[6] L. Jonker and D. Rand, *Bifurcations in one dimension*, I: *The nonwandering set*, Invent. Math. 62 (1981), 347–365.
[7] J. Milnor and W. Thurston, *On iterated maps of the interval*, preprint (1977).
[8] M. Misiurewicz, *Horseshoes for mappings of an interval*, Bull. Acad. Pol. Sci., Ser. sci. math., astron. and phys. 27 (1979), 167–169.
[9] M. Misiurewicz and J. Smital, *Smooth chaotic maps with zero topological entropy*, preprint (1987).
[10] M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. 67 (1980), 45–63.
[11] S. Newhouse, *Continuity properties of entropy*, preprint (1986).

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Received 1 April 1987;
in revised form 7 December 1987

The Lipschitz condition for the conjugacies of Feigenbaum-like mappings

by

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Abstract. For a map f in the stable manifold $W^s(g)$ of the Feigenbaum function g the conjugacies h, h^{-1}: $h \circ f \circ h^{-1} = g$ are Lipschitz continuous maps at points of the Cantor set attractors. Moreover, h and h^{-1} occur to be Lipschitz continuous on the whole interval $[-1, 1]$ if and only if the products of derivatives of f as taken over periodic orbits are all equal.

§ 0. Introduction. In this paper, we study some properties of mappings topologically conjugate to Feigenbaum's fixed point, i.e. a concave analytic solution $g\colon [-1, 1] \to [-1, 1]$ of the functional equation $Tg = g$ with $Tf(x)$ defined as in Section 1.

We are interested in even analytic functions f conjugate to g and such that for inductively defined $T^n f = T(T^{n-1} f)$ we have $T^n f \to g$ with exponential rate.

For f chosen like above a conjugacy h: $g \circ h = h \circ f$ is uniquely given by the kneading invariant. Furthermore, there exists an f-invariant Cantor set attractor, such that $\lim_{n\to\infty} \operatorname{dist}(f^n(x), J(f)) = 0$ for every x which is not eventually periodic.

We show (§ 1) that h considered as a mapping with the domain restricted to $J(f)$ is a Lipschitz continuous function. Using this, we also prove (§ 2) that there exists a constant $\mathscr{K}$ such that h fulfils the Lipschitz condition with this constant at arbitrarily chosen point $x \in J(f)$ with respect to any point $y \in [-1, 1]$, when regarded h as a function from $[-1, 1]$ into itself. This leads us to deal with general question when $h\colon [-1, 1] \to [-1, 1]$ can be Lipschitz continuous on the whole interval. The answer as mentioned in the abstract is given in § 3.

The results of this paper are an expanded version of §§ 1, 2 of my Masters Thesis written in 1985 under the supervision of professor Michał Misiurewicz; I would like to thank him for calling my interest to the problem and encouragement.

After this paper was written I have learnt that D. Sullivan obtained the result covering the statement of Theorem 1.

Finally, in § 3 there is stated the question of analyticity of h, which seems to be an interesting direction of further work, by similarity to the known results for expanding mappings of the circle (cf. [7], [8]).

§ 1. **The conjugacy is a Lipschitz continuous mapping on the Cantor set attractor.** Let us consider an even function f of the interval $[-1, 1]$ into itself such that:

1. $f(0) = 1$,
2. $f(x) = f_t(|x|^{1+t})$ where f_t is an analytic mapping bounded in some open domain, $\Omega \supset [0, 1]$, $t = 1$ or t-sufficiently small and $f_t|_{\Omega\cap R}$ is a real function,
3. $x \cdot f'(x) < 0$ for all $x \neq 0$,
4. $0 < f^3(1) < -f(1) < f(2)$.

Doubling operator T is defined as follows:

$$Tf(x) = \frac{1}{f(1)} f^2(f(1)\cdot x)$$

For a survey of results see [1], [5], [6]. For the convenience of the reader we recall some relevant facts about T.

For t chosen as above, there exists a function g_t with negative Schwarzian derivative such that $Tg_t = g_t$. Put $\overline{g_t}$ for appropriate analytic function on some $\Omega \supset [0, 1]$ so as to have Condition 2. fulfilled with $f = g_t$, $f_t = \overline{g_t}$. Given fixed t and Ω like above, consider the Banach space $\boldsymbol{B}$ formed by functions which are bounded and analytic on Ω, real on the set $\Omega \cap \boldsymbol{R}$ and vanishing at 0 to the second order. $\boldsymbol{B}$ is equipped with the natural topology of uniform convergence on compact subsets of Ω. Then $\hat{g_t} = \overline{g_t} - 1$ belongs to $\boldsymbol{B}$ and lies on some codimension 1 submanifold (denoted by $W^s(g_t)$) of a special interest. This manifold consists of functions $\hat{f} = \bar{f} - 1$ where $\bar{f}$ is such that:

(i) for $f(x) = \bar{f}(|x|^{1+t})$ Conditions 1–4 are satisfied and $T^n f$ is inductively defined,

(ii) each $T^n f$ is of the form $T^n f(x) = \overline{T^n f}(|x|^{1+t})$, $\overline{T^n f} - 1$ belongs to $\boldsymbol{B}$,

(iii) $\overline{T^n f} - 1$ converges to $\hat{g_t}$ with exponential rate in $\boldsymbol{B}$,

It will cause no confusion if we say that f itself belongs to $W^s(g_t)$ and continue to write g for g_t everywhere in the sequel.

