) \to H_1(S_0)$ is the boundary homomorphism. Assume that $M_0 \setminus S_0 = N_1 \cup N_2$, where N_i, $i = 1, 2$ are disjoint closed subsets of $M_0 \setminus S_0$. Using a description of the group $H_2(M_0, S_0)$ by means of suitably chosen open coverings of the pair (M_0, S_0), one sees that ζ can be described as $\zeta_1 + \zeta_2$, where $\overline{N}_i$ is a carrier of ζ_i. Then $\partial_*(\zeta_i) = r_i\gamma$, where $r_i \in Q$; and therefore $r_i \neq 0$ implies that $\overline{N}_i$ is a membrane of γ spanned on S_0. Since $\partial_*(\zeta) = \gamma$ and no proper compact subset of M_0 can be a membrane spanned on S_0, we conclude that one of the sets N_1, N_2 must be empty.

PROPOSITION 2. *Let Y be a compact subset of X and let a simple closed curve $S \subset X$ be the union of two arcs $I \cup J$ with common end-points x_1, x_2 such that $I \subset Y$, $\mathring{J} \subset X \setminus Y$. Assume that there is a membrane $N \subset X$ spanned on S. Then there is a connected set $C \subset N \cap \mathrm{Bd}_X(Y)$ joining x_1 and x_2.*

Proof. Let γ be a generator of $H_1(S)$. There is a $\zeta \in H_2(N, S)$ such that $\partial_*(\zeta) = \gamma$, where $\partial_*\colon H_2(N, S) \to H_1(S)$ is the boundary homomorphism. Consider the relative Mayer–Vietoris sequence of the pairs $(N \cap Y, I)$ and $(\overline{N\setminus Y}, J)$ (cf. [18], p. 190):

$$\ldots \to H_2(N \cap Y, I) \oplus H_2(\overline{N\setminus Y}, J) \to H_2\big((N \cap Y) \cup \overline{N\setminus Y}, I \cup J\big)$$
$$= H_2(N, S) \to H_1(\overline{N\setminus Y} \cap Y, I \cap J) = H_1(\overline{N\setminus Y} \cap Y, \{x_1\} \cup \{x_2\}) \to \ldots$$

The boundary homomorphism of this sequence maps ζ to a non-zero element of $H_1(\overline{N\setminus Y} \cap Y, \{x_1\} \cup \{x_2\})$. Consequently, there is a connected set $C \subset \overline{N\setminus Y} \cap Y \subset N \cap \mathrm{Bd}_X(Y)$ joining x_1 and x_2.

In the next part of this section, we shall prepare the proof of Theorem 2 and of its two corollaries which will be utilized in Section 5. We shall assume that X is a (compact) 2-dimensional ANR and $\varepsilon_0 > 0$ is a given number such that any subset A of X with $\operatorname{diam} A < 2\varepsilon_0$ is contractible in X. A membrane $M \subset X$ will be called *sufficiently small* if $\operatorname{diam} M < \varepsilon_0$ for a fixed ε_0. This implies — as above — that such a membrane is unique. If we need a generalization to locally compact ANR's, then this definition should be changed to assume that there are two compact subsets F_1, F_2 of X such that $M \subset \mathrm{Int}(F_1) \subset F_1 \subset F_2$ and any subset A of F_1 with $\operatorname{diam} A < 2\varepsilon_0$ is contractible in F_2. Moreover, in the proofs, instead of considering the regions (i.e. open and connected sets) in X, only the regions contained in $\mathrm{Int}(F_1)$ should be considered.

Given a sufficiently small membrane of an element γ of the first homology group, we describe a method of constructing an approximation of γ in a given open covering of the membrane, and we prove a lemma concerning this construction.

Notice that essentially the same proof can be used to obtain the n-dimensional version of Theorem 2, where $\dim X = n$ and $\gamma \in H_{n-1}(S)$. However, in order not to complicate the description, we limit ourselves to the case when $\dim X = 2$.

Thus assume that (Y, Y_0) is a compact pair in X, $\operatorname{diam} Y < \varepsilon_0$, $0 \neq \gamma \in H_1(Y_0)$, $\zeta \in H_2(Y, Y_0)$ and $\partial_*(\zeta) = \gamma$. Let F be a compact neighborhood of Y in X such that $\operatorname{diam} F < \varepsilon_0$. Consider a covering $\mathcal{G}$ of Y by regions in X contained in F such that $\operatorname{ord}\mathcal{G} \leqslant 2$ and so fine that, for any $G \in \mathcal{G}$, if S is a continuous image of S^1 contained in the star of G in $\mathcal{G}$, then there is a membrane M spanned on S and contained in F. Let $N(N_0)$ denote the subcomplex of the nerve of $\mathcal{G}$ consisting of those simplexes whose carriers intersect $Y(Y_0)$. Denote by ζ', γ' the images of ζ and γ under the natural projections $H_2(Y, Y_0) \to H_2(N, N_0)$, $H_1(Y_0) \to H_1(N_0)$ respectively. Then $\partial_* \zeta' = \gamma'$.

Now, we shall construct some compact subsets of X corresponding to simplexes of N in the following way: First, for each 0-simplex of N, choose a point belonging to its carrier in the covering $\mathcal{G}$. Next, for each 1-simplex $[vw]$ of N, choose an arc joining the points corresponding to the vertices v and w, and lying in the union of their carriers in $\mathcal{G}$. Finally, for each 2-simplex Δ of N, choose a membrane spanned on the curve corresponding to the boundary of Δ under this construction, and lying in F. Since $\operatorname{diam} F < \varepsilon_0$, this membrane is unique.

Further, choose cycles representing ζ' and γ', denoted by the same letters and such that $\partial(\zeta') = \gamma'$. Replace the simplexes occurring in them with non-zero coefficients by the corresponding subsets of X. Thus we obtain two compact subsets of X: Z, corresponding to ζ' (and therefore also to ζ), and Z_0, corresponding to γ' (and therefore also to γ). The construction also determines an element δ of $H_1(Z_0)$ corresponding to γ'. One sees from the construction that:

(3) There are a 1-subpolyhedron P of N_0, $\gamma'' \in H_1(P)$ and a map f of P onto Z_0 which is an embedding on each 1-simplex of P and satisfies $\delta = f_*(\gamma'')$.

This element δ will be called an *approximation* of γ in the covering $\mathcal{G}$ (by means of γ'), constructed in F.

The following observation explains the sense of this definition. Let V be any compact neighborhood of Y_0 in X contained in F. If the covering $\mathcal{G}$ is sufficiently fine, then the elements of $\mathcal{G}$ intersecting Y_0 lie in V, and therefore $Z_0 \subset V$. Let $\varphi\colon Y_0 \to V$, $\psi\colon Z_0 \to V$ denote the inclusions. Notice that:

(4) If the covering $\mathcal{G}$ is sufficiently fine, then $\varphi_*(\gamma) = \psi_*(\delta)$.

Indeed, find a sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of coverings of Y (of order at most 2) by regions in X such that $\mathcal{G}_{i+1}$ is a refinement of $\mathcal{G}_i$ and $\lim_{i\to\infty} \operatorname{diam}\mathcal{G}_i = 0$. Using these coverings, find a description of the group $H_1(Y_0)$ as an inverse limit and let $\gamma = \{\gamma_i\}$. As above, find an approximation γ_i' of γ in the covering $\mathcal{G}_i$ by means of γ_i, constructed in F. Now, it remains to observe that, if i_0 is sufficiently great, then the natural homomorphisms transform γ_i' and γ_{i_0}' to the same element of $H_1(V)$ for each $i \geqslant i_0$.

To see this, consider the 1-simplexes occurring in γ_i and their images under the projection $\pi_i\colon N_i \to N_{i_0}$, where $N_i(N_{i_0})$ is the nerve of $\mathcal{G}_i(\mathcal{G}_{i_0})$. Passing to the approximations, consider the corresponding arcs and join their respective end-points by arcs which lie in the elements of $\mathcal{G}_{i_0}$. Thus we obtain several curves for which — if i_0 is sufficiently great — there are membranes spanned on them and contained in V. Consequently, we can construct membranes of $\gamma_i' - \gamma_{i_0}'$ contained in V, for all $i \geqslant i_0$, which proves (4).

Assume now that Y is a membrane of γ spanned on Y_0 and let δ be an approximation of γ in the covering $\mathcal{G}$, constructed in F. If $\partial_*(\zeta) = \gamma$ and Z, Z_0 are the subsets of F corresponding to ζ, γ resp. constructed as above, then one can prove that $g_*(\delta) = 0$, where $g\colon Z_0 \to Z$ is the inclusion map. This implies that Z contains a membrane M of δ spanned on Z_0. Assume that a neighborhood V of Y_0 satisfies the condition given in (4). Then $M \cup V$ contains a membrane of γ spanned on Y_0. Since $Y, Z, V \subset F$ and $\operatorname{diam} F < \varepsilon_0$, we infer from the definition of ε_0 that:

(5) $Y \setminus V = M \setminus V$ and therefore $M \subset Y \cup V$.

Suppose now that M_0 is any membrane of $\alpha \in H_1(S_0)$ spanned on S_0. Let U be an open subset of $M_0 \setminus S_0$ with $\operatorname{diam} U < \varepsilon_0$. By (1), there is a $\gamma \in H_1(\overline{U} \setminus U)$ such that $\overline{U}$ is a membrane of γ spanned on $\overline{U} \setminus U$. Substituting in the above consideration $Y = \overline{U}$, $Y_0 = \overline{U} \setminus U$, and applying (3), (4) and (5), we conclude that the following lemma is true:

LEMMA. *Let X be a 2-dimensional* ANR, *S_0 a compact subset of X and M_0 a membrane of $\alpha \in H_1(S_0)$ spanned on S_0. Then, for any sufficiently small open subset U of $M_0 \setminus S_0$ and for any compact neighborhood V of $\overline{U} \setminus U$ in X, there are a 1-polyhedron P, a $\beta \in H_1(P)$ and a map $f\colon P \to V$ which is an imbedding on each 1-simplex of P, such that there is a membrane M of $\delta = f_*(\beta)$, spanned on $f(P)$ and satisfying $M \subset \overline{U} \cup V$, $M \setminus V = \overline{U} \setminus V$.*

THEOREM 2. *Assume that X is a 2-dimensional* ANR, *S is a compact subset of X, $\alpha \in H_1(S)$ and M is a membrane of α spanned on S. Then $M \setminus S$ is locally connected.*

Proof. Assume that $M \setminus S$ is not locally connected. Thus, there are an open subset U of $M \setminus S$ with $\overline{U} \subset M \setminus S$ and $\operatorname{diam} U < \varepsilon_0$ (where ε_0 is defined as above) and an $x_0 \in U$ such that, if C_0 denotes the component of $\overline{U}$ containing x_0, then $x_0 \in \operatorname{Bd}_U(C_0)$. Find a compact neighborhood F of $\overline{U}$ in X with $\operatorname{diam} F < \varepsilon_0$. Find also a sequence $\mathcal{V}_i = \{V_{i1}, \ldots, V_{ik_i}\}$, $i = 1, 2, \ldots$, of finite coverings of $\overline{U}$ by open and disjoint subsets of X such that each $V_{ij} \cap \overline{U}$ is both open and closed subset of $\overline{U}$, $\bigcup \mathcal{V}_1 \subset F$, $\mathcal{V}_{i+1}$ is a refinement of $\mathcal{V}_i$ and for each component C of $\overline{U}$ there is a sequence $V_{1j_1}, V_{2j_2}, \ldots$ with $C = \bigcap_{s=1}^{\infty} V_{sj_s} \cap \overline{U}$.

To obtain a convenient description of the groups $H_1(\overline{U} \setminus U)$ and $H_2(\overline{U}, \overline{U} \setminus U)$, find a sequence $\mathcal{G}_i$, $i = 1, 2, \ldots$, of finite coverings (of order at most 2) of $\overline{U}$ by regions in X such that $\bigcup \mathcal{G}_1 \subset F$, $\mathcal{G}_{i+1}$ is a refinement of $\mathcal{G}_i$ and $\lim \operatorname{diam}\mathcal{G}_i = 0$.

Moreover, we shall assume that, for any $G \in \mathcal{G}_i$ intersecting a set V_{is}, the star of G in $\mathcal{G}_i$ is contained in V_{is}, and even, if C is a continuous image of S^1 contained in this star, then there is a membrane spanned on C and contained in V_{is}.

Now, represent the groups $H_2(\overline{U}, \overline{U} \setminus U)$ and $H_1(\overline{U} \setminus U)$ as inverse limits:

$$(*) \qquad H_2(\overline{U}, \overline{U} \setminus U) = \varprojlim \{H_2(N_i, N_i'), \pi_i\},$$
$$H_1(\overline{U} \setminus U) = \varprojlim \{H_1(N_i'), \pi_i'\},$$

where N_i and N_i' are the respective subcomplexes of the nerve of $\mathcal{G}_i$, and π_i: $H_2(N_{i+1}, N_{i+1}') \to H_2(N_i, N_i')$, π_i': $H_1(N_{i+1}') \to H_1(N_i')$ are the respective homomorphisms. Observe that the groups $H_2(N_i, N_i')$ and $H_1(N_i')$ can be represented as direct sums

$$H_2(N_i, N_i') = H_2(N_{i1}, N_{i1}') \oplus \ldots \oplus H_2(N_{ik_i}, N_{ik_i}'),$$
$$H_1(N_i') = H_1(N_{i1}') \oplus \ldots \oplus H_1(N_{ik_i}'),$$

where N_{ij} (N_{ij}') denotes the subcomplex of N_i (N_i') determined by those elements of $\mathcal{G}_i$ which lie in V_{ij}.

Since M is a membrane of $\alpha \in H_1(S)$ spanned on S, there is a $\gamma \in H_1(\overline{U} \setminus U)$ such that $\overline{U}$ is a membrane of γ spanned on $\overline{U} \setminus U$. Consequently, there is a $\zeta \in H_2(\overline{U}, \overline{U} \setminus U)$ such that $\partial_*(\zeta) = \gamma$. Let $\zeta = \{\zeta_i\}$, $\gamma = \{\gamma_i\}$ and $\zeta_i = \zeta_{i1} \oplus \ldots \oplus \zeta_{ik_i}$, $\gamma_i = \gamma_{i1} \oplus \ldots \oplus \gamma_{ik_i}$ under the above representations. We can assume that $\partial_*(\zeta_{ij}) = \gamma_{ij}$ and that there is a $k_i' \leqslant k_i$ such that ζ_{ij}, γ_{ij} for $j \leqslant k_i'$ are all non-zero summands in these representations.