It is known (see [1], [2]) that for every f in $W^s(g)$ a unique homeomorphism $h\colon [-1, 1] \to [-1, 1]$ conjugates f to g. Furthermore, there exists the Cantor set attractor

$$J(f) = \bigcap_{n\geq 1} J_n(f), \qquad \text{where } J_1(f) = [f(1), f^3(1)] \cup [f^2(1), 1],$$

$$J_n(f) = \Sigma_f^{-1}(J_{n-1}(Tf)) \qquad \text{for } \Sigma_f \text{ defined as follows:}$$

$$\Sigma_f(x) = \begin{cases} \dfrac{x}{f(1)}, & \text{for } x \in [f(1), -f(1)], \\[2ex] \dfrac{f(x)}{f(1)}, & \text{for } x \in [c(f), 1] \end{cases}$$

where $c(f) \in [-f(1), f^2(1)]$ and $f(c(f)) = -f(1)$.

Adopting the notation used in [3] we shall write $[c(f), 1] = L(f)$, $[f^2(1), 1] = K(f)$, $[f(1), -f(1)] = f(L(f))$, $[f(1), f^3(1)] = f(K(f))$, $\mathscr{L}(f) = L(f) \cup f(L(f))$. It is worth mentioning that one can replace $J_1(f)$ by $\mathscr{L}(f)$ in the definition of $J(f)$. Every point from $J(g)$ is an image — under the conjugacy h — of the corresponding point from $J(f)$ with the same code (see the notation below).

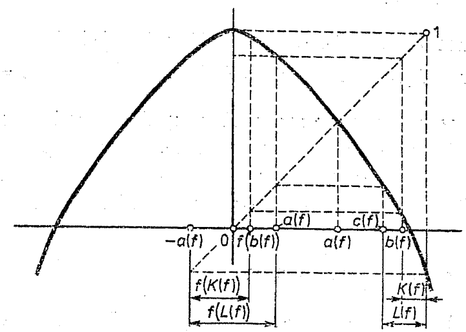

Fig. 1

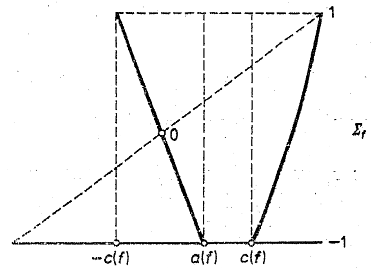

Fig. 2

Here we want to prove that $h|_{J(f)}\colon J(f) \to J(g)$ satisfies the Lipschitz condition with some finite constant $\mathscr{K}$.

We start with some notation and a brief idea of the proof.

A sequence uniformly convergent on compact subsets of Ω converges in the topology of $C^p[-1, 1]$, $p = 0, 1, 2, \ldots$, when restrict functions to $[-1, 1]$. Since $T^n f \to g$ there exists $n_0 \in N$ and $c < 1$ such that for all $n \geq n_0$ $T^n f$ is close enough to g to have

(1.1) $$\inf_{x\in\mathscr{L}(T^n f)} |(\Sigma_{T^n f})'(x)| \geq \frac{1}{c} > 1, \quad \text{i.e.}$$

(1.1′) $$\sup_{x\in[-1,1]} |(\Sigma_{T^n f}^{-1})'(x)| \leq c < 1, \quad \text{on each branch},$$

as it holds for g. Both Cantor sets $J(f)$, $J(g)$ are intersections of descending families of intervals. $J_n(f)$ is a family consisting of 2^n intervals naturally coded by n-element 0-1 sequences. For an interval $V \in J_n(f)$ we define

$$c_0(V) = \begin{cases} 0, & \text{when } V \subset f(L(f)), \\ 1, & \text{when } V \subset L(f), \end{cases}$$

$$c_i(V) = \begin{cases} 0, & \text{when } \Sigma_{T^{i-1}f} \circ \Sigma_{T^{i-2}f} \circ \ldots \circ \Sigma_f(V) \subset T^i f(L(T^i f)), \\ 1, & \text{when } \Sigma_{T^{i-1}f} \circ \Sigma_{T^{i-2}f} \circ \ldots \circ \Sigma_f(V) \subset L(T^i f), \qquad i = 1, 2, \ldots, n-1. \end{cases}$$

Therefore, we can denote every element of $J_n(f)$ by $J_n^{\cdots}(f)$ with its own code as the upper index, and this notation expands to infinite codes for points of $J(f)$. For every interval W from the family $J_{n+1}(f)$, there exists the unique interval V from $J_n(f)$ such that $W \subset V$, so their codes agree on positions $0, 1, \ldots, n-1$ and they differ only in their lengths:

$$c(W) = \{c_0(W), \ldots, c_{n-1}(W), c_n(W)\},$$

$$c(V) = \{c_0(V), \ldots, c_{n-1}(V)\} \quad \text{and} \quad c_i(V) = c_i(W) \quad \text{for } 0 \leqslant i \leqslant n-1.$$

Thus for each such a couple of intervals $W \in J_{n+1}(f)$, $V \in J_n(f)$, $W \subset V$, we continue to write $V = J_n^{c(W)}$ instead of $J_n^{c(V)}$ when no confusion can arise (it means we "forget" the last symbol of $c(W)$ in the case). By the above if $W \subset V$ we have

$$V = J_n^{c(W)}(f) = (\Sigma_f^{-1})^{c_0(W)} \circ (\Sigma_{Tf}^{-1})^{c_1(W)} \circ \ldots \circ (\Sigma_{T^{n-2}f}^{-1})_1^{c_{n-2}(W)}(J_1^{c_{n-1}(W)}(T^{n-1}f)),$$

$$W = J_{n+1}^{c(W)}(f) = (\Sigma_f^{-1})^{c_0(W)} \circ \ldots \circ (\Sigma_{T^{n-1}f}^{-1})^{c_{n-1}(W)}(J_1^{c_n(W)}(T^n f)).$$

Here and subsequently, for any $\varphi \in W^s(g)$, $(\Sigma_\varphi^{-1})^0$ stands for the inverse of $\Sigma_\varphi|_{\varphi(L(\varphi))}$ and $(\Sigma_\varphi^{-1})^1$ for the inverse of $\Sigma_\varphi|_{L(\varphi)}$. Likewise, $J_1^0(\varphi) = \varphi(L(\varphi))$ and $J_1^1(\varphi) = L(\varphi)$. We shall be investigating the ratio of lengths

$$\frac{|J_{n+1}^{c(W)}(f)|}{|J_n^{c(W)}(f)|}$$

and the one for corresponding (having the same codes) intervals from $J_{n+1}(g)$, $J_n(g)$

$$\frac{|J_{n+1}^{c(W)}(g)|}{|J_n^{c(W)}(g)|}.$$

Our procedure will be to compare both the ratios. We claim that for sufficiently large n they are "almost equal". This "almost equality" can be expressed as the upper estimation (depending only on n, not on the code) for the cross-ratio

$$\frac{|J_{n+1}^{c(W)}(f)|}{|J_n^{c(W)}(f)|} \div \frac{|J_{n+1}^{c(W)}(W)(g)|}{|J_n^{c(W)}(g)|}$$

by a factor of order

$$\left(1 + \frac{\varepsilon}{2\left[\frac{n}{r-1}\right]}\right)\left(1 + a^{-\left[\frac{n}{r-1}\right]}\right),$$

where ε, a, r are some positive constants; $a, r > 1$. In the end, we will find a lower bound L of the ratio of measure of an arbitrarily chosen interval $W = J_{m+1}^{c(W)}(f)$ from the $(m+1)$th family to the length of $J_m^{c(W)}(f)$. L occurs to be greater then 0 and uniform both regarding the code and the number m. That is all we need to obtain the desired conclusion, as the same is true for gaps.

Actually, if we take $x, y \in J(f)$, $\bar{x}, \bar{y} \in J(g)$, $\bar{x} = h(x)$, $\bar{y} = h(y)$ and codes for x, y differ for the first time in the mth position, then both belong to the same interval from $J_m(f)$ and to two different intervals from $J_{m+1}(f)$. But the length of any interval from $J_m(f)$ differs from the length of corresponding interval from $J_m(g)$ by a factor non-greater then

$$\prod_{n=1}^{m}\left(1 + \frac{\varepsilon}{2\left[\frac{n}{r-1}\right]}\right)\left(1 + a^{-\left[\frac{n}{r-1}\right]}\right) \leqslant \prod_{n=1}^{\infty}\left(1 + \frac{\varepsilon}{2\left[\frac{n}{r-1}\right]}\right)\left(1 + a^{-\left[\frac{n}{r-1}\right]}\right) \leqslant K$$

for some finite constant K. Accordingly, we have $\operatorname{dist}(\bar{x}, \bar{y}) \leqslant \frac{K}{L} \operatorname{dist}(x, y)$ and constants K, L do not depend on codes. Analogously we can estimate $\operatorname{dist}(x, y) \leqslant \text{constant} \cdot \operatorname{dist}(\bar{x}, \bar{y})$. So we claim the following:

THEOREM 1. *Let f be any function such that $T^n f \xrightarrow[n\to\infty]{} g$ exponentially in the space $C^1[-1, 1]$ and the absolute values of the second derivatives $|f''|$, $|(Tf)''|$, $|(T^2 f)''|$, ... are all bounded by the same finite constant $\mathscr{M}$ (for example, $f \in W^s(g)$).*

If h is a homeomorphism such that $g \circ h = h \circ f$ then $h|_{J(f)}$ is a Lipschitz continuous function; also $h^{-1}|_{J(g)}$ is Lipschitz continuous.

Proof. Throughout the proof we shall deal with the case of f close to g; if it is not the case, one can take $\hat{f} = T^{n_0} f$ with some large n_0. The general case will be consider at the end. Due to the definition of Σ for f sufficiently close to g we have inequalities (1.1), (1.1′) satisfied for all non-negative integers n, and there exist $0 < \beta < 1$, $0 < \gamma < 1$, such that

$$\sup_{W \in J_2(f)} \frac{|J_2^{c(W)}(f)|}{|J_2^{c(W)}(g)|} \leqslant 1 + \gamma \quad \text{and} \quad 1 - \beta \leqslant \inf_{V \in J_1(f)} \frac{|J_1^{c(V)}(f)|}{|J_1^{c(V)}(g)|}.$$

Then for $W \in J_2(f)$

$$\frac{|J_2^{c(W)}(f)|}{|J_1^{c(W)}(f)|} \div \frac{|(J_2^{c(W)}(g)|}{|(J_1^{c(W)}(g)|} \leqslant \frac{1+\gamma}{1-\beta}.$$

Set $1 + \varepsilon_0 = \dfrac{1+\gamma}{1-\beta}$. We are now led to assert the main lemma.