Furthermore, by the method described above, we find an approximation δ_i of γ in the covering $\mathcal{G}_i$ by means of γ_i, constructed in F. Moreover, we construct two compact subsets of X: D_i, corresponding to γ_i, and Z_i, corresponding to ζ_i, such that $\delta_i \in H_1(D_i)$, $\varphi_{i*}(\delta_i) = 0$, where φ_i: $D_i \to Z_i$ is the inclusion. One sees from the assumptions concerning the coverings $\mathcal{G}_i$, from the construction of the approximation, and because the membranes are unique, that $D_i = \bigcup \{D_{ij}\colon j \leqslant k_i'\}$, $Z_i = \bigcup Z_{ij}$, where $D_{ij} \subset Z_{ij} \subset V_{ij}$; therefore $\delta_i = \delta_{i1} \oplus \ldots \oplus \delta_{ik_i'}$, where $\delta_{ij} \in H_1(D_{ij})$. Moreover, $\varphi_{ij*}(\delta_{ij}) = 0$, where φ_{ij}: $D_{ij} \to Z_{ij}$ is the inclusion map. Consequently, Z_{ij} contains a membrane M_{ij} of δ_{ij} spanned on D_{ij}. Since $M_{ij} \subset Z_{ij} \subset V_{ij}$ and $V_{ij} \cap V_{ij'} = \emptyset$ for $j \neq j'$, it follows that $M_{ij} \cap M_{ij'} = \emptyset$ for $j \neq j'$.

Recall now that $x_0 \in U \subset \overline{U} \subset \mathrm{Int}_X(F) \subset F$ and find an open neighborhood W of x_0 in X such that $F \setminus W$ is a neighborhood of $\overline{U} \setminus U$ in X. Observe that for each $m \geqslant 1$ there is an $n \geqslant 1$ such that at least m of the membranes $M_{n1}, \ldots, M_{nk_n'}$ intersect W. Indeed, $M_n = M_{n1} \cup \ldots \cup M_{nk_n'}$ is a membrane of δ_n spanned on D_n, and therefore, by (4), if n is sufficiently great, then $\varphi_*(\gamma) = \psi_*(\delta_n)$, where $\varphi\colon \overline{U} \setminus U \to F \setminus W$, ψ: $D_n \to F \setminus W$ are the inclusion maps. Since $\overline{U} \cup M_n \subset F$ and $\operatorname{diam} F < \varepsilon_0$, we infer from (5) that $M_n \cap W = \overline{U} \cap W$. Since $M_{nj} \subset V_{nj}$, we conclude from the definition of the $\mathcal{V}_i$'s that, actually, for n sufficiently great at least m of the M_{nj}, $j \leqslant k_n'$, intersect W (since at least m of the $V_{nj} \cap \overline{U}$ intersect W).

Next, denote by $\mathcal{G}_i'$ the subfamily of $\mathcal{G}_i$ consisting of those $G \in \mathcal{G}_i$ which intersect $\overline{U} \setminus U$. Find an index i_0 sufficiently great so that, for any curve C which is a continuous image of S^1 and is contained in the star of a $G \in \mathcal{G}_{i_0}'$ in $\mathcal{G}_{i_0}'$, there is a membrane spanned on C and contained in $F \setminus W$. Denote by l the number of generators of the group $H_1(N_{i_0}')$ (determined by polyhedral simple closed curves in N_{i_0}') and find an index $n > i_0$ such that at least $m = 2^{2l}$ of the membranes $M_{n1}, \ldots, M_{nk_n'}$ intersect W, say $M_{n1}, \ldots, M_{nm}$. Consider the homomorphism

$$\pi'\colon H_1(N_{n1}') \oplus \ldots \oplus H_1(N_{nk_n}') = H_1(N_n') \to H_1(N_{i_0}')$$

given by $(*)$ and let $\pi'(\gamma_{nj}) = \gamma_{nj}'$ for $j \leqslant k_n'$. Thus one sees that either there is a $j \leqslant m$ such that $\gamma_{nj}' = 0 \in H_1(N_{i_0}')$, or at least $2^l = \sqrt{m}$ of the γ_{nj}' with $j \leqslant m$, say $\gamma_{n1}', \ldots, \gamma_{n\sqrt{m}}'$, contain the same generators of $H_1(N_{i_0}')$ with non-zero coefficients. To complete the proof of the theorem, we shall show that both these cases are impossible.

Assume first that $\gamma_{nj}' = 0 \in H_1(N_{i_0}')$ and choose a cycle representing this element (denoted by the same letter). Find a chain $c' \in C_2(N_{i_0}')$ such that $\partial c' = \gamma_{nj}'$. Taking into consideration the properties of $\mathcal{G}_{i_0}'$ and using the method described before this theorem, one can construct a compact subset D' of $\bigcup \mathcal{G}_{i_0}'$, a $\delta_{nj}' \in H_1(D')$ corresponding to γ_{nj}', and a compact set C' corresponding to c' such that $C' \subset F \setminus W$ and $\varrho_*(\delta_{nj}') = 0$, where ϱ: $D' \to C'$ is the inclusion map. Moreover, one sees as in the proof of (4) that $\sigma_*(\delta_{nj}) = \sigma_*'(\delta_{nj}')$, where σ: $D_{nj} \to F \setminus W$, σ': $D' \to F \setminus W$ are the inclusions. Consequently, $\sigma_*(\delta_{nj}) = 0$, where σ: $D_{nj} \to F \setminus W$ is the inclusion. Therefore there is a membrane M_{nj}' of δ_{nj} spanned on D_{nj} and contained in $F \setminus W$. However, this is a contradiction, because M_{nj} is also a membrane of δ_{nj} spanned on D_{nj} and contained in F, but intersecting W.

Finally, assume that the elements $\gamma_{n1}', \ldots, \gamma_{n\sqrt{m}}'$ of $H_1(N_{i_0}')$ contain the same generators, say $\beta_1, \ldots, \beta_k$ with $k \leqslant l$, of $H_1(N_{i_0}')$ with non-zero coefficients. Since $\sqrt{m} = 2^l \geqslant 2^k$, it is easy to prove by induction with respect to k that there are non-zero integers $r_1, \ldots, r_{\sqrt{m}}$ such that $r_1\gamma_{n1}' + \ldots + r_{\sqrt{m}}\gamma_{n\sqrt{m}}' = 0 \in H_1(N_{i_0}')$. Since M_{nj}, for $j \leqslant m$, are the membranes of δ_{nj} spanned on D_{nj}, which are disjoint, contained in F and intersect W, the set $\bigcup \{M_{nj}\colon j \leqslant \sqrt{m}\}$ is a membrane of

$$\delta = r_1\delta_{n1} + \ldots + r_{\sqrt{m}}\delta_{n\sqrt{m}}$$

spanned on $D = \bigcup \{D_{nj}\colon j \leqslant \sqrt{m}\}$, which intersects W and is contained in F. By the method used in the preceding case, one can construct another membrane M' of δ spanned on D and contained in $F \setminus W$. This is a contradiction, which concludes the proof of the theorem.

By a theorem of Mazurkiewicz–Moore–Menger, from Proposition 1 and Theorem 2 we obtain:

COROLLARY 1. *Under the assumptions of Theorem 2, the set $M \setminus S$ is locally arcwise connected. Moreover, if M is a membrane spanned on a compact set S such that $H_1(S) = Q$, then $M \setminus S$ is arcwise connected.*

COROLLARY 2. *In the lemma given before the theorem, we can require the map f to have values belonging to $V \cap M_0$.*

Indeed, we can assume in the proof of the lemma that $\overline{U} \subset M_0 \setminus S_0$, and we can choose the covering $\mathcal{G}$ of $Y = \overline{U}$ so that the sets $G \cap M_0 = G \cap M_0 \setminus S_0$ for $G \in \mathcal{G}$ are connected. Consequently, by Corollary 1, when constructing the approximation of γ in the covering $\mathcal{G}$, we can construct the arcs replacing the respective 1-simplexes of the nerve of $\mathcal{G}$ in such a way that they lie in M_0.

3. Topological transformation groups. The following well-known theorem on homogeneous spaces has been proved by Effros and Hagopian (cf. [6] and [9]):

THEOREM OF EFFROS AND HAGOPIAN. *Let X be a metric separable topologically complete space and (G, d) a metric separable topologically complete group acting continuously and transitively (from the left) on X. Then for every $\varepsilon > 0$ there is an open covering $\{U_t\}_{t \in T}$ of X such that for any $t \in T$ and $x, y \in U_t$ there is a $g \in G$ with $d(g, e) < \varepsilon$ and $gx = y$.*

We shall need the following more general version of this theorem, which is its adaptation to the case when X is not necessarily topologically complete (but rather σ-compact).

THEOREM 3. *Let X be a metric space and (G, d) a metric separable topologically complete group acting continuously and transitively (from the left) on X. Moreover, let F be a topologically complete subset of X such that, for any $x \in F$, there are a neighborhood U_0 of x in F and a neighborhood M_0 of the identity e in G with $M_0 U_0 \subset F$. Then there are an $x_0 \in F$ and an open neighborhood M of e in G such that $C_0 = Mx_0$ is a dense G_δ subset of an open neighborhood W_0 of x_0 in F (where $M = G$ if $F = X$), satisfying the condition:*

(*) *For any $\varepsilon > 0$, there is an open covering $\{U_t\}_{t \in T}$ of C_0 such that, if $y_1, y_2 \in U_t$, then there exists an $f \in G$ with $d(f, e) < \varepsilon$ and $fy_1 = y_2$.*

Moreover, given any dense sequence $g_0 = e, g_1, g_2, \ldots$ in G, there is a countable covering $\{C_i\}_{i=0}^{\infty}$ of X by topologically complete sets such that each C_i is a dense G_δ set in an open subset W_i of $F_i = g_i F$, satisfying the condition (*) with C_0 replaced by C_i.

Proof. First, let us show that:

(1) There is a point $x_0 \in F$ such that, for any neighborhood M of the identity e in G, the set Mx_0 contains a dense G_δ set in a certain open subset of F containing x_0.

First, consider any open neighborhood P of e in G and let $\{g_i\}_{i=0}^{\infty}$ be a given dense subset of G. Then $G = \bigcup_{i=0}^{\infty} Pg_i$. Indeed, for each $g \in G$ there is an open set Q in G such that $g \in Q$, $QQ^{-1} \subset P$. If $g_i \in Q$, then $g \in Pg_i$.

Choose any $x \in F$. Since the orbit Gx of x in X is equal to X, it follows that $X = Gx = \bigcup_{i=0}^{\infty} Pg_i x$. Thus the sets $F_i = (Pg_i x) \cap F$, $i = 0, 1, \ldots$, cover F. Consider a fixed i and let $y = g_i x$. Let $A = \{g \in G: gy = y\}$ denote the stabilizer of y in G. The set $Py = Pg_i x$ is analytic in X. Consequently, F_i is analytic in F, and therefore it has the Baire property in F (cf. ibidem, p. 56); i.e. there are an open subset V_i of F and first category sets K_i, L_i in F such that $F_i = (V_i \setminus K_i) \cup L_i$. Since the set $F = \bigcup_{i=0}^{\infty} F_i = \bigcup_{i=0}^{\infty} (V_i \setminus K_i \cup L_i)$ is topologically complete, there is a dense G_δ subset B_1 of F such that $B_1 \subset \bigcup \{V_i \setminus K_i : V_i \neq \emptyset\}$. Notice that, if $y_1, y_2 \in V_i \setminus K_i \subset F_i \subset Pg_i x$, then there is an $h \in PP^{-1}$ such that $hy_1 = y_2$. Indeed, there are $h_1, h_2 \in P$ such that $h_1 g_i x = y_1$, $h_2 g_i x = y_2$, whence $hy_1 = y_2$, where $h = h_2 h_1^{-1}$.

Now, let $P_1, P_2, \ldots$ be a base of open neighborhoods of e in G. By the above reasoning applied to P_j instead of P, one can find a dense G_δ subset B_j of F and a countable covering $\{V_{ji} \setminus K_{ji}\}_{i=0}^{\infty}$ of B_j, where $V_{ji} \neq \emptyset$ is an open subset of F and K_{ji} is a first category subset of F, such that, for any $y_1, y_2 \in V_{ji} \setminus K_{ji}$, there is an $h \in P_j P_j^{-1}$ with $hy_1 = y_2$.

Let $x_0 \in \bigcap_{j=1}^{\infty} B_j$. Then x_0 satisfies the required condition (1). Indeed, if M is a neighborhood of e in G, then there are a j such that $P_j P_j^{-1} \subset M$ and an i such that $x_0 \in V_{ji} \setminus K_{ji}$. Evidently, $V_{ji} \setminus K_{ji}$ contains a dense G_δ subset of V_{ji} and $V_{ji} \setminus K_{ji} \subset Mx_0$.

Let x_0 satisfy condition (1). We shall now prove that:

(2) There is an open neighborhood M of e in G with $Mx_0 \subset F$ such that, for each $g \in M$, there is an open neighborhood V of x_0 in F such that gV is an open neighborhood of gx_0 in F.

Indeed, by the assumption, there are an open neighborhood M_0 of e in G and an open neighborhood U_0 of x_0 in F such that $M_0 U_0 \subset F$. One can find an open neighborhood M of e in G such that $MM^{-1} \subset M_0$ and $Mx_0 \subset U_0$. Consider now a fixed $g \in M$ and let U_0' be an open subset of X such that $U_0' \cap F = U_0$. Since $x = gx_0 \in Mx_0 \subset U_0 = U_0' \cap F$, it follows that there is an open neighborhood V' of x_0 in X such that $V' \subset U_0'$ and $gV' \subset U_0'$. Let $V = V' \cap F$, $W' = gV'$ and $W = W' \cap F$. To complete the proof of (2), it suffices to show that $gV = W$. Indeed, $gV \subset gV'$ and, since $g \in M \subset M_0$, it follows that $gV = g(V' \cap F) \subset g(U_0' \cap F) = gU_0 \subset F$, whence $gV \subset gV' \cap F = W$. On the other hand, $W \subset W' = gV'$ and, since $g^{-1} \in M^{-1} \subset M_0$ and $W = gV' \cap F \subset U_0' \cap F = U_0$, it follows that $g^{-1} W \subset F$, whence $W \subset gF$. Consequently, $W \subset gV' \cap gF = g(V' \cap F) = gV$, which completes the proof that $gV = W$.

Further, let M be a neighborhood of e in G satisfying (2). Evidently, if $F = X$, then we can take $M = G$. Suppose that $x \in X$, $g \in G$ and $gx \in Mx_0$. Let N be an open neighborhood of g in G. We show that:

(3) There are an open neighborhood U of $y = gx$ in F and a dense G_δ subset B of U such that $B \subset Nx$.