LEMMA 1.1. *There exists a positive integer r such that for each $k \geq 1$ the inequality*

$$\frac{|J_{k+1}^{c(W)}(T^{(k-1)r-k+1}f)|}{|J_k^{c(W)}(T^{(k-1)r-k+1}f)|} \div \frac{|J_{k+1}^{c(W)}(g)|}{|J_k^{c(W)}(g)|} \leq 1+\frac{\varepsilon_0}{2^{k-1}} \tag{1.2}$$

holds with any W from $J_{k+1}(T^{(k-1)r-k+1}f)$.

The point of the lemma is that the estimate (1.2) does not depend on the code.

Proof of the lemma. By the definition of ε_0 the case of $k = 1$ is immediate, so we take $k \geq 2$. $T^n f$ converges to g exponentially, so one can choose $0<\delta<1$ and a positive constant B so as to have

$$\operatorname{dist}((T^n f)^i(1), g^i(1)) \leq B\cdot\delta^n, \quad \text{for } i = 0, 1, 2, 3,\ n = 0, 1, 2, \ldots \tag{1.3}$$

and

$$\|\Sigma_{T^n f}^{-1}-\Sigma_g^{-1}\|_{\sup} \leq B\cdot\delta^n \quad \text{on each branch.} \tag{1.4}$$

Observe that having $\operatorname{dist}(x, y) \leq A$ yields

$$\operatorname{dist}(\Sigma_{T^n f}^{-1}(x), \Sigma_g^{-1}(y)) \leq A+B\cdot\delta^n. \tag{1.5}$$

It is clear from (1.1′) and (1.4) because

$$\operatorname{dist}(\Sigma_{T^n f}^{-1}(x), \Sigma_g^{-1}(y)) \leq \operatorname{dist}(\Sigma_{T^n f}^{-1}(x), \Sigma_{T^n f}^{-1}(y))+\operatorname{dist}(\Sigma_{T^n f}^{-1}(y), \Sigma_g^{-1}(y))$$

and

$$\operatorname{dist}(\Sigma_{T^n f}^{-1}(x), \Sigma_{T^n f}^{-1}(y)) \leq \operatorname{dist}(x, y)$$

as $|(\Sigma_{T^n f}^{-1})'| < 1$ and also

$$\operatorname{dist}(\Sigma_{T^n f}^{-1}(y), \Sigma_g^{-1}(y)) \leq \|\Sigma_{T^n f}^{-1}-\Sigma_g^{-1}\|_{\sup}.$$

For an arbitrarily chosen positive integer r and an interval W from $J_{k+1}(T^{(k-1)r-k+1}f)$ we have

$$J_{k+1}^{c(W)}(T^{(k-1)r-k+1}f) = (\Sigma_{T^{(k-1)r-k+1}f}^{-1})^{c_0(W)} \circ \ldots \circ (\Sigma_{T^{(k-1)r}f}^{-1})^{c_{k-1}(W)}(J_1^{c_k(W)}(T^{(k-1)r+1}f)),$$

$$J_k^{c(W)}(T^{(k-1)r-k+1}f) = (\Sigma_{T^{(k-1)r-k+1}f}^{-1})^{c_0(W)} \circ \ldots \circ (\Sigma_{T^{(k-1)r-1}f}^{-1})^{c_{k-2}(W)}(J_1^{c_{k-1}(W)}(T^{(k-1)r}f)).$$

Let us consider an endpoint of an interval from $J_1(T^{(k-1)r+1}f)$ and the corresponding endpoint of the interval from $J_1(g)$. By inequality (1.3) and k-times repeated application of (1.5) we obtain

$$\operatorname{dist}\big((\Sigma_{T^{(k-1)r-k+1}f}^{-1})^{c_0(W)} \circ \ldots \circ (\Sigma_{T^{(k-1)r}f}^{-1})^{c_{k-1}(W)}((T^{(k-1)r+1}f)^i(1)),$$
$$(\Sigma_g^{-1})^{c_0(W)} \circ \ldots \circ (\Sigma_g^{-1})^{c_{k-1}(W)}(g^i(1))\big)$$
$$\leq B\cdot\delta^{(k-1)r+1}+\ldots+B\cdot\delta^{(k-1)r-(k-1)} \quad \text{for } i = 0, 1, 2, 3.$$

Thus, considering both endpoints, we have

$$|J_{k+1}^{c(W)}(T^{(k-1)r-k+1}f)| \leq J_{k+1}^{c(W)}(g)+2B(\delta^{(k-1)r-(k-1)}+\ldots+\delta^{(k-1)r+1}).$$

Similarly,

$$|J_k^{c(W)}(T^{(k-1)r-k+1}f)| \geq J_k^{c(W)}(g)-2B(\delta^{(k-1)r-(k-1)}+\ldots+\delta^{(k-1)r}).$$

Set

$$\frac{1}{\nu} = \sup_{n\geq 0}\Big(\sup_{x\in\mathscr{P}(T^n f)} |(\Sigma_{T^n f})'|\Big).$$