Since $y \in Mx_0$, there is an $h \in M$ such that $y = hx_0$. Since $heh^{-1}g = g \in N$, it follows that there is a neighborhood P of e in G such that $hPh^{-1}g \subset N$. In virtue of (1), there are an open neighborhood W of x_0 in F and a dense G_δ subset B_0 of W such that $B_0 \subset Px_0$. Since $h \in M$, using (2), we infer that there is an open neighborhood U of y in F such that the set $B = hB_0 \cap U$ is a dense G_δ subset of U. Then $B \subset hB_0 \subset hPx_0 = hPh^{-1}y = hPh^{-1}gx \subset Nx$, and therefore B is the required subset of U.

Next, let us prove that:

(4) For any $x \in Mx_0$ and for any neighborhood P of e in G, there is a neighborhood U' of x in Mx_0 such that $U' \subset Px$.

First, find an open neighborhood Q of e in G such that $Q^{-1}Q \subset P \cap M$. Applying (3) to the given point x, $g = e$ and this neighborhood Q of e, one can find an open neighborhood U of x in F and a dense G_δ subset B of U such that $B \subset Qx$. Let $U' = U \cap Mx_0$. We shall show that $Q^{-1}U' \subset Q^{-1}Qx$. Then $U' = eU' \subset Q^{-1}U' \subset Px$, and therefore U' is the required neighborhood of x.

Thus assume that there exists a $y \in Q^{-1}U'$ with $y \notin Q^{-1}Qx$. Find $g \in G$ such that $y = gx$. Since $y = gx \notin Q^{-1}Qx$, it follows that $Qgx \cap Qx = \emptyset$. Thus $Qgx \cap U' = Qgx \cap Mx_0 \cap U \subset U \setminus Qx \subset U \setminus B$. To obtain a contradiction, we shall prove that $Qgx \cap U'$ contains a dense G_δ set in an open subset of F.

For this purpose, let $\tau\colon G \to X$ denote the map given by the formula $\tau(h) = hx$ and let U'' be an open subset of X such that $U'' \cap F = U$. Then $\tau^{-1}(U'')$ is an open subset of G, $\tau^{-1}(U'')x = U''$ and, since $Mx_0 \subset F$, we have $Qgx \cap U' = Qgx \cap \cap U \cap Mx_0 = Qgx \cap U'' \cap Mx_0 = Qgx \cap \tau^{-1}(U'')x \cap Mx_0 \supset (Qg \cap \tau^{-1}(U''))x \cap \cap Mx_0$. The set $Qg \cap \tau^{-1}(U'')$ is an open subset of G, which is non-empty, because $y = gx \in Q^{-1}U'$, and therefore there is an $h \in Q$ such that $hy = hgx \in U' = U \cap Mx_0 = U'' \cap Mx_0$, whence $hg \in Qg \cap \tau^{-1}(U'')$. Since $hgx \in Mx_0$, applying (3) we infer that there are an open neighborhood V_1 of hgx in F and a dense G_δ subset B_1 of V_1 such that $B_1 \subset (Qg \cap \tau^{-1}(U''))x$. Substituting, in (3), x_0 instead of x, M instead of N and taking into consideration that there is an $f \in M$ with $fx_0 = hy = hgx \in Mx_0$, we infer that there are an open neighborhood V_2 of hy in F and a dense G_δ subset B_2 of V_2 such that $B_2 \subset Mx_0$. Consequently, $B_1 \cap B_2$ contains a dense G_δ subset of an open non-empty subset of F, contained in $(Qg \cap \tau^{-1}(U''))x \cap Mx_0 \subset Qgx \cap U'$. This is the desired contradiction, which completes the proof of (4).

To complete the proof of the first assertion of the theorem, consider the map $\sigma\colon M \to Mx_0$ given by the formula $\sigma(g) = gx_0$. Condition (4) implies that the map σ is open. Indeed, let N be an open subset of M and let $x = gx_0 \in \sigma(N) = Nx_0 \subset Mx_0$, where $g \in N$. There is an open neighborhood P of e in G such that $Pg \subset N$. By (4), there is an open neighborhood U' of x in Mx_0 such that $U' \subset Px = Pgx_0 \subset Nx_0$, which proves that $Nx_0 = \sigma(N)$ is open in Mx_0.

Since M, being an open subset of G, is topologically complete, so is Mx_0 (cf. [8], p. 442). Consequently, since each topologically complete set is an absolute G_δ set, it follows from (3) that there is an open subset W_0 of F such that Mx_0 is a dense G_δ subset of W_0. Moreover, given any $\varepsilon > 0$, we can find a neighborhood P of e in G such that $PP^{-1} \subset \{f \in G\colon d(f, e) < \varepsilon\}$. By (4), for any $x \in C_0 = Mx_0$, there is an open neighborhood U' of x in Mx_0 such that $U' \subset Px$. Consequently, for any $y_1, y_2 \in U'$, there are $f_1, f_2 \in P$ with $f_jx = y_j$ for $j = 1, 2$; this implies that $fy_1 = y_2$ and $d(f, e) < \varepsilon$, where $f = f_2f_1^{-1}$.

To prove the second assertion, consider a given dense sequence $\{g_i\}_{i=0}^{\infty}$ in G. As at the beginning of the proof, the sequence $\{g_iM\}_{i=0}^{\infty}$ is a covering of G. Thus the sequence $\{C_i\}_{i=0}^{\infty}$, where $C_i = g_iMx_0$, is a covering of X. The C_i's, being homeomorphic images of Mx_0, are topologically complete and, moreover, for each i there is an open subset W_i of $F_i = g_iF$ such that C_i is a dense G_δ subset of W_i. It follows from (4) that, for each $y \in C_i$ and for each neighborhood Q of e in G, there is a neighborhood V' of y in C_i such that $V' \subset Qy$ (it suffices to find $U' \subset Px$, where $x = g^{-1}y \in Mx_0$, $P \subset g_i^{-1}Qg_i$, and take $V' = g_iU'$). As before, this implies that C_i satisfies the condition (*) (with C_0 replaced by C_i), which completes the proof.

4. Proof of Theorem 1 in the case when X has a locally separating points. First, we shall consider the case when X has a separating point. We shall prove Lemma 1 below, using the theory of cyclic elements for Peano continua, as described in [13] and [15]. Recall that a non-degenerate cyclic element of X is a maximal subset of X which is disconnected by no point. If, in this lemma, X is locally compact, instead of being compact, then any point of X has a neighborhood in X which is a Peano continuum. Considering the cyclic elements of such neighborhoods and reasoning as in our proof of the lemma, one can show that the set of separation points of X is locally finite.

LEMMA 1. *Let X be a $\frac{1}{2}$-homogeneous Peano continuum of dimension greater than 1 which has a separation point. Then X has exactly one separation point.*

Proof. Let A consist of the points $x \in X$ which separate X, and let $B = X \setminus A$. Since $\dim X \geqslant 2$, it follows that X contains at least one non-degenerate cyclic element Z and $Z \cap A$ is an at most countable set of points which bound components of $X \setminus Z$. Then $B \cap Z \neq \emptyset$, and since $Z \neq X$ and Z is closed in X, $A \cap Z \neq \emptyset$. By the $\frac{1}{2}$-homogeneity of X, this implies that each $x \in X$ belongs to a non-degenerate cyclic element of X.

Assume that A contains more than one point. Then there is a non-degenerate cyclic element Z_0 of X containing at least two points of A. Since, for any non-degenerate cyclic element Z of X, no $x \in Z \cap B$ belongs to a cyclic element of X different from Z, it follows from the $\frac{1}{2}$-homogeneity of X that $Z \cap A$ must also contain at least two points. Order all the non-degenerate cyclic elements of X into a sequence $Z_1, Z_2, \ldots$ Observe that, for each i, each component C of $X \setminus Z_i$ must contain infinitely many Z_j's, because otherwise one can find a $Z_k \subset \bar{C}$ such that $Z_k \cap A$ consists of exactly one point.

Now, let us construct a subsequence $Z_{i_1}, Z_{i_2}, \ldots$ of the sequence $\{Z_i\}_{i=1}^{\infty}$ and a descending sequence $C_1, C_2, \ldots$ of compact subsets of X such that $\overline{X \setminus C_j}$ is connected. Let $Z_{i_1} = Z_1$ and let C_1 be the closure of any component of $X \setminus Z_1$. If Z_{i_j} and C_j are constructed, let i_{j+1} denote the first index greater than i_j such that $Z_{i_{j+1}} \subset C_j$ and let C_{j+1} denote the closure of any component of $X \setminus Z_{i_{j+1}}$ contained in C_j. Such a component exists, because the set $\overline{X \setminus C_j}$ is connected, and therefore it intersects $Z_{i_{j+1}}$ it at most one point of the (at most) countable set $Z_{i_{j+1}} \cap A$. Evidently, $\overline{X \setminus C_{j+1}}$ is connected and one sees from the construction that, for each j, $Z_{i_j} \cap C_{j+1} = \emptyset$. Consequently, if x_0 is a common point of the sets C_j, $j = 1, 2, \ldots$, then x_0 cannot belong to any non-degenerate cyclic element of X. This is a contradiction, which completes the proof of the lemma.

COROLLARY. *Each connected $\frac{1}{2}$-homogeneous 2-dimensional* ANR-*space which has a separation point is a polyhedron. Moreover, we can obtain a full classification of all such spaces.*

Proof. Define A and B as in the preceding lemma and let $A = \{x_0\}$. Let C be a component of $B = X \setminus \{x_0\}$. Then C is a homogeneous, locally compact and locally contractible, 2-dimensional (metric separable) space, which implies, by the Bing–Borsuk theorem [5], that C is an open 2-manifold. We shall show that $\bar{C} = C \cup \{x_0\}$ is a polyhedron (cf. [10]) and that, in fact, it is a pseudomanifold. Indeed, $\bar{C}$, being a retract of X, is an ANR. It can have at most one locally separating point, namely x_0. Since x_0 does not separate $\bar{C}$, there is a region U in $\bar{C}$ such that $x_0 \in U$, $\bar{C} \setminus U \neq \emptyset$ and for any region V in $\bar{C}$, where $x_0 \in V \subset U$, the set $V \setminus \{x_0\}$ has the same number $n \geqslant 1$ of components (cf. [16], p. 276). Moreover, we can assume that the region U is so small that each simple closed curve $S \subset U$ is contractible in $\bar{C}$. It follows that x_0 has an open neighborhood homeomorphic with the plane E^2 in the closure of any component of $U \setminus \{x_0\}$. Consequently, $\bar{C}$ must be a closed 2-pseudomanifold with at most one irregular point, obtained from a closed 2-manifold by identifying some n points. Evidently, if $n = 1$, then $\bar{C}$ is a manifold.

It follows from the $\frac{1}{2}$-homogeneity of X that all the non-degenerate cyclic elements of X (i.e. the closures of the components of B) must be homeomorphic pseudomanifolds, and since $X \in$ ANR, their number is finite. Thus X is a bouquet of $k > 1$ homeomorphic pseudomanifolds described above, and therefore a polyhedron. Conversely, every such bouquet is evidently a $\frac{1}{2}$-homogeneous connected 2-dimensional ANR having a separation point.

Next, we shall consider the case when X has locally separating points, but is a cyclic space, i.e. no point $x \in X$ separates X. Denote by L_X the set of these points which locally separate X. We shall use the theory of strongly cyclic elements, which has been described in [16] for the spaces $X \in \alpha$, i.e. for Peano continua satisfying the condition: there is an $\varepsilon > 0$ such that no simple closed curve $S \subset X$ with $\operatorname{diam} S < \varepsilon$ is a retract of X. Recall that a non-degenerate strongly cyclic element of X is a maximal subset of X which is disconnected by no finite set.

If X is locally compact, instead of being compact, then the definition of the class α can be adapted to this case, by the assumption that there is an open covering $\mathcal{G}$ of X such that no simple closed curve $S \subset G \in \mathcal{G}$ is a retract of X. Moreover, in the definition of strongly cyclic elements of X and in their properties proved in [16], the word "finite" must be replaced by "locally finite". Using these remarks, one can adapt our proof of Lemma 2 so as to obtain the conclusion that X is the union of its non-degenerate strongly cyclic elements, where the family of these elements, as well as the set L_X, are locally finite.

LEMMA 2. *Let $X \in \alpha$ be a $\frac{1}{2}$-homogeneous cyclic space of dimension greater than 1 such that $L_X \neq \emptyset$. Then there is a positive integer m such that $X = \bigcup_{i=1}^{m} E_i$, where E_i, for $i \leqslant m$, are all non-degenerate strongly cyclic elements of X. Moreover, the E_i's are homeomorphic and, in particular, there is a positive integer n such that for each i the set $E_i \cap L_X$ consists of exactly n points.*

Proof. Let $A = L_X$ and $B = X \setminus L_X$. Since X is a cyclic space, it follows from [16] that A is closed in X and $B \neq \emptyset$, because X cannot be a local dendron. A non-degenerate strongly cyclic element of X is the closure of a component of $B = X \setminus A$. By [16], for any such element E of X, the set $E \cap L_X$ is finite, it does not separate E and contains $\operatorname{Bd}(E) \cup L_E$. Moreover, if $E \neq X$, then $\operatorname{Bd}(E)$ contains at least two points. By the $\frac{1}{2}$-homogeneity of X, each $x \in X$ must belong to such a non-degenerate element E of X and all these elements must be homeomorphic. Consequently, there is a positive integer n such that, for each such element E of X, the set $E \cap A$ consists of exactly n points. Moreover (cf. [16], p. 276), any $x \in E$ locally disconnects E into a finite number of components, i.e. for any sufficiently small region U in E containing x, the set $U \setminus \{x\}$ has the same finite number of components.

It remains to prove that the number of non-degenerate strongly cyclic elements of X must be finite. Assume that this is not the case. By [16], the family of these elements is countable and their diameters converge to 0. Since $2 \leqslant \operatorname{card} \operatorname{Bd}(E) \leqslant \operatorname{card}(E \cap A) = n$, it follows that $\operatorname{card} A = \aleph_0$. Since A is closed in X, it contains its accumulation points. Thus, by the $\frac{1}{2}$-homogeneity of X, the compact set A is countable and dense in itself, which is a contradiction.

COROLLARY. *Each connected, cyclic, $\frac{1}{2}$-homogeneous, 2-dimensional* ANR-*space X such that $L_X \neq \emptyset$ is a polyhedron. Moreover, we can obtain a full classification of all such spaces by means of homogeneous multigraphs.*

Proof. In virtue of Lemma 2, there are positive integers m and n such that $X = \bigcup_{i=1}^{m} E_i$ and $\operatorname{card}(E_i \cap L_X) = n$. Since $E_i \cap L_X \supset \operatorname{Bd}(E_i)$, the sets E_i for $i \leqslant m$ can intersect one another only at points belonging to L_X. By the $\frac{1}{2}$-homogeneity of X, the sets $E_i \setminus L_X$ are homogeneous and, applying the Bing–Borsuk theorem, we infer as in the proof of the preceding corollary that they are open 2-manifolds. Moreover, one can prove that the E_i's are closed 2-dimensional pseudomanifolds with the finite set $L_{E_i} \subset E_i \cap L_X$ of irregular points. Thus one sees that $X = \bigcup \{E_i\colon i \leqslant m\}$ is a polyhedron.

By the $\frac{1}{2}$-homogeneity of X, there is a positive integer p such that $E_i \cap L_X = \bigcup_{j=1}^{p} E_{ij}$, where E_{ij} consists of those points $x \in E_i \cap L_X$ which locally disconnect E_i into the same number q_j of components and $1 \leqslant q_1 \leqslant \ldots \leqslant q_p$. Evidently, $\operatorname{card} E_{ij} = \operatorname{card} E_{i'j}$ for $i, i' \leqslant m$, $j \leqslant p$. If $m = 1$, then $p = 1$ and $q_1 > 1$. If $m > 1$, then $\operatorname{card} \operatorname{Bd}(E_i) = n \geqslant 2$.

Now, the structure of the polyhedron X can be described by means of the following multigraph $G = (V, E)$: The set V of vertices of G is in a one-to-one correspondence with L_X. Then $V = \bigcup_{i=1}^{m} V_i$, where V_i corresponds to $E_i \cap L_X$. The set V_i, in turn, is the union of p disjoint subsets V_{ij}, $j \leqslant p$, corresponding to the E_{ij}'s. Two different vertices $v, w \in V$ such that $v, w \in V_i$ for some i are joined by an edge $e \in E$. In this way we obtain a multigraph G which is the union of m subgraphs $G_i = (V_i, E_i)$, where any two different vertices $v, w \in V_i$ are joined by an edge $e \in E_i$. By the $\frac{1}{2}$-homogeneity of X, the multigraph G is homogeneous in the following sense: For any two vertices $v, w \in V$ there is an autoisomorphism ψ of G such that $\psi(v) = w$ and, for each $i \leqslant m$, there is an $i' \leqslant m$ with $\psi(G_i) = G_{i'}$ and $\psi(V_{ij}) = V_{i'j}$ for all $j \leqslant p$. Moreover, for any $i, i' \leqslant m$ there is an autoisomorphism ψ of G of this type such that $\psi(G_i) = G_{i'}$. Since X is connected and cyclic, the multigraph G is also connected and cyclic.

Conversely, it is easy to see that for any multigraph G of this type and for any closed 2-dimensional pseudomanifold P whose irregular points $x \in L_P$ satisfy the above conditions, corresponding to the structure of G, there is exactly one $\frac{1}{2}$-homogeneous 2-dimensional $X \in \mathrm{ANR}$ whose non-degenerate strongly cyclic elements are homeomorphic to P, whose intersections are described by G, and such that L_X corresponds to V. The conditions put on P and G imply that X satisfies the assumptions of the corollary. Thus there is a one-to-one correspondence between such ANR-spaces and the pairs (P, G), which completes the proof.

5. Proof of Theorem 1 in the case when X has no locally separating points. Let X be a connected 2-dimensional, $\frac{1}{2}$-homogeneous ANR-space such that $L_X = \emptyset$. We begin as in Bing and Borsuk's paper [5]. Since $X \in \mathrm{ANR}$, there is an $\varepsilon_0 > 0$ such that any subset Y of X with $\operatorname{diam} Y < 2\varepsilon_0$ is contractible in a proper subset of X. Moreover, there exists an $\varepsilon_1 > 0$ such that for any compact subset Z of X with $\operatorname{diam} Z < \varepsilon_1$ there is a compact subset Y of X with $\operatorname{diam} Y < \varepsilon_0$ such that the inclusion $i\colon Z \to Y$ induces the null-homomorphism of the homology groups. To prove that X is a polyhedron, we can assume that X is not a 2-manifold. It follows from Young's characterization of 2-manifolds (cf. [22]) that there is a simple closed curve $S_0 \subset X$ with $\operatorname{diam} S_0 < \varepsilon_1$ which does not separate X. Consequently, there is a membrane M_0 spanned on S_0 such that $\operatorname{diam} M_0 < \varepsilon_0$. Since S_0 does not separate X and $X \setminus M_0 \neq \emptyset$, the set $A_0 = \overline{X \setminus M_0} \cap (M_0 \setminus S_0)$ is not empty.

Denote by A the class of all points $x \in X$ such that there is an autohomeomorphism h of X with $x \in h(A_0)$, and let $B = X \setminus A$. Observe that, as in [5], the set A is of the first Baire category in X. To see this, find a sequence $h_0 = \mathrm{id}, h_1, h_2, \ldots$ of autohomeomorphisms of X which is dense in the group $H(X)$ of all autohomeomorphisms of X. Let $M_i = h_i(M_0)$, $S_i = h_i(S_0)$ and $A_i = h_i(A_0)$ for $i = 0, 1, 2, \ldots$ It is clear from the definition that each of the A_i's is of the first Baire category in X. Moreover, one sees as in [5] that $A \subset \bigcup_{i=0}^{\infty} A_i$, because, if h and h_i are sufficiently close to each other, then the membranes $h(M_0)$ and $h_i(M_0) = M_i$ must coincide outside a small neighborhood of S_i. (Indeed, it suffices to apply condition (2) of Section 2 to the membrane M_0 and the homeomorphism $h^{-1}h_i$.) Thus $A = \bigcup_{i=0}^{\infty} A_i$ is an F_σ set of the first category in X, and therefore $B = X \setminus A$ is a dense G_δ set in X. Evidently, for each autohomeomorphism $h \in H(X)$, we have $h(A) = A$ and $h(B) = B$, and therefore A and B are the two orbits of the action of $H(X)$ on X, mentioned in the Introduction.

We shall first prove that:

(1) There are an $i \geqslant 0$ and an open subset $U \neq \emptyset$ of A_i such that, for any j, $\overline{D_{ij}} \supset U \cap A_j$, where $D_{ij} = \operatorname{Int}_{A_i}(A_i \cap A_j)$.

Assume that (1) is not true. Thus, for each $i \geqslant 0$ and for any open subset U of A_i, there are a j and an open subset V of $A_i \cap A_j$ such that $V \subset U$ and V is a boundary subset of A_i. Observe that V can be assumed to be open in A_j. Indeed, since each A_i is locally compact, applying Baire's theorem to the covering $\{A_i \cap A_0\}_{i=0}^{\infty}$ of A_0, we infer that there is an $x \in A_0$ such that, for each i, if $x \in A_i$, then A_i contains a neighborhood of x in A_0. Since the image of A_0 by any autohomeomorphism of X locally coincides with one of the A_i's, it follows from the $\frac{1}{2}$-homogeneity of X that for each $y \in A$ there is a set A_k containing y such that, for each i, if $y \in A_i$, then A_i contains a neighborhood of y in A_k. In particular, if $y \in V \subset A_i \cap A_j$, as above, then V contains a neighborhood of y in A_k. Consequently, replacing j by k and shrinking V if necessary, we can assume that V is indeed an open subset of A_j.

Now, let us define an (infinite) sequence $i_0 = 0 < i_1 < \ldots$ of indices and a descending sequence $V_0 \supset V_1 \supset \ldots$ of compact sets, as follows: V_0 is any compact subset of A_0 such that $\operatorname{Int}_{A_0}(V_0) \neq \emptyset$. If i_m and V_m are defined, then i_{m+1} is the least index j such that there is an open subset V of A_j which is a boundary subset of A_{i_m} and satisfies $\overline{V} \subset V_m$. Then V_{m+1} is the closure of V. Let $x \in \bigcap_{j=0}^{\infty} V_j$. Thus $x \in A$, and therefore there is a k such that $x \in A_k$ and, for any j, if $x \in A_j$, then A_j contains a neighborhood of x in A_k. Consequently, each of the sets $A_{i_0}, A_{i_1}, \ldots$ contains a neighborhood of x in A_k, and therefore, by the construction, k is one of the indices $i_0, i_1, \ldots$ This is a contradiction, which proves (1).

Since the image of each A_j by any autohomeomorphism of X locally coincides again with one of these sets, it follows that the index i given by (1) can be assumed to be 0. If $U \subset A_0$ is given by (1), then, since A is an orbit of the action of $H(X)$ on X, we have $A = \bigcup_{i=0}^{\infty} h_i(U)$. Consequently, we can assume that $U = A_0$. Indeed, there is an open subset V of $M_0 \setminus S_0$ such that $V \cap A_0 = U$. Since M_0 is a membrane spanned on S_0, it follows from (1) of Section 2 that there is a $\gamma \in H_1(\overline{V} \setminus V)$ such that $\overline{V}$ is a membrane of γ spanned on $\overline{V} \setminus V$. Replacing, in our definition of the orbit A, the membrane M_0 by $\overline{V}$, S_0 by $\overline{V} \setminus V$ and A_0 by $U = V \cap A_0 = \overline{X \setminus \overline{V}} \cap V$, we can in fact assume that $U = A_0$. Consequently, we infer from (1) that $\overline{D_{ij}} \supset A_i \cap A_j$ for any i, j. This implies that $E_{ij} = D_{ij} \cap D_{ij}$ is an open and dense subset of $A_i \cap A_j$, and it is open both in A_i and in A_j.

Now, consider the set A_0 and the family $\{F_i\}_{i=1}^{\infty}$ of subsets of A_0, where $F_i = A_0 \cap A_i \setminus E_{0i}$. Since each A_i is locally compact, we infer from Baire's theorem that there is an $x_0 \in A_0$ such that, for each i, if $x_0 \in A_i$, then $x_0 \in E_{0i}$. Consequently, for any i, j, if $x_0 \in A_i \cap A_j$, then $x_0 \in E_{ij}$, i.e. there is a common neighborhood of x_0 both in A_i and A_j. It follows from the $\frac{1}{2}$-homogeneity of X that:

(2) For any $x \in A$ and for any i, j, if $x \in A_i \cap A_j$, then there is a common open neighborhood of x both in A_i and A_j.

Now, we shall prove that:

(3) There is an $x_0 \in A_0$ such that A_0 is locally connected at x_0.

First, consider the case when any open subset U of A_0 has at most countably many components. Let U_1 be any open subset of A_0 such that $\operatorname{diam} U_1 \leqslant 1$. Using Baire's theorem, find a component C_1 of U_1 containing an interior point of A_0. In this way, we can construct inductively a sequence $\{C_i\}_{i=1}^{\infty}$ of subsets of A_0 such that C_i is connected, $\overline{C_{i+1}} \subset \operatorname{Int}_{A_0}(C_i)$ and $\operatorname{diam} C_i \leqslant \frac{1}{i}$. Evidently, A_0 is locally connected at the common point x_0 of the C_i's.

Thus assume that there are an open subset U_0 of A_0 and a family $\{C_t\}_{t \in T}$, where $\operatorname{card} T > \aleph_0$, consisting of components of U_0. We shall prove that this is impossible. For this purpose, for any $t \in T$, choose an $x_t \in C_t$. Since the set of points which are accessible (by arcs) from $X \setminus M_0$ is dense in $\operatorname{Bd}(X \setminus M_0) \supset A_0 = \overline{X \setminus M_0} \cap (M_0 \setminus S_0)$ (cf. [13], p. 194), there is an $x_0 \in A_0$ which is accessible from $X \setminus M_0$. By the $\frac{1}{2}$-homogeneity of X, for any $x \in A$ there is an $i \geqslant 0$ such that x is accessible from $X \setminus M_i$. Thus, replacing if necessary the family $\{C_t\}_{t \in T}$ by an uncountable subfamily, we can assume that there exists an $i_0 \geqslant 0$ such that, for any $t \in T$, there is an arc I_t with $x_t \in \dot{I}_t$, $I_t \setminus \{x_t\} \subset X \setminus M_{i_0}$. Find a $y = x_{t_0}$ which is a condensation point of $\{x_t\}_{t \in T}$; i.e. in any neighborhood of y there are uncountably many of x_t's (cf. [12], p. 140). Using (2) and replacing U_0 if necessary by a smaller open subset of A_0, we can assume that U_0 is also an open neighborhood of y in $A_{i_0} \subset M_{i_0} \setminus S_{i_0}$. Now, find a neighborhood U of y in X such that $U \cap S_{i_0} = \emptyset$ and $U \cap A_{i_0} \subset U_0$. Since $X \in \mathrm{ANR}$ and by Theorem 2, there is a region $V \subset U$ in X such that $y \in V$, the set $V \cap M_{i_0} = V \cap (M_{i_0} \setminus S_{i_0})$ is connected and, moreover, for any simple closed curve $S \subset V$ there is a membrane spanned on S and contained in U. Since $V \cap A_{i_0} \cap \{x_t\}_{t \in T}$ is uncountable, there are a $\delta > 0$ and uncountably many x_t's such that $x_t \in V$ and $\varrho(y_t, M_{i_0}) > \delta$, where y_t denotes the end-point of I_t different from x_t. Thus one sees that there are two different points $p_1 = x_{t_1}$, $p_2 = x_{t_2} \in V$ and an arc I joining them such that $\dot{I} \subset V \setminus M_{i_0}$. By Corollary 1 to Theorem 2, there is an arc $J \subset V \cap M_{i_0}$ joining p_1 with p_2. By the assumption on V, there is a membrane N spanned on the simple closed curve $S = I \cup J$ and contained in U. Consequently, by Proposition 2 from Section 2, there is a connected set $C \subset N \cap \overline{X \setminus M_{i_0}} \cap M_{i_0}$ joining p_1 and p_2. Since $C \subset N \subset U \subset X \setminus S_{i_0}$, it follows that $C \subset U \cap A_{i_0} \subset U_0$. Thus we obtain a contradiction with the fact that $p_1 = x_{t_1}$ and $p_2 = x_{t_2}$ lie in different components of U_0, and therefore (3) is proved.

Using (2), we infer from (3) that, for each $i \geqslant 0$ such that $x_0 \in A_i$, the set A_i is locally connected at x_0. By the $\frac{1}{2}$-homogeneity of X, we conclude that:

(4) Each A_i, $i = 0, 1, \ldots$, is locally connected.

In the next part of the proof, we are going to apply Theorem 3 with X replaced by A, G replaced by $H(X)$ and F replaced by A_0. Since A is an orbit of the action of $H(X)$ on X, it follows that $H(X)$ acts continuously and transitively on A. Thus to verify the assumptions of Theorem 3, it remains to prove that:

(5) For any $x_0 \in A_0$ there are a neighborhood U_0 of x_0 in A_0 and a neighborhood Q_0 of e in $H(X)$ such that $Q_0 U_0 \subset A_0$.