Then $|J_{k+1}^{c(W)}(g)| \geq 2\cdot\nu^{k+1}$ and $|J_k^{c(W)}(g)| \geq 2\cdot\nu^k$ as $|[-1, 1]| = 2$. Therefore,

$$\frac{|J_{k+1}^{c(W)}(T^{(k-1)r-(k-1)}f)|}{|J_k^{c(W)}(T^{(k-1)r-(k-1)}f)|} \div \frac{|J_{k+1}^{c(W)}(g)|}{|J_k^{c(W)}(g)|} \leq \frac{1+\frac{B}{\nu^{k+1}}\delta^{(k-1)r-(k-1)}(1+\delta+\ldots+\delta^{k+1})}{1-\frac{B}{\nu^k}\delta^{(k-1)r-(k-1)}(1+\delta+\ldots+\delta^k)}$$
$$\leq \frac{1+\frac{C}{\nu^{k+1}}\cdot\delta^{(k-1)r-k+1}}{1-\frac{C}{\nu^k}\cdot\delta^{(k-1)r-k+1}} \quad \text{with } C = B\cdot\sum_{i=0}^{\infty}\delta^i$$

provided that the denominator on the right-hand side of the inequality is positive. Particularly, we may choose r so as to have the following:

$$0< \frac{1+\frac{C}{(1-\delta)\nu^3}\cdot\delta^{r-1}}{1-\frac{C}{\nu^2}\cdot\delta^{r-1}} \leq 1+\frac{\varepsilon_0}{2} \tag{1.6}$$

and

$$\frac{1}{\nu}\cdot\delta^{r-1} < \frac{1}{4}. \tag{1.7}$$

After these preparations we proceed by induction.

For $k = 2$ we have insured the statement of our lemma on account of (1.6). Assuming (1.2) to hold for some $k \geq 2$, we have for $k+1$

$$\frac{1+\frac{C}{\nu^{k+2}}\cdot\delta^{kr-k}}{1-\frac{C}{\nu^{k+1}}\cdot\delta^{kr-k}} = \frac{1+\left(\frac{C}{\nu^{k+1}}\cdot\delta^{(k-1)r-k+1}\right)\cdot\frac{1}{\nu}\cdot\delta^{r-1}}{1-\left(\frac{C}{\nu^k}\cdot\delta^{(k-1)r-k+1}\right)\cdot\frac{1}{\nu}\cdot\delta^{r-1}}$$
$$\leq \frac{1+\frac{1}{4}\left(\frac{C}{\nu^{k+1}}\cdot\delta^{(k-1)r-k+1}\right)}{1-\frac{1}{4}\left(\frac{C}{\nu^k}\cdot\delta^{(k-1)r-k+1}\right)}.$$

But it is easy to verify that for positive a, b $a, b < 1$ the inequality $\frac{1+a}{1-b} \leqslant 1+\varepsilon$ implies $\frac{1+\frac{1}{4}a}{1-\frac{1}{4}b} \leqslant 1+\frac{\varepsilon}{2}$ and so we are done. ■

Now we shall prove another lemma; it is an easy and very useful one.

LEMMA 1.2. *Suppose that I, J are intervals such that $J \subset I$ and $\varphi\colon I \to \mathbf{R}$ is a mapping of class C^2, with its first derivative separated from 0 $|\varphi'| \geqslant \eta > 0$, and second derivative bounded by some finite constant $|\varphi''| \leqslant \alpha < +\infty$.*

Then

$$e^{-\lambda\frac{\alpha}{\eta}} \leqslant \frac{|\varphi(J)|}{|\varphi(I)|} \div \frac{|J|}{|I|} \leqslant e^{\lambda\frac{\alpha}{\eta}}. \tag{1.8}$$

Proof of the lemma. For any $x, y \in I$ we have

$$|\log|\varphi'(x)| - \log|\varphi'(y)|| \leqslant \frac{\alpha}{\eta}|x-y|, \quad \text{as} \quad (\log|\varphi'|)' = \frac{|\varphi''|}{|\varphi'|} \leqslant \frac{\alpha}{\eta}.$$

Thus

$$\sup_{z\in I}|\varphi'(z)| \leqslant e^{\lambda\frac{\alpha}{\eta}} \cdot \inf_{z\in I}|\varphi'(z)|$$

and obviously

$$\inf_{z\in I}|\varphi'(z)| \leqslant \frac{|\varphi(I)|}{|I|} \leqslant \sup_{z\in I}|\varphi'(z)|.$$

Also

$$\inf_{z\in I}|\varphi'(z)| \leqslant \frac{|\varphi(J)|}{|J|} \leqslant \sup_{z\in I}|\varphi'(z)|,$$

so

$$e^{-\lambda\frac{\alpha}{\eta}} \leqslant \frac{|\varphi(J)|}{|\varphi(I)|} \div \frac{|J|}{|I|} \leqslant e^{\lambda\frac{\alpha}{\eta}},$$

which is the desired conclusion. ■

Now consider a positive integer n of the form $n = r\cdot k$. From the proof of Lemma 1.1 it is clear that

$$\frac{|J_{k+1}^{c(W)}(T^{rk-k}f)|}{|J_k^{c(W)}(T^{rk-k}f)|} \div \frac{|J_{k+1}^{c(W)}(g)|}{|J_k^{c(W)}(g)|} \leqslant 1+\frac{\varepsilon_0}{2^{k-1}}$$

— independently on code of an interval W from the family $J_{k+1}(T^{rk-k}f)$ — since $T^{rk-k}f$ is closer to g than $T^{(k-1)r-k+1}f$.

Our next purpose is to compare the cross-ratio

$$\frac{|J_{n+1}^{c(W)}(f)|}{|J_n^{c(W)}(f)|} \div \frac{|J_{n+1}^{c(W)}(g)|}{|J_n^{c(W)}(g)|} \tag{$*$}$$

with the one

$$\frac{|J_{k+1}^{\widehat{c(W)}}(T^{n-k}f)|}{|J_k^{\widehat{c(W)}}(T^{n-k}f)|} \div \frac{|J_{k+1}^{\widehat{c(W)}}(g)|}{|J_k^{\widehat{c(W)}}(g)|}. \tag{$**$}$$

Here W is an interval from $J_{n+1}(f)$ and $\widehat{c(W)}$ in ($**$) is the code of W when restricted to the last $k+1$ symbols, so as to have

$$J_{n+1}^{c(W)}(f) = (\Sigma_f^{-1})^{c_0(W)} \circ \ldots \circ (\Sigma_{T^{n-k-1}f}^{-1})^{c_{n-k-1}(W)}(J_{k+1}^{\widehat{c(W)}}(T^{n-f}f))$$

and

$$J_n^{c(W)}(f) = (\Sigma_f^{-1}) \circ \ldots \circ (\Sigma_{T^{n-k-1}f}^{-1})^{c_{n-k-1}(W)}(J_k^{\widehat{c(W)}}(T^{n-k}f)).$$

The meaning of $J_n^{c(W)}(\ldots)$ where $c(W)$ has $n+1$ symbols, and of $J_k^{\widehat{c(W)}}(\ldots)$ where $\widehat{c(W)}$ has $k+1$ symbols was explained before.

Now our idea is to replace acting of Σ^{-1} in the terms above by the linearized maps. Recall that intervals in question are not only short but their lengths grows exponentially shorter — with coefficient at least as c given in (1, 1′) — at succeeding steps; it makes the thing reasonable. Define

$$\beta = \frac{\max\left(\sup_n\left(\sup_{x\in[-1,1]}|(\Sigma_{T^nf}^{-1})''(z)|\right), \sup_{z\in[-1,1]}|(\Sigma_g^{-1})''(z)|\right)}{\nu}.$$

It is tacitly understood that both branches of Σ^{-1} are submitted in the definition; ν is as in the proof of Lemma 1.1.

The qth family of the form $J_q(\ldots)$ consists of 2^q intervals and by (1.1′) each one has the length non greater then $2c^q$.

Thus $(r-1)k$-times repeated application of Lemma (1.2) enables us to write

$$\begin{aligned}\frac{|J_{n+1}^{c(W)}(f)|}{|J_n^{c(W)}(f)|} &\leqslant \frac{|J_{k+1}^{\widehat{c(W)}}(T^{n-k}f)|}{|J_k^{\widehat{c(W)}}(T^{n-k}f)|} \cdot e^{2\beta c^k} \ldots e^{2\beta c^{rk-1}} \\ &\leqslant \frac{|J_{k+1}^{\widehat{c(W)}}(T^{n-k}f)|}{|J_k^{\widehat{c(W)}}(T^{n-k}f)|} \cdot \prod_{q=k}^{\infty} e^{2\beta c^q} = \frac{|J_{k+1}^{\widehat{c(W)}}(T^{n-k}f)|}{|J_k^{\widehat{c(W)}}(T^{n-k}f)|} \cdot e^{2\beta Rc^k}\end{aligned} \tag{1.9}$$

for $R = \frac{c^q}{1-c}$.

Likewise

$$\frac{|J_{n+1}^{c(W)}(f)|}{|J_n^{c(W)}(f)|} \geqslant \frac{|J_{k+1}^{\widehat{c(W)}}(T^{n-k}f)|}{|J_k^{\widehat{c(W)}}(T^{n-k}f)|} \cdot e^{-2\beta Rc^k}. \tag{1.10}$$

For g we have as well

$$e^{-2\beta Rc^k}\frac{|J_{k+1}^{\widehat{c(W)}}(g)|}{|J_k^{\widehat{c(W)}}(g)|} \leqslant \frac{|J_{n+1}^{c(W)}(g)|}{|J_n^{c(W)}(g)|} \leqslant \frac{|J_{k+1}^{\widehat{c(W)}}(g)|}{|J_k^{\widehat{c(W)}}(g)|} \cdot e^{2\beta Rc^k}. \tag{1.11}$$

Combining (1.9), (1.11) and (1.2), we obtain

$$\text{(1.12)}\qquad \frac{|J_{n+1}^{c(W)}(f)|}{|J_{n}^{c(W)}(f)|} \div \frac{|J_{n+1}^{c(W)}(g)|}{|J_{n}^{c(W)}(g)|} \leqslant \frac{|\widehat{J_{k+1}^{c(W)}}(T^{n-k}f)|}{|\widehat{J_{k}^{c(W)}}(T^{n-k}f)|} \div \frac{|\widehat{J_{k+1}^{c(W)}}(g)|}{|\widehat{J_{k}^{c(W)}}(g)|} \cdot e^{4\beta Rc^k}$$

$$\leqslant \left(1+\frac{\varepsilon_0}{2^{k-1}}\right)\cdot e^{4\beta Rc^k}.$$

This estimate does not depend on code of W. Because we replaced a finite number of factors by the infinite product, we also have

$$\text{(1.12')}\qquad \frac{|J_{rk+l}^{c(W)}(f)|}{|J_{rk+l-1}^{c(W)}(f)|} \div \frac{|J_{rk+l}^{c(W)}(g)|}{|J_{rk+l-1}^{c(W)}(g)|} \leqslant \left(1+\frac{\varepsilon_0}{2^{k-1}}\right)\cdot e^{4\beta Rc^k}$$

for all positive integers l and for any W from $(rk+l)$th family.