Let $x_0 \in A_0 = (M_0 \setminus S_0) \cap \overline{X \setminus M_0}$ and let V be a compact neighborhood of S_0 in X such that $x_0 \notin V$. Since $\operatorname{diam} M_0 < \varepsilon_0$, it follows from condition (2) of Section 2 that there is a neighborhood P of $e = \mathrm{id}$ in $H(X)$ such that, if $h \in P$, then the membranes M_0 and $h(M_0)$ coincide outside V; therefore $h(M_0) \setminus V \subset M_0 \setminus S_0$. Find an open neighborhood U of x_0 in X and a neighborhood Q of e in $H(X)$ such that $U \subset X \setminus V$, $Q \subset P$ and $QU \subset X \setminus V$. Next, find a neighborhood Q_0 of e in $H(X)$ such that $Q_0 Q_0^{-1} \subset Q$ and let $U_0 = U \cap A_0$. Thus, if $x \in U_0$, $h \in Q_0$, then $h(x) \in h(M_0) \cap X \setminus V \subset M_0 \setminus S_0$. If $y \in U \setminus M_0$, $h \in Q_0$, then $h(y) \notin M_0$, because otherwise $y = h^{-1}h(y) \in h^{-1}(M_0) \cap U \subset h^{-1}(M_0) \setminus V \subset M_0$, since $h^{-1} \in Q_0^{-1} \subset Q \subset P$. Thus $y \in \overline{U \setminus M_0}$, $h \in Q_0$ imply $h(y) \in \overline{X \setminus M_0}$, and therefore $x \in U_0$, $h \in Q_0$ yield that $h(x) \in (M_0 \setminus S_0) \cap \overline{X \setminus M_0} = A_0$, which completes the proof of (5).

Thus we infer from Theorem 3 that:

(6) For each $i = 0, 1, \ldots$ there are an open subset V_i of $A_i = h_i(A_0)$ and a dense G_δ subset C_i of V_i satisfying the condition (*) in Theorem 3 with C_0 replaced by C_i and such that $A = \bigcup_{i=0}^{\infty} C_i$.

Further, we shall establish that:

(7) No set A_i, $i = 0, 1, \ldots$, contains a triod (i.e. the union of three arcs, as in the letter T).

For this purpose, fix any $p \in C_0 \subset V_0 \subset h_0(A_0) = A_0$. As we have already seen in the proof of (3), there is a k such that $p \in A_k \subset M_k \setminus S_k$ and p is accessible from $X \setminus M_k$. It follows from (5) that there is an open neighborhood Q of the identity e in $H(X)$ such that, for any $h \in Q$, the point $h(p)$ belongs to A_k and is accessible from $X \setminus M_k$. Consequently, we infer from (2) and (6) that there is an open neighborhood U_0 of p, both in V_0 and in A_k, such that any $x \in U_0 \cap C_0$ is accessible from $X \setminus M_k$.

Now, assume that (7) is not true. It follows from the $\frac{1}{2}$-homogeneity of X and from (2) that for each i any $x \in A_i$ is a ramification point of A_i. In particular, there is a triod $T = \bigcup_{j=1}^{3} I_j$ contained in U_0, where $I_1 \cap I_2 \cap I_3 = (p) = \dot{I}_1 \cap \dot{I}_2 \cap \dot{I}_3$. To obtain a contradiction, we are going to prove that:

(8) There are a membrane N_0 spanned on a set S, a point $x \in N_0 \setminus S$ and an arc $I \subset U_0$ such that $x \in \dot{I}$ and $I \setminus \{x\} \subset X \setminus N_0$.

For this purpose, first find an open neighborhood O_1 of p in X such that $\operatorname{diam} O_1 < \varepsilon_0$ and each of the sets $I_j \setminus \overline{O_1}$ for $j = 1, 2, 3$ is non-empty. Next, find an open neighborhood O_2 of p in X such that, for any curve $R \subset O_2$ which is a continuous image of S^1, there is a membrane spanned on R and contained in O_1. Further, let us construct an open neighborhood O_3 of p in X such that, for any curve $R \subset O_3$ which is a continuous image of S^1, there is a membrane spanned on R, contained in O_2 and, moreover, satisfying the following condition:

(9) For any $x, y \in O_3 \cap C_0$ there is an arc J joining these points such that $J \subset O_2$ and $\dot{J} \subset X \setminus M_k$.

To find this neighborhood O_3, one can apply (6) and the facts that $p \in U_0 \cap C_0$ and that any $x \in U_0 \cap C_0$ is accessible from $X \setminus M_k$.

Now, we shall construct an arc $I_4 \subset U_0$ issuing from the triod $T = \bigcup_{j=1}^{3} I_j$ such that $I_4 \cap T$ is an end-point q of I_4, $q \in O_3$ and $I_4 \setminus \overline{O_1} \neq \emptyset$, possibly modifying the triod. We can do it, for instance, in a small neighborhood of the arc I_3, joining a subtriod of T with two points lying outside O_1 by two disjoint arcs, as in a lemma given in [13] (p. 241). The construction can be made, because A_k is locally compact and locally connected, each $x \in A_k$ is a ramification point of A_k and, moreover, no point of A_k locally disconnects A_k. This last statement follows from the $\frac{1}{2}$-homogeneity of X and from the theorem of Whyburn (cf. [21] and [13], p. 223) that all points which locally disconnect a continuum X, except at most countably many, have order (in the sense of Menger–Urysohn) equal to 2.

Having constructed the arc I_4, denote by I_0 the only arc contained in T and joining p with q. We can assume that $I_0 \subset O_3$. By [12] (p. 128), we can find an open neighborhood W of p in M_k such that $I_0 \subset W$, $\overline{W} \subset O_3$ and the boundary of W in M_k intersects $T \cup I_4$ at exactly four points, each belonging to an arc I_j, where $j \leqslant 4$. Next, find a compact neighborhood V of $\overline{W} \setminus W$ in X such that $V \subset O_3$, $I_0 \subset W \setminus V$ (cf. Fig. 1). Since M_k is a membrane spanned on S_k, we can apply to M_k

the lemma and Corollary 2 from Section 2, with W instead of U. Thus there are a 1-polyhedron P, a map $f\colon P \to V \cap M_k$ and a membrane N of the corresponding $\delta \in H_1(V \cap M_k)$ spanned on $f(P)$, as in the lemma. We can assume that the four points at which $\overline{W} \setminus W$ intersects $T \cup I_4$ correspond to some vertices of P, and, moreover, that the intersection $f(P) \cap (T \cup I_4)$ lies in four disjoint sets, corresponding to small neighborhoods of these vertices in P.

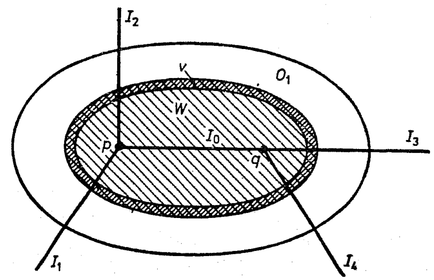


Fig. 1

Now, observe that there is a simple closed curve $P_1 \subset P$ defining a generator (belonging to a natural base) of the group $H_1(P)$ and such that p belongs to a membrane $N_1 \subset O_2$ spanned on $f(P_1)$. Indeed, since $f(P) \subset V \subset O_3$, it follows from the definition of O_3 that for any simple closed curve $R \subset P$ defining a generator belonging to a base of $H_1(P)$, there is a membrane spanned on $f(R)$ and contained in O_2. The union of these membranes contains a membrane of δ spanned on $f(P)$ and contained in $O_2 \subset O_1$, which must coincide with N, because $N \subset \overline{W} \cup V \subset O_3 \subset O_1$ and $\operatorname{diam} O_1 < \varepsilon_0$. Since $p \in W \setminus V = N \setminus V$, we conclude that there exist the desired simple closed curve P_1 and the membrane $N_1 \subset O_2$ spanned on $f(P_1)$ and such that $p \in N_1 \setminus V \subset N_1 \setminus f(P_1)$.

Further, notice that, if (8) is not true, then each of the arcs I_1, I_2, I_3, I_4 must intersect $f(P_1) \subset f(P) \subset V \cap M_k$ at a point different from p and q. Indeed, each of these arcs contains a point which does not belong to N_1, because $N_1 \subset O_2 \subset O_1$, and each of them contains a point which does not belong to O_1 (see Fig. 1). Since $I_0 \subset X \setminus V \subset X \setminus f(P_1)$, we conclude that either (8) is true, or there exist four points r_j, $j \leqslant 4$, where $r_j \neq p, q$, at which the arcs I_j's intersect $f(P_1)$. We shall show that in this last case (8) is also true.

In fact, considering the assumptions about the intersection $f(P) \cap (T \cup I_4)$, we can assume that the points r_i, $i \leqslant 4$, are ordered so that, on the simple closed curve P_1 the pair of sets $f^{-1}(r_1), f^{-1}(r_3)$ interlaces the pair $f^{-1}(r_2), f^{-1}(r_4)$ (more exactly, some arcs containing these sets have this property). Moreover, there are four disjoint neighborhoods of the r_i's, $i \leqslant 4$, in $V \cap M_k$, the union of which contains $f(P_1) \cap (T \cup I_4)$. Since $r_1, r_3 \in T \cup I_4 \subset U_0 \subset A_k \cap V_0$, $r_1, r_3 \in V \subset O_3$ and C_0

is a dense G_δ subset of V_0, it follows from (9) that there is an arc $J \subset O_2$ joining r_1 and r_3 and intersecting M_k only in the neighborhoods of these points, chosen above (cf. Fig. 2).

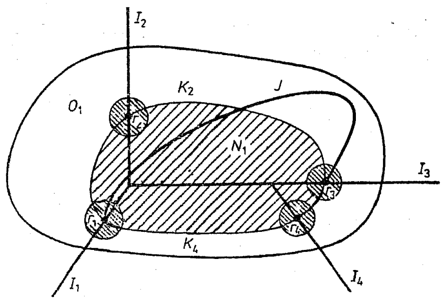


Fig. 2

Find two arcs with common end-points belonging to $f^{-1}(r_1)$ and $f^{-1}(r_3)$ respectively, whose union is the simple closed curve P_1, and denote by K_2 and K_4 their images under f, where $r_2 \in K_2$, $r_4 \in K_4$. Since $K_2, K_4, J \subset O_2$, there are two membranes: N_2, spanned on $K_2 \cup J$, and N_4, spanned on $K_4 \cup J$, both contained in O_1. The union $N_2 \cup N_4$ contains a membrane spanned on $f(P_1)$, which must coincide with N_1, because diam $O_1 < \varepsilon_0$. Consequently, $p \in N_2 \cup N_4$. Assume, for instance that $p \in N_2$. Since $N_2 \subset O_1$, the arc $I_0 \cup I_4 \subset U_0$ does not intersect $K_2 \cup J$ and I_4 contains a point which does not belong to O_1, we conclude that condition (8) must be satisfied, with N_2 instead of N_0 and a suitable subarc of $I_0 \cup I_4$ instead of I.

Having established condition (8), using (6) we get a contradiction, which completes the proof of (7). Indeed, let $x \in I \cap N_0 \setminus S$ and $I \subset U_0$, as in (8). Since $U_0 \subset A_k \subset A = \bigcup_{i=0}^{\infty} C_i$, there is an l such that $x \in C_l$, C_l being a dense G_δ subset of an open subset V_l of A_l. By (2), there is a common open neighborhood W of x, both in $U_0 \subset A_k$ and in V_l. Since $x \in N_0 \setminus S$, where N_0 is a membrane spanned on S, it follows as in the proof of (5) that there is a neighborhood P of the identity e in $H(X)$ such that, for each $h \in P$, we have $h(x) \in N_0$. On the other hand, since $x \in C_l$, $I \setminus \{x\} \subset X \setminus N_0$ and I contains a subarc with end-point x contained in W, it follows from (6) that there are an $h_0 \in P$ and a $y \in X \setminus N_0$ such that $h_0(x) = y$. This is the desired contradiction, which completes the proof of (7).

In the next part of the proof, we shall need the following fact:

(10) There are a point $y_0 \in A_0$, an arc $I \subset A_0$ and a region O_0 in X such that $y_0 \in \mathring{I} \subset O_0$ and, for any region $O \subset O_0$ containing y_0, the set $O \setminus I$ has at least three components.

To prove (10), recall that $A_0 = (M_0 \setminus S_0) \cap \overline{X \setminus M_0} = \mathrm{Bd}_X(M_0) \cap (M_0 \setminus S_0)$, where M_0 is a membrane of an $\alpha \in H_1(S_0)$ spanned on S_0. It follows from (4) and (7) that the components of A_0 form a locally finite (in $M_0 \setminus S_0$) family consisting of open arcs and/or simple closed curves (cf. [13], p. 218 and 220). Consequently, one can find a point $y_0 \in A_0$, an arc $I \subset A_0$ and a region O_1 in X such that $y_0 \in \mathring{I} \subset O_1 \subset X \setminus S_0$, $\dot{I} \subset X \setminus O_1$, diam $O_1 < \varepsilon_0$, $O_1 \cap A_0 \subset I$ and $O_1 \setminus M_0$ has at least one component whose boundary is contained in I and contains y_0. Find also a region O_2 and an open neighborhood V of $\overline{O_2} \setminus O_2$ in X such that $y_0 \in O_2 \subset \overline{O_2} \subset O_1$, $y_0 \notin \overline{V}$ and, for any curve $R \subset O_2 \cup V$ which is a continuous image of S^1, there is a membrane spanned on R and contained in O_1. Let $M = \overline{O_2 \cap M_0}$, $S = M \setminus O_2$. In virtue of (1) from Section 2, there is a $\gamma \in H_1(S)$ such that M is a membrane of γ spanned on S.

Assume that (10) is not true for y_0 chosen above. Observe that $(M \setminus S) \setminus I$ is open in X, because $\mathrm{Bd}_X(M) \setminus S \subset \mathrm{Bd}_X(M_0) \cap O_1 \subset A_0 \cap O_1 \subset I$. Since the membrane M can be replaced by the closure of its open subset, we can assume, shrinking also the neighborhood V if necessary, that $(M \cup V) \setminus I$ is connected. Now, apply the lemma and Corollary 2 from Section 2 to $\gamma \in H_1(S)$ and the neighborhood V of $\overline{O_2} \setminus O_2 \supset S$. Thus there are a 1-polyhedron P, a map $f\colon P \to \overline{V} \cap M_0$ with $f(P) \subset V$ which is an imbedding on each 1-simplex of P, and there are a $\delta \in H_1(P)$ and a membrane N of $\beta = f_*(\delta)$ spanned on $f(P)$ and satisfying $N \subset M \cup V$, $M \setminus V = N \setminus V$.

Thus $y_0 \in N \setminus V$. Let $P_1, \ldots, P_k$ be simple closed curves in P which determine a base of $H_1(P)$. Since $f(P_i) \subset V$ for $i \leqslant k$, there is a membrane N_i spanned on $f(P_i)$ and contained in O_1. Then $\bigcup_{i=1}^{k} N_i \cup f(P)$ must contain a membrane of β spanned on $f(P)$, which must coincide with N, because diam $O_1 \leqslant \varepsilon_0$. Consequently, there is a $j \leqslant k$ such that $y_0 \in N_j$. We shall assume that $j = 1$. Observe now that:

(11) If $R \subset X \setminus \{y_0\}$ is a curve which is a continuous image of S^1 and intersects at most one component of $I \setminus \{y_0\}$, then, if Q is a membrane spanned on R and contained in O_1, then $y_0 \notin Q$.