Particularly, (1.12′) is satisfied with $l = 1, 2, \ldots, r$.

For $l = r+1$ we drop the estimate (1.12′) and rewrite (1.12) with $n = r(k+1)$ which gives the better estimation

$$\text{(1.12'')}\qquad \frac{|J_{r(k+1)+1}^{c(W)}(f)|}{|J_{r(k+1)}^{c(W)}(f)|} \div \frac{|J_{r(k+1)+1}^{c(W)}(g)|}{|J_{r(k+1)}^{c(W)}(g)|} \leqslant \left(1+\frac{\varepsilon_0}{2^{k}}\right)\cdot e^{4\beta Rc^{k+1}}.$$

We are now in a position to consider the general case of an arbitrary positive integer $n = r\cdot m+s$ $0 \leqslant s \leqslant r-1$. For any V from $(n+1)$th family, proceeding step by step from (1.12) through (1.12′) to (1.12″), we can obtain

$$\frac{|J_{n+1}^{c(V)}(f)|}{|J_{n+1}^{c(V)}(g)|} = \frac{|J_{n+1}^{c(V)}(f)|}{|J_{n}^{c(V)}(f)|} \div \frac{|J_{n+1}^{c(V)}(g)|}{|J_{n}^{c(V)}(g)|} \cdot \frac{|J_{n}^{c(V)}(f)|}{|J_{n}^{c(V)}(g)|} = \ldots$$

$$\ldots = \left(\frac{|J_{n+1}^{c(V)}(f)|}{|J_{n}^{c(V)}(f)|} \div \frac{|J_{n+1}^{c(V)}(g)|}{|J_{n}^{c(V)}(g)|}\right)\ldots\left(\frac{|J_{2}^{c(V)}(f)|}{|J_{1}^{c(V)}(f)|} \div \frac{|J_{2}^{c(V)}(g)|}{|J_{1}^{c(V)}(g)|}\right)\cdot\frac{|J_{1}^{c(V)}(f)|}{|J_{1}^{c(V)}(g)|}$$

$$\leqslant \underbrace{\left[\left(e^{4\beta Rc^m}\cdot\left(1+\frac{\varepsilon_0}{2^{m-1}}\right)\right)\ldots\left(e^{4\beta Rc^m}\cdot\left(1+\frac{\varepsilon_0}{2^{m-1}}\right)\right)\right]}_{s\text{-times}} \times$$

$$\times \underbrace{\left[\left(e^{4\beta Rc^{m-1}}\cdot\left(1+\frac{\varepsilon_0}{2^{m-2}}\right)\right)\ldots\left(e^{4\beta Rc^{m-1}}\cdot\left(1+\frac{\varepsilon_0}{2^{m-2}}\right)\right)\right]}_{r\text{-times}}\ldots$$

$$\ldots \underbrace{\left[\left(e^{4\beta Rc}\cdot(1+\varepsilon_0)\right)\ldots\left(e^{4\beta Rc}\cdot(1+\varepsilon_0)\right)\right]}_{r\text{-times}} \times$$

$$\times \frac{2}{\min\big(|K(f)|,\,|f(K(f))|,\,|K(g)|,\,|g(K(g))|\big)}$$

$$\leqslant \frac{2}{\min\big(|K(f)|,\,|f(K(f))|,\,|K(g)|,\,|g(K(g))|\big)}$$

$$\times \prod_{m=1}^{\infty}\left(1+\frac{\varepsilon_0}{2^{m-1}}\right)^r (e^{4\beta Rc^m})^r = \hat{\mathscr{K}} < +\infty.$$

Of course $\hat{\mathscr{K}}$ depends on f.

Obviously, a lower estimate analogous to Lemma 1.1 allows us to prove in the same way that

$$\frac{|J_{n+1}^{c(V)}(f)|}{|J_{n+1}^{c(V)}(g)|} \geqslant \hat{\hat{\mathscr{K}}} > 0,$$

Now we drop the assumption that f is close to g. Taking any f and n_0 large enough the proof goes with $\hat{f} = T^{n_0}f$. For any $V \in J_{n+1}(f)$ and $n \geqslant n_0$ we have

$$\Sigma_{T^{n_0-1}f}\circ\ldots\circ\Sigma_f(J_{n+1}^{c(V)}(f)) \in J_{n-n_0+1}(T^{n_0}f),$$

$$\underbrace{\Sigma_g\circ\ldots\circ\Sigma_g}_{n_0\text{-times}}(J_{n+1}^{c(V)}(g)) \in J_{n-n_0+1}(g).$$

Consequently,

$$\frac{|\Sigma^{T^{n_0-1}f}\circ\ldots\circ\Sigma_f(J_{n+1}^{c(V)}(f))|}{\Sigma_g\circ\ldots\circ\Sigma_g(J_{n+1}^{c(V)}(g))|} \leqslant \hat{\mathscr{K}},$$

and so

$$\frac{|J_{n+1}^{c(V)}(f)|}{|J_{n+1}^{c(V)}(g)|} \leqslant \hat{\mathscr{K}}\cdot P,\qquad P = \left[\frac{\sup\limits_{z\in\mathscr{L}(g)}|(\Sigma_g)'(z)|}{\min\limits_{0\leqslant i\leqslant n_0-1}\big(\inf\limits_{z\in\mathscr{L}(T^if)}|(\Sigma_{T^if})'(z)|\big)}\right]^{n_0}.$$