Indeed, suppose that $y_0 \in Q$. Then $y_0 \in Q \setminus R$. Since $N \subset O_1$ and $\dot{I} \subset X \setminus O_1$, there are a $y \in \mathring{I} \cap (Q \setminus R)$ and a subarc J of I such that $y \in \mathring{J}$, $(J \setminus \{y\}) \cap Q = \emptyset$. Since $y \in Q \setminus R$, we infer as in (5) that each autohomeomorphism of X sufficiently close to the identity maps y to a point belonging to $Q \setminus R$. On the other hand, since $J \subset I \subset A_0$, it follows from (2) and (6) that there is an autohomeomorphism h of X, arbitrarily close to the identity, such that $h(y) \in J \setminus \{y\}$. This is a contradiction, which proves (11).

Thus, since $y_0 \in N_1 \setminus f(P_1)$, we infer from (11) that the curve $P_1' = f(P_1) \subset V$ intersects both components of $I \setminus \{y_0\}$. Consequently, there is a finite family of arcs (or points) K_i, for $i \leqslant 4l$, such that $P_1 = \bigcup_{j=1}^{4l} K_j$; if i is odd, then $f(\mathring{K}_i) \cap I = \emptyset$ and $f(\mathring{K}_i)$ intersects both components of $I \setminus \{y_0\}$ and, if i is even, then $f(K_i)$ intersects only one component of $I \setminus \{y_0\}$. Moreover, the arcs K_i, K_{i+1} (where the indices are considered mod $4l$) have only a common end-point (see Fig. 3).

Since, for each i, $K_i' = f(K_i) \subset V$ and $(M \cup V) \setminus I$ is a region in X, one can con-

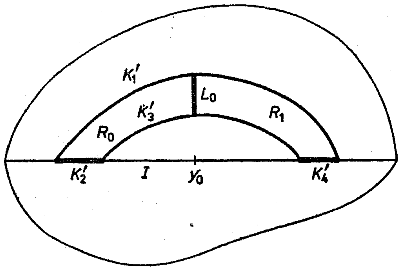


Fig. 3

struct some arcs $L_0, \ldots, L_{l-1} \subset (M \cup V) \setminus I$ such that L_i joins a point belonging to $\mathring{K}'_{4l+1}$ with a point belonging to $\mathring{K}'_{4l+3}$ (cf. Fig. 3). Thus one obtains $l+1$ curves $R_0, R_1, \ldots, R_l \subset (M \cup V) \setminus \{y_0\}$, each being (in a natural way) a continuous image of S^1, whose union contains P_1', but none of which intersects both components of $I \setminus \{y_0\}$. Since $R_i \subset M \cup V \subset O_2 \cup V$, there is a membrane Q_i spanned on R_i and contained in O_1. Then $\bigcup_{i=0}^{l} Q_i \subset O_1$ contains a membrane spanned on the curve P_1', and therefore $\bigcup Q_i \supset N_1$, because $\operatorname{diam} O_1 < \varepsilon_0$. It follows from (11) that $y_0 \notin \bigcup Q_i$. Since $y_0 \in N_1$, we obtain a contradiction, which completes the proof of (10).

Next, let us consider the structure of the set $A = \bigcup_{i=0}^{\infty} A_i$. By (4) and (7), each A_i is locally an (open) arc, and therefore A is the union of countably many arcs. We shall consider the following three cases:

1°. The components of A are non-compact.
2°. The components of A are compact and infinitely many.
3°. The components of A are compact and finite in number.

Evidently, by the homogeneity of A, in cases 2° and 3° any component R of A is a simple closed curve and, if 1° or 2° holds, then A, being non-compact, is dense in X.

In case 1°, the orbit A is the union of at most countably many disjoint sets A_i', $i = 1, 2, \ldots$, each of which is a biunique continuous image of the line E^1 and it is maximal with respect to this property. Evidently, A_i' are composants of A, and therefore in this case the orbit A has the structure described in the assumptions of Theorem 1. Thus in the sequel of this proof we shall assume that this case does not hold, and we are going to prove that case 2° cannot hold.

Thus consider case 2° and order all components of A into a sequence $R_1, R_2, \ldots$ To prove that 2° cannot hold, we shall consider successively the following subcases:

(i) Each component R_i of A disconnects the whole space X.
(ii) Case (i) does not hold, but any component R_i of A locally disconnects X into at least three components.

(iii) Case (i) does not hold, but any component R_i of A locally disconnects A into exactly two components.
(iv) No component R_i disconnects X locally or globally.

First, consider (i). To obtain a contradiction, we shall show that:

(12) There are a $b_0 \in B$ and a continuum $E \subset B$ such that $b_0 \in E$ and for any neighborhood U of E in X there is a component R_i of A such that the component of $X \setminus R_i$ containing E is contained in U.

To prove (12), we shall construct inductively a subsequence $R_{i_1}, R_{i_2}, \ldots$ and a descending sequence $E_1, E_2, \ldots$ of continua as follows: Let $R_{i_1} = R_1$ and let E_1 be the closure of any component of $X \setminus R_1$. If i_j and E_j are defined, let i_{j+1} denote the least i such that $i > i_j$ and $R_i \subset E_j$. Then E_{j+1} is the closure of any component C of $X \setminus R_{i_{j+1}}$ contained in E_j. Let $E = \bigcap_{j=1}^{\infty} E_j$ and let b_0 be any point of E. It follows from the construction that E is a continuum contained in B and satisfying (12).

To obtain a contradiction, we shall now find a point $b \in B$ which does not have the property (12) with b_0 replaced by b. For this purpose, we also construct inductively a subsequence $R_{j_1}, R_{j_2}, \ldots$ and a descending sequence $F_1, F_2, \ldots$ of continua in X such that $\operatorname{Bd}_X(F_k) \subset \bigcup_{j=1}^{k} R_{j_i}$. Let $j_1 = 1$ and let F_1 be the closure of any component of $X \setminus R_1$. If j_k and F_k are defined, let j_{k+1} denote the least j such that $j > j_k$ and $R_j \subset F_k$. Then F_{k+1} is the closure of any component C of $X \setminus \bigcup_{i=1}^{k+1} R_{j_i}$ such that $\overline{C} \subset F_k$ and $\operatorname{Bd}_X(C) \cap \operatorname{Bd}_X(F_k) \neq \emptyset$. Let $F = \bigcap_{k=1}^{\infty} F_k$. Since F is a continuum, $F \setminus \bigcup_{k=1}^{\infty} R_{j_k}$ is non-empty and it is clearly contained in B. Let b be any point of this set. Then b does not have the property (12). Indeed, let $E \subset B$ be any continuum such that $b \in E$. Choose a component C_k of $X \setminus R_{j_k}$ for $k = 1, 2$ such that $C_k \cap F = \emptyset$, and let C denote the component of $(X \setminus R_{j_1}) \setminus R_{j_2}$ whose closure contains F. Let U be a region in X such that $\overline{C} \subset U$ and $C_k \setminus U \neq \emptyset$ for $k = 1, 2$. Then, for any component R_i of A contained in U, the component D of $X \setminus R_i$ containing b (and therefore also containing E) intersects $X \setminus U$, and therefore (12) is not satisfied by b. Thus we obtain a contradiction with the fact that B is an orbit of the action of $H(X)$ on X, which implies that (i) cannot hold.

Consider now (ii). For $n = 1, 2, \ldots$ let $\mathscr{D}_n$ denote the decomposition of X whose non-degenerate elements are the curves $R_1, R_2, \ldots, R_n$. Let $Y_n = X/\mathscr{D}_n$ and let $\pi_n\colon X \to Y_n$ denote the natural projection. Since $X \in \mathrm{LC}^1$ and each element of this decomposition belongs to $\mathrm{LC}^0 \cap C^0$, we infer by the well-known Smale theorem (cf. [17]) that $Y_n \in \mathrm{LC}^1$. Consequently, $Y_n \in \alpha$, where α is the class defined in Section 4. Let $y_i = \pi_n(R_i)$ for $i \leqslant n$. By the assumptions of (ii), each point y_i, $i \leqslant n$, locally disconnects Y_n into at least three components; however, no point disconnects the whole space Y_n, which implies that Y_n is a cyclic space in the sense of Section 4. Using the theory of strongly cyclic elements described in [16], it is easy to show (by induction with respect to n) that the rank of $H_1(Y_n)$ is at least $n+1$. Moreover, there are simple closed curves $S_1, \ldots, S_{n+1}$ in Y_n determining the generators of this group. These curves can be chosen so that there exist simple closed curves $T_1, \ldots, T_{n+1}$ in X such that $\pi_n(T_i) = S_i$ for $i \leqslant n+1$. Consequently, the rank of $H_1(X)$ cannot be finite, which contradicts the assumption that $X \in \mathrm{ANR}$. Thus (ii) also cannot hold.

Next, onsider subcase (iii). As in (ii), consider the decomposition $\mathcal{D}_n$ of X and the spaces $Y_n = X/\mathcal{D}_n$ for $n = 1, 2, \ldots$ Let $\pi_n\colon X \to Y_n$ and $\sigma_{mn}\colon Y_m \to Y_n$ for $m \leqslant n$ denote the natural projections. Consider the set L_{Y_n} of all points locally disconnecting Y_n, and the strongly cyclic elements of Y_n. The set L_{Y_n} consists of exactly n points, which must belong to non-degenerate strongly cyclic elements of Y_n, which are the closures of components of $Y_n \setminus L_{Y_n}$. Consider the following two possibilities: (1) for each m, there is an n such that Y_n has at least m locally disconnecting points, each belonging to the interior of the non-degenerate strongly cyclic element of Y_n containing this point (cf. Fig. 4); (2) for each m, there is an n, such that Y_n contains at least m non-degenerate strongly cyclic elements whose boundaries contain more than two points (cf. Fig. 5). If one of these possibilities holds, then one can prove, as in (ii), that the rank of $H_1(X)$ is infinite, which is a contradiction.

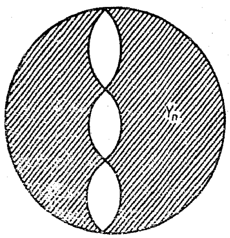


Fig. 4

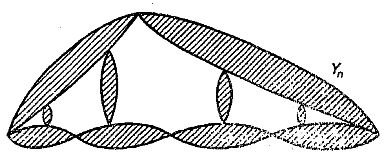


Fig. 5

Therefore we can assume that none of these possibilities holds. Thus there are a space Y_{n_0} and a non-degenerate strongly cyclic element E of Y_{n_0} such that for each $n \geqslant n_0$ and for any non-degenerate strongly cyclic element E' of Y_n contained

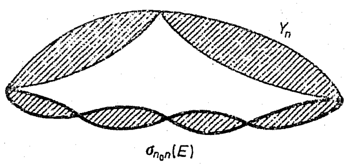


Fig. 6

in $\sigma_{n_0n}(E)$ the set $\mathrm{Bd}_{Y_n}(E')$ consists of exactly two points and no point $y \in \mathrm{Int}_{Y_n}(E')$ locally disconnects Y_n (cf. Fig. 6). Let $\mathrm{Bd}_{Y_{n_0}}(E) = \{p_0\ q_0\}$, $F = \pi_{n_0}^{-1}(E)$ and $R_{i_0} = \pi_{n_0}^{-1}(p_0)$, $R_{j_0} = \pi_{n_0}^{-1}(q_0)$. Choose $p \in R_{i_0}$, $q \in R_{j_0}$. One sees that no proper

subset of a component $R_i \subset (F \setminus R_{i_0}) \setminus R_{j_0}$ of A disconnects F, because otherwise there are points of R_i arbitrarily close to each other such that one of them cannot be mapped to the other by a small autohomeomorphism $h \in H(X)$. Thus F is the closure of a component of $(X \setminus R_{i_0}) \setminus R_{j_0}$ and each curve R_i contained in $\mathrm{Int}_X(F) = (F \setminus R_{i_0}) \setminus R_{j_0}$ disconnects F irreductively into two components, the first containing p and the second containing q. Evidently $F \in \mathrm{ANR}$, since it is a neighborhood retract of X.

Observe now that for any $b \in B \cap F$ there is a continuum $T \subset B \cap F$ such that $b \in T$ and T disconnects F between p and q. To find T, construct inductively a sequence $T_1, T_2, \ldots$ of subcontinua of F, bounded by the curves R_i and containing b, as follows: Let $T_1 = F$. If T_n is defined. let k denote the least i such that $R_i \subset \mathrm{Int}_X(T_n)$ and let l denote the least index greater than k such that $R_l \subset \mathrm{Int}_X(T_n)$ and the component C of $(T_n \setminus R_k) \setminus R_l$ satisfying $\mathrm{Bd}_X(C) = R_k \cup R_l$ contains b. Such k and l always exist, because $b \in \mathrm{Int}_X(T_n)$ and $A = \bigcup_{i=1}^{\infty} R_i$ is dense in X. Then T_{n+1} is the closure of C. The set $T = \bigcap_{n=1}^{\infty} T_n$ is evidently a continuum containing b and disconnecting F between p and q. Moreover, T must coincide with the component of the orbit B containing b. Thus $\mathrm{Int}_X(F)$ is the union of disjoint continua, each disconnecting F between p and q (and therefore also between R_{i_0} and R_{j_0}). It follows from the theory of the separation of connected spaces described in [13] (p. 98) that there is a map f of F onto the interval $\langle 0, 1\rangle$ such that $f(R_{i_0}) = 0$, $f(R_{j_0}) = 1$, for any $t \in \langle 0, 1\rangle$ the set $f^{-1}(t)$ is either a component R_i of A or a component of B contained in F and, if $t_1 < t_2 < t_3$, then $f^{-1}(t_2)$ disconnects F between $f^{-1}(t_1)$ and $f^{-1}(t_3)$. Moreover, since A is dense in X, we infer that $A \cap F$ is dense in F, and therefore

$$\{t \in \langle 0, 1\rangle\colon f^{-1}(t) \subset A\}$$

is dense in $\langle 0, 1\rangle$. Observe also that, for any $t_0 \in \langle 0, 1\rangle$ and any neighborhood of $f^{-1}(t_0)$ in X, there are $t_1, t_2 \in \langle 0, 1\rangle$ such that $f^{-1}(\langle t_1, t_2\rangle)$ is a neighborhood of $f^{-1}(t_0)$ contained in U. In particular, if $f^{-1}(t_0) \subset A$, then there are $t_1 < t_0 < t_2$ such that $f^{-1}(t_0)$ is a retract of $f^{-1}(\langle t_1, t_2\rangle)$.

Consider now a fixed curve $R_{k_0} \subset \mathrm{Int}_X(F)$ and let U be a region in X such that $U \cap R_{k_0} \neq \emptyset$ and, for any simple closed curve $S \subset U$, there is a membrane M spanned on S with $\operatorname{diam} M < \mu$, where $\mu = \min\big(\varepsilon_0, \operatorname{diam} R_{k_0}, \varrho(R_{k_0}, \mathrm{Bd}_X(F))\big)$. Then, for any region V intersecting R_{k_0} and such that $\overline{V} \subset U$, we have:

(13) If R_{k_0} disconnects $\overline{V}$ between some points $x, y \in V \setminus R_{k_0}$, then R_{k_0} also disconnects U between these points.

Indeed, suppose that this is not true. Then we can construct a simple closed curve $S \subset U$ such that $S \cap R_{k_0}$ is a small arc (or point) contained in V and S intersects two different components of $\overline{V} \setminus R_{k_0}$. Let M be a membrane spanned on S with $\operatorname{diam}(M) < \mu$. Then $M \cap R_{k_0} = S \cap R_{k_0}$, since otherwise there is a $z \in (M \setminus S) \cap \overline{R_{k_0} \setminus M}$, and this yields a contradiction as in the proof of (11).

Thus consider the decomposition $M = (M \cap \overline{V}) \cup (M \setminus V)$, where $M \cap \overline{V} \setminus V \subset \overline{V} \setminus R_{k_0}$, since $M \cap R_{k_0} = S \cap R_{k_0} \subset V$. Taking into consideration that S intersects two different components of $\overline{V} \setminus R_{k_0}$, one can prove that S is a retract of M (cf. [16], the proof of (3.1)). Since M is a membrane spanned on S, this is a contradiction, which proves (13).

Now, let $\mathfrak{M}$ denote the number of components of $U \setminus R_{k_0}$. It follows from (10) and (13) that $3 \leqslant \mathfrak{M} \leqslant \aleph_0$. Evidently, $\mathfrak{M}$ does not depend on the choice of the region U (satisfying the above mentioned assumptions) and, if $\mathfrak{M} = \aleph_0$, then almost all components of $U \setminus R_{k_0}$ must lie in an arbitrarily small neighborhood of R_{k_0}.

Next, observe that there is a component C of $F \setminus R_{k_0}$ intersecting at least two components of $U \setminus R_{k_0}$, because $F \setminus R_{k_0}$ has exactly two components and $\mathfrak{M} > 2$. For any curve $R_{k_0} \subset \mathrm{Int}_X(F)$, we choose a component C of $F \setminus R_{k_0}$ such that the number $\mathfrak{N}$ of components of $C \cap U$ is greater than or equal to the number of components of $C' \cap U$, where C' is the other component of $F \setminus R_{k_0}$. This C will be called the *distinguished component* of $F \setminus R_{k_0}$. Evidently, $2 \leqslant \mathfrak{N} \leqslant \aleph_0$ and this number is the same for all curves $R_k \subset \mathrm{Int}_X(F)$. Replacing F if necessary by $F' = f^{-1}(\langle t_1, t_2 \rangle)$ (where $f^{-1}(t_1), f^{-1}(t_2) \subset A$), we can assume that the set of $t \in \langle 0, 1 \rangle$ such that $f^{-1}(t) \subset A$ and the distinguished component of $F \setminus f^{-1}(t)$ contains $R_{j_0} = f^{-1}(1)$ is dense in $\langle 0, 1 \rangle$. We shall also find a neighborhood W of R_{k_0} in $\overline{C}$, which will be called a *typical neighborhood* of R_{k_0} (in the closure of the distinguished component C of $F \setminus R_{k_0}$). To construct W, find a finite number $W_1, \ldots, W_l$ of regions in $\overline{C}$ such that $\overline{W}_i \cap \overline{W}_j \neq \emptyset$ iff $|i-j| \leqslant 1$ (where the indices are considered modulo l), $\overline{W}_i \cap R_{k_0}$ is an arc and, moreover, for any simple closed curve $S \subset W_{i-1} \cup W_i \cup W_{i+1}$ for some $i \leqslant l$ there is a membrane M spanned on S with $\operatorname{diam} M < \mu$ (where μ is defined for R_{k_0} as before). Considering the intersections of the components of the sets $W_i \setminus R_{k_0}$ for $i \leqslant l$, one sees that $W = \bigcup_{i=1}^{l} W_i$ is a neighborhood of R_{k_0} in $\overline{C}$ which is — in a sense — an $\mathfrak{N}$-band which runs along R_{k_0} $\mathfrak{N}$ times, but is not disconnected by R_{k_0}. Observe that the arc $\overline{W}_i \cap R_{k_0}$ is contained in the boundary of any component of $W_i \setminus R_{k_0}$, just as we have shown that R_{k_0} is the boundary in F of both components of $F \setminus R_{k_0}$. The components of the sets $W_i \setminus R_{k_0}$ for $i \leqslant l$ will be called *parts* of this typical neighborhood W of R_{k_0}, and the parts contained in the consecutive regions W_i will be called *consecutive parts.*

In the next part of the proof that (iii) cannot hold, we shall use the Čech homology groups with coefficients in the group Z of the integers. These groups for a pair (Y, Y_0) will be denoted by $H_n(Y, Y_0; Z)$, to distinguish them from the groups $H_n(Y, Y_0)$, where the coefficients are in the field Q. To simplify notation, in the sequel we shall identify elements of homology groups with their images by natural homomorphisms, which should be clear by indicating homology groups in which some relations between these elements hold.

Since $F \in \mathrm{ANR}$, it follows that the group $H_1(F; Z)$ is finitely generated and therefore, since F contains infinitely many of the curves R_k, there are a finite sequence

$R_{k_1}, \ldots, R_{k_l}$ and a linear combination $m_1\gamma_1 + \ldots + m_l\gamma_l$, where m_i is a non-zero integer and γ_i is a generator of $H_1(R_{k_i}; Z)$, such that $m_1\gamma_1 + \ldots + m_l\gamma_l = 0$ in $H_1(F; Z)$.

Assume first that $l = 1$ and let $R_{k_1} = f^{-1}(t_1)$, $F_0 = f^{-1}(\langle 0, t_1 \rangle)$, $F_1 = f^{-1}(\langle t_1, 1 \rangle)$. Since F_0, F_1, $F = F_0 \cup F_1$ and $R_{k_1} = F_0 \cap F_1$ are all ANR-spaces, their Čech homology groups coincide with the singular ones, and therefore they satisfy the Exactness Axiom. Thus there is a $\zeta \in H_2(F, R_{k_1}; Z)$ such that $\partial_*(\zeta) = m_1\gamma_1$. Using the relative Mayer–Vietoris sequence for the pairs (F_0, R_{k_1}) and (F_1, R_{k_1}), one sees that at least one of F_0, F_1, say F_1, satisfies the conditions: $\mathrm{Int}_X(F_1) \neq \emptyset$ and there is a non-zero integer n_1 such that $n_1\gamma_1 = 0$ in $H_1(F_1; Z)$. The described structure of F and of the map $f\colon F \to \langle 0, 1 \rangle$ implies that there is a $t_2 > t_1$ such that R_{k_1} is a retract of $F_2 = f^{-1}(\langle t_1, t_2 \rangle)$ and therefore no non-zero multiple of γ_1 is zero in $H_1(F_2; Z)$. We can assume that $f^{-1}(t_2)$ is a curve R_{k_2}, and let γ_2 be a generator of $H_1(R_{k_2}; Z)$. Considering the decomposition $F_1 = f^{-1}(\langle t_1, 1 \rangle) = f^{-1}(\langle t_1, t_2 \rangle) \cup f^{-1}(\langle t_2, 1 \rangle)$ and using the same methods as before, one can prove that there is a non-zero integer n_2 such that $n_1\gamma_1 + n_2\gamma_2 = 0$ in $H_1(F_2; Z)$. Let γ_i' denote the image of γ_i under the natural homomorphism $H_1(R_{k_i}; Z) \to H_1(R_{k_i}; Q) = H_1(R_{k_i})$. Then, evidently, $n_1\gamma_1' + n_2\gamma_2' = 0$ in $H_1(F_2)$; however, no non-zero multiple of γ_1 or γ_2 (γ_1' or γ_2') is 0 in $H_1(F_2; Z)$ (in $H_1(F_2)$).

In the case when $l > 1$, using the same methods, one can also find two curves $R_{k_1} = f^{-1}(t_1)$, $R_{k_2} = f^{-1}(t_2)$ such that $n_1\gamma_1 + n_2\gamma_2 = 0$ in $H_1(F_2; Z)$, where n_i is a non-zero integer, γ_i is a generator of $H_1(R_{k_i}; Z)$ and $F_2 = f^{-1}(\langle t_1, t_2 \rangle)$. We can also assume that no non-zero multiple of γ_1 or γ_2 (resp. γ_1' or γ_2') is zero in $H_1(F_2; Z)$ (resp. $H_1(F_2)$).

Observe also that, for any curve $R_k = f^{-1}(t) \subset \mathrm{Int}(F_2)$, there are non-zero integers m, n such that $n_1\gamma_1 + m\gamma = 0$ in $H_1(f^{-1}(\langle t_1, t \rangle); Z)$ and $n\gamma + n_2\gamma_2 = 0$ in $H_1(f^{-1}(\langle t, t_2 \rangle); Z)$, where γ is a generator of $H_1(R_k; Z)$. However, no non-zero multiple of γ is zero in $H_1(F_2; Z)$ and the same holds for γ', where γ' is the image of γ under the natural homomorphism $H_1(R_k; Z) \to H_1(R_k)$.

Finally, to obtain a contradiction showing that (iii) cannot hold, we shall show that a non-zero multiple of γ_1' is 0 in $H_1(F_2^*)$ for some $F_2^* \supset F_2$. Indeed, since the set of $t \in \langle 0, 1 \rangle$ such that the distinguished component of $F \setminus f^{-1}(t)$ contains $R_{j_0} = f^{-1}(1)$ is dense in $\langle 0, 1 \rangle$, we can assume, replacing if necessary the curve R_{k_1} by another one, that $t_1 > 0$ and that the distinguished component of $F \setminus R_{k_1}$ contains R_{j_0}. For the same reasons, for any natural number m, we can find a sequence of curves $R_{l_1} = R_{k_1}$, $R_{l_2} = f^{-1}(s_2), \ldots, R_{l_m} = f^{-1}(s_m)$ such that $t_1 < s_2 < \ldots < s_m < t_2$, the distinguished component of $F \setminus R_{l_i}$ contains R_{j_0}, and R_{l_j}, for $j > 1$, is contained in some typical neighborhoods of the curves R_{l_i} with $i < j$. Let δ_i for $2 \leqslant i \leqslant m$ be a generator of $H_1(R_{l_i}; Z)$ and let δ_i' denote its image under the natural homomorphism $H_1(R_{l_i}; Z) \to H_1(R_{l_i})$. We can assume that these typical neighborhoods of the curves R_{l_i} are chosen so that, for any simple closed curve S contained in the union of three consecutive parts of this neighborhood, there is a membrane N spanned on S with $\operatorname{diam} N < \max(\varepsilon_0, \operatorname{diam} R_{l_i})$ and $N \subset F_2^*$, where $F_2^* = f^{-1}(\langle t_1^*, t_2 \rangle)$ for some t_1^*, $0 < t_1^* < t_1$, chosen so close to t_1 that γ_1' is not 0 in $H_1(F_2^*)$. It follows that the typical neighborhood of $R_{l_1} = R_{k_1}$ cannot run along $R_{l_1}\aleph_0$ times. Indeed, otherwise R_{l_2} intersects only a finite number of parts of the chosen typical neighborhood of R_{l_1}. Then, constructing a finite number of arcs, each joining two points of R_{l_2} and contained in a part of this neighborhood, one obtains a finite number of simple closed curves such that the union of some membranes spanned on them contains a membrane $M \subset F_2^*$ of δ_2' spanned on R_{l_2}. However, this is impossible, because — as noticed before — δ_2' cannot be 0 in $H_1(F_2)$ nor in $H_1(F_2^*)$.

Thus there is a natural number $n \geqslant 2$ such that any typical neighborhood of $R_k \subset \mathrm{Int}(F)$ runs n times along R_k. One sees, by a similar construction of membranes, that $\delta_2' - n\gamma_1' = 0$ and, moreover, for each $1 < i < m$ we have $\delta_{i+1}' - n\delta_i' = 0$ in $H_1(F_2^*)$, where the chosen generator δ_i must be replaced by $(-1)\delta_i$ if necessary. Consequently, $\delta_m' - n^{m-1}\gamma_1' = 0$ in $H_1(F_2^*)$.

On the other hand, as noticed before, there is a non-zero integer p_m such that $n_1\gamma_1' + p_m\delta_m' = 0$ in $H_1(F_2)$ (and therefore also in $H_1(F_2^*)$), where n_1 is the fixed integer from the formula $n_1\gamma_1 + n_2\gamma_2 = 0$. Consequently, $(n_1 + p_m n^{m-1})\gamma_1' = 0$ in $H_1(F_2^*)$. Since one can construct the sequences $R_{l_1}, \ldots, R_{l_m}$ with arbitrarily great m and since $n \geqslant 2$, one sees that the integer $n_1 + p_m n^{m-1}$ can be made non-zero. As noticed before, this is a contradiction, which shows that (iii) cannot hold.

Finally, consider (iv). Then, using (10) as in (iii), we can find a number $3 \leqslant \mathfrak{N} \leqslant \aleph_0$ and for any curve R_i, $i = 1, 2, \ldots$, we can construct a typical neighborhood U of R_i in X, which runs along R_i $\mathfrak{N}$ times. We shall consider separately the cases $\mathfrak{N} = \aleph_0$ and $\mathfrak{N} < \aleph_0$.

Suppose first that $\mathfrak{N} = \aleph_0$. Then, for any positive integer m, we shall construct a polyhedron P_m and a map $f_m\colon X \to P_m$ such that $H_1(P_m)$ is a free group with m generators and f_m sends homeomorphically the curves $R_1, \ldots, R_m$ onto curves determining a base of $H_1(P_m)$. To construct P_m, find a typical neighborhood U_i of R_i for $i \leqslant m$ such that $\overline{U}_i \cap \overline{U}_j = \emptyset$ for $i \neq j$. For each $i \leqslant m$ find a finite number of parts of U_i whose closures form a band F_i running a finite number of times along R_i and such that the remaining parts are contained in $\mathrm{Int}_X(U_i)$. Let $X' = \overline{X \setminus \bigcup \{U_i\colon i \leqslant m\}} \cup \bigcup \{V_i\colon i \leqslant m\}$ and consider the closed covering of X' by $\overline{X \setminus \bigcup U_i}$ and the closures of those parts of the U_i's which lie in the V_i's. The desired polyhedron P_m is constructed by replacing the closures of these parts by 2-simplexes with a common vertex v corresponding to the set $\overline{X \setminus \bigcup U_i}$. The 1-faces of those simplexes which do not contain v lie on simple closed curves corresponding to the R_i's for $i \leqslant m$ and P_m is a bouquet of m polyhedra $P_{m1}, \ldots, P_{mm}$ with the common point v. The intersections of the remaining 1-faces of the 2-simplexes lying in P_{mi} correspond to the intersections of the corresponding parts of the U_i's. Then it is easy to show that $H_1(P_m)$ is a free group with m generators and one can construct the desired map f_m, first defining it on X' and then extending it to X. Consequently, since $X \in \mathrm{ANR}$, we infer that the case $\mathfrak{N} = \aleph_0$ cannot hold.

Suppose now that $3 \leqslant \mathfrak{N} < \aleph_0$. Then we proceed similarly, with the change that the groups $H_1(P_m; Z)$ and $H_1(X; Z)$ must be considered. For each positive integer m, we construct similarly a polyhedron P_m and a map $f_m\colon X \to P_m$ such that $H_1(P_m; Z)$ is isomorphic to the direct sum of m groups $Z_{\mathfrak{N}} \oplus \ldots \oplus Z_{\mathfrak{N}}$ and f_m maps homeomorphically the curves $R_1, \ldots, R_m$ onto curves determining a base of $H_1(P_m; Z)$. Since $H_1(X; Z)$ is finitely generated, we obtain a contradiction, which completes the proof that (iv) cannot hold. Thus we conclude that case 2° cannot hold, and therefore 3° remains, which proves that:

(14) The set A has a finite number of components, say $R_1, \ldots, R_{n_0}$, which are simple closed curves.

Thus $B = X \setminus A$ is locally compact and therefore it follows from the homogeneity of the orbit B and the Bing–Borsuk theorem [5] that B is an open 2-manifold. By (10), for any $x \in A$ there is a region U in X containing x such that $\overline{U} \cap A$ is an arc and $U \setminus A$ has at least three components. Since $X \in \mathrm{ANR}$, we infer as in the proof of the corollary to Lemma 1 that this region can be chosen so that the closure of any component of $U \setminus A$ is a disk and, moreover, the number of these components is finite. Since, as proved by Kosiński in [10], the property of being a 2-polyhedron is a local one, we conclude that:

(15) The space X is a 2-polyhedron (except the described case).

Thus it remains to give a full classification of $\frac{1}{2}$-homogeneous connected 2-polyhedra X which are not 2-manifolds and which have no locally disconnecting points. Let $\mathscr{D}$ denote the decomposition of X whose non-degenerate elements are the components of A, let $Y = X/\mathscr{D}$ and let $\pi\colon X \to Y$ denote the natural projection. Evidently, Y is a $\frac{1}{2}$-homogeneous polyhedron. We shall consider the following three cases:

I. No $y \in \pi(A)$ locally disconnects Y.
II. There is a $y \in \pi(A)$ which disconnects Y.
III. Neither I nor II holds.

Assume first that I holds. Then Y is a closed 2-pseudomanifold with no locally separating points, and therefore a closed 2-manifold. Let $y_i = \pi(R_i)$ for $i \leqslant n_0$. Find disjoint disks $D_1, \ldots, D_{n_0}$ such that D_i is a neighborhood of y_i in Y and let $M = \overline{Y \setminus \bigcup \{D_i\colon i \leqslant n_0\}}$. Evidently, M is a connected 2-manifold with n_0 boundary components and $\pi^{-1}(M) = M'$ is homeomorphic to M. Let $D_i' = \pi^{-1}(D_i)$. Then $R_i = \pi^{-1}(y_i)$ does not disconnect D_i'. However, for any $x \in R_i$, there are an $m_0 \geqslant 3$ and a region U in D_i' containing x such that $U \setminus R_i$ has m_0 components. Evidently, m_0 does not depend on x and on i. Thus one sees that D_i' must be homeomorphic to the pseudo-projective band of order m_0 obtained from the annulus

$$N = \{x \in R^2\colon \tfrac{1}{2} \leqslant |x| \leqslant 1\}$$

by contraction (to points) of the subsets of the curve $S^1 \subset N$ of the form $p^{-1}(x)$ for $x \in S^1$, where $p\colon S^1 \to S^1$ is the covering projection of order m_0. Evidently, the space $X = M' \cup \bigcup \{D_i': i \leqslant n_0\}$ is homeomorphic to the manifold M, where the same contractions are made on each component of the boundary ∂M. Conversely, given any (compact connected) 2-manifold M with $n_0 \geqslant 1$ boundary components and a number $m_0 \geqslant 3$, the space X obtained by the above-described contractions on the boundary ∂M is a $\frac{1}{2}$-homogeneous 2-polyhedron for which the assumptions of the case I are satisfied. This completes the proof in this case.

Consider now case II. It follows from Lemma 1 and its corollary that in this case $n_0 = 1$ and there is a $k > 1$ such that Y is a bouquet of k homeomorphic pseudo-manifolds $P_1, \ldots, P_k$ intersecting at $y_1 = \pi(R_1)$. Moreover, there are a number $l \geqslant 1$, k homeomorphic closed 2-manifolds $M_1, \ldots, M_k$ and maps $\varphi_i: M_i \to P_i$ for $i \leqslant k$ such that $\varphi_i^{-1}(y_1)$ consists of l points and $\varphi_i^{-1}(y)$ consists of exactly one point if $y \neq y_1$. Considering a neighborhood Q_i of y_1 in P_i — which is a bouquet of l disks intersecting at y_1 — and the set $\varphi_i^{-1}(Q_i)$, one can construct a polyhedron M_i' and maps $\varphi_i': M_i' \to P_i'$, $\pi_i': M_i' \to M_i$, where $P_i' = \pi^{-1}(P_i)$, such that the diagram

$$\begin{array}{ccc} M_i' & \xrightarrow{\varphi_i'} & P_i' \\ {\scriptstyle \pi_i'}\downarrow & & \downarrow{\scriptstyle \pi|P_i'} \\ M_i & \xrightarrow{\varphi_i} & P_i \end{array}$$

commutes. The maps φ_i' and π_i' are one-to-one on $M_i' \setminus \varphi_i'^{-1}(R_1)$, φ_i' is a homeomorphism on each component of $\varphi_i'^{-1}(R_1)$, and π_i' maps each of these components to a point. As in case I, one sees that there are numbers $m_1, \ldots, m_l$, where $1 \leqslant m_1 \leqslant \ldots \leqslant m_l$, such that M_i' is homeomorphic to a polyhedron obtained from M_i by removings the interiors of l disjoint disks $D_1, \ldots, D_l$ and by identifications of those points belonging to the boundary $\dot{D}_i$ of D_i which correspond to one another under the covering projection of order m_i. Since $P_1', \ldots, P_k'$ must be homeomorphic, the numbers $m_1, \ldots, m_l$ do not depend on i and — since X is not a manifold — if $k = 2$, then either $l > 1$ or $m_l > 1$. Conversely, given a closed 2-manifold M and numbers $k, l, m_1, \ldots, m_l$ satisfying the above-described conditions, one can construct a $\frac{1}{2}$-homogeneous polyhedron satisfying the assumptions of case II, considering manifolds $M_1, \ldots, M_k$ homeomorphic to M and then constructing successively by the above-described identifications polyhedra $M_1', \ldots, M_k'$, polyhedra $P_1', \ldots, P_k'$ and finally the desired space X. This completes the proof in case II.

Finally, it remains to consider case III. In this case the space Y satisfies the assumptions of Lemma 2. We shall describe it as in the corollary to Lemma 2, using the same notation, except that we replace there X by Y. Thus $Y = \bigcup_{i=1}^{m} E_i$, where E_i is a pseudomanifold — being a strongly cyclic element of Y — and $E_i \cap L_Y = \bigcup_{j=1}^{p} E_{ij}$, where any $y \in E_{ij}$ locally disconnects E_i into q_j components and $1 \leqslant q_1 < \ldots < q_p$. The intersection scheme of the E_i's is given by a homogeneous multigraph $G = (V, E)$, where $V = \bigcup \{V_i: i \leqslant m\}$, $V_i = \bigcup \{V_{ij}: j \leqslant p\}$, as described in that corollary. As in case II, we construct manifolds M_i for $i \leqslant m$ and maps $\varphi_i: M_i \to E_i$.

Moreover, for each $i \leqslant m$ we construct the corresponding polyhedron M_i' and maps $\varphi_i': M_i' \to E_i' = \pi^{-1}(E_i)$, $\pi_i': M_i' \to M_i$ such that the diagram

$$\begin{array}{ccc} M_i' & \xrightarrow{\varphi_i'} & E_i' \\ {\scriptstyle \pi_i'}\downarrow & & \downarrow{\scriptstyle \pi|E_i'} \\ M_i & \xrightarrow{\varphi_i} & E_i \end{array}$$

commutes. Evidently, $L_Y = \pi(\bigcup \{R_i: i \leqslant n_0\})$. For any $y \in E_{ij} \subset E_i \cap L_Y$, the set $\varphi_i^{-1}(y)$ consists of q_j points whose counter-images under π_i' are simple closed curves. We shall divide these q_j points according to the orders of the pseudo-projective bands which are neighborhoods of the respective curves in M_i'. Thus, if $y_{ijk} \in E_{ij}$ and Q, a bouquet of q_j disks, is its neighborhood in E_i, then there are pairs of numbers $(r_{jk1}, t_{jk1}), \ldots, (r_{jks_{jk}}, t_{jks_{jk}})$ such that $\varphi_i'^{-1}\pi^{-1}(Q)$ consists of r_{jkl} bands of order t_{jkl}, where $1 \leqslant l \leqslant s_{jk}$, $r_{jk1} + \ldots + r_{jks_{jk}} = q_j$, $1 \leqslant t_{jk1} < \ldots < t_{jks_{jk}}$. Here, the index i is omitted, because the polyhedra E_i', for $i \leqslant m$, are homeomorphic, and therefore their structures are isomorphic. If the multigraph G is the union of m subgraphs $G_i = (V_i, E_i)$ for $i \leqslant m$ as in the corollary to Lemma 2, then we assign this sequence of pairs to that vertex $v_{ijk} \in V_{ij} \subset V_i$ of G_i which corresponds to y_{ijk}. By the $\frac{1}{2}$-homogeneity of X, for any two vertices $v_1, v_2 \in V$ there is an autoisomorphism ψ of the multigraph G such that $\psi(v_1) = v_2$, for each $i \leqslant m$ there is an $i' \leqslant m$ with $\psi(G_i) = G_{i'}$, $\psi(V_{ij}) = V_{i'j}$ for all $j \leqslant p$ and, for any $v \in V_i$, the sequences assigned to v in G_i and to $\psi(v)$ in $\psi(G_i)$ are the same. Moreover, for any $i, i' \leqslant m$ there is an autoisomorphism ψ of G as above such that $\psi(G_i) = G_{i'}$. The multigraph G is connected and cyclic and, moreover, since X is not a manifold in a neighborhood of any point $x \in A = \pi^{-1}(L_Y)$, we have: If $m = 1$, then $p = 1$, $q_1 \geqslant 2$ and if $q_1 = 2$, then $t_{1ks_{1k}} > 1$ for any $v_{11k} \in V_{11}$. If $m > 1$ and any vertex $v \in V$ belongs to exactly two of the G_i's, say G_{i_1}, G_{i_2}, then a similar condition must be satisfied by the numbers q_j and the suitable sequences of pairs assigned to v in at least one of the graphs G_{i_1}, G_{i_2}.

Conversely, given a closed 2-manifold M and a multigraph G with the above-described structure and with assigned sequences of pairs of numbers, where G is homogeneous in the above-described sense, we can construct successively the manifolds M_i for $i \leqslant m$ homeomorphic to M, then the polyhedra M_i' and E_i', $i \leqslant m$, and finally the space X. The constructed space is evidently a $\frac{1}{2}$-homogeneous polyhedron which satisfies the assumptions of case III. This completes the proof of Theorem 1.

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The space $(\omega^*)^{n+1}$ is not always a continuous image of $(\omega^*)^n$

by

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Abstract. It is shown that the following statement is relatively consistent with ZFC: "For all $n \in \omega$, the space $(\omega^*)^{n+1}$ is not a continuous image of the space $(\omega^*)^n$".

§ 1. Introduction. By ω^* we denote the remainder of the Čech–Stone compactification of ω, the countable discrete space, and by $(\omega^*)^n$ the product of n copies of ω^*. It was shown in [vD] that the spaces $(\omega^*)^n$ and $(\omega^*)^m$ are not homeomorphic whenever $n \neq m$. Clearly, if $n < m$, then $(\omega^*)^n$ is a continuous image of $(\omega^*)^m$. Moreover, if the Continuum Hypothesis holds, then $(\omega^*)^n$ is a continuous image of ω^* for every n (see [P]), and hence it is relatively consistent with ZFC that $(\omega^*)^n$ is a continuous image of $(\omega^*)^m$ for arbitrary $m, n \geqslant 1$.

Naturally, the question arises whether one can prove in ZFC alone that $(\omega^*)^{n+1}$ is a continuous image of $(\omega^*)^n$ for some $n \geqslant 1$.

In order to answer the above question we first translate it into the language of Boolean algebras.

Let $n, k \in \omega$. By $I_{k,n}$ we denote the subset of ω^n defined as

$$I_{k,n} = \{\langle x_0, \ldots, x_n\rangle : \exists i < n \ (x_i < k)\}$$

and let

$$J_n = \{X \in \mathscr{P}(\omega^n) : \exists k \in \omega \ (X \subset I_{k,n})\} = \bigcup_{k \in \omega} \mathscr{P}(I_{k,n}).$$

Then J_n is a proper non-principal ideal in the Boolean algebra $\mathscr{P}(\omega^n)$ of all subsets of the set ω^n.

By $\mathscr{B}_n$ we denote the subalgebra of $\mathscr{P}(\omega^n)$ generated by the family

$$\{X_0 \times X_1 \times \ldots \times X_{n-1} : \forall i < n \ (X_i \subseteq \omega)\}.$$

Obviously, the set J_n^- defined as $J_n^- = J_n \cap \mathscr{B}_n$ is an ideal in $\mathscr{B}_n$ and it is not hard to see that the Stone space of the Boolean algebra $\mathscr{B}_n/J_n^-$ is homeomorphic to $(\omega^*)^n$.

Therefore the question stated above dualizes as follows: "Is it provable in ZFC that for some $n \geqslant 1$ the Boolean algebra $\mathscr{B}_{n+1}/J_{n+1}^-$ is isomorphic to a subalgebra of $\mathscr{B}_n/J_n^-$?"