Analogously,

$$\frac{|J_{n+1}^{c(V)}(f)|}{|J_{n+1}^{c(V)}(g)|} \geqslant \hat{\hat{\mathscr{K}}}\cdot\tilde{P},\qquad \tilde{P} = \left[\frac{\inf\limits_{z\in\mathscr{L}(g)}|(\Sigma_g)'(z)|}{\max\limits_{0\leqslant i\leqslant n_0-1}\big(\sup\limits_{z\in\mathscr{L}(T^if)}|(\Sigma_{T^if})'(z)|\big)}\right]^{n_0}.$$

For $n \leqslant n_0$ we have an obvious estimate

$$\tilde{P}_1 \leqslant \frac{|J_{n}^{c(V)}(f)|}{|J_{n}^{c(V)}(g)|} \leqslant P_1 \quad\text{with}$$

$$P_1 = P\cdot\max_{0\leqslant i\leqslant n_0-1}\left(\max\left(\frac{|K(T^if)|}{|K(g)|},\ \frac{|T^if(K(T^if))|}{|g(K(g))|}\right)\right),$$

$$\tilde{P}_1 = \tilde{P}\cdot\min_{0\leqslant i\leqslant n_0-1}\left(\min\left(\frac{|K(T^if)|}{|K(g)|},\ \frac{|T^if(K(T^if))|}{|g(K(g))|}\right)\right)$$

because for any $V \in J_n(f)$

$$\text{(1.13)}\qquad \Sigma_{T^{n-2}f}\circ\ldots\circ\Sigma_f(J_n^{c(V)}(f)) \in J_1(T^{n-1}f) \quad\text{and}\quad \underbrace{\Sigma_g\circ\ldots\circ\Sigma_g}_{(n-1)\text{-times}}(J_n^{c(V)}(g)) \in J_1(g).$$

Hence, setting $\hat{\mathscr{K}}_1 = \max(\hat{\hat{\mathscr{K}}}\cdot P, P_1)$, $\tilde{\mathscr{K}}_1 = \min(\hat{\hat{\mathscr{K}}}\cdot \tilde{P}, \tilde{P}_1)$, we obtain

$$0<\tilde{\mathscr{K}}_1<\frac{|J_n^{c(V)}(f)|}{|J_n^{c(V)}(g)|}<\hat{\mathscr{K}}_1<+\infty \quad \text{for each } n.$$

Finally, we have to estimate the minimal value L of the ratio of length of an interval "removed" at any consecutive step of forming the Cantor set, to the length of its predecessor (i.e. the interval including the removed one). We want this ratio to be separated from 0. But it is an easy consequence of inductive using of Lemma 1.2 (the bounded distortion in other words).

Put

$$\hat{L}=\inf_{i\geqslant 0}\left(\frac{|[T^i f(1),1]|-|K(T^i f)|-|T^i f(K(T^i f))|}{|[T^i f(1),1]|}\right).$$

By assumptions of Theorem 1 all $\Sigma_{T^i f}^{-1}$ $i=0,1,2,\ldots$ and Σ_g^{-1} fulfil the hypothesis on φ from Lemma 1.2 with some common constants η, α (as usual we consider both branches). Since the maximal length of an interval from $J_q(f)$ or $J_q(g)$ is not greater than $2\cdot c^q$, thus due to (1.13) and Lemma 1.2 we at once obtain

$$L\geqslant \hat{L}\cdot\prod_{q=1}^{\infty} e^{-\frac{\alpha}{\eta}\cdot 2c^q}>0.$$

For any $x, y\in J(f)$ there exists n such that x, y belong to the same interval V from $J_n(f)$ and to two different intervals from $J_{n+1}(f)$.

Thus

$$\operatorname{dist}(x,y)\leqslant\frac{\hat{\mathscr{K}}_1}{L}\cdot\operatorname{dist}(h(x),h(y)), \quad \text{and}$$

$$\operatorname{dist}(x,y)\geqslant\tilde{\mathscr{K}}_1\cdot L\cdot\operatorname{dist}(h(x),h(y)),$$

which proves the theorem. ■

Remark 1. A slight change in the proof allows us to show the theorem under weaker hypothesis on smoothness. Nevertheless, it is not worth doing, as Feigenbaum's theory is developed just for maps of at least C^2-class of smoothness.

§ 2. **Complementary results.** We have shown above that $h|_{J(f)}$ is a Lipschitz continuous mapping. Actually we can sharpen the statement of Theorem 1 as follows:

THEOREM 2. *There exists a finite constant K such that for each $x\in J(f)$, $y\in[-1,1]$ the inequality*

(2.1) $$\operatorname{dist}(h(x),h(y))\leqslant K\cdot\operatorname{dist}(x,y)$$

holds. The same is satisfied for h^{-1}, $x\in J(g)$.

Theorem 2 states that h regarded as a function defined on the whole interval $[-1,1]$ satisfies the Lipschitz condition at all points of the Cantor set attractor

with common constant K. We carry out the proof under hypothesis that x is an endpoint of some interval from $J_n(f)$ (i.e. $x=f^l(1)$ for some $l\in N$) and y lies inside of one of the open intervals from the family forming $[-1,1]\setminus J(f)$. Then the general conclusion is a consequence of this case and of Theorem 1; it is enough to take the point z from $J(f)$, $z\in[x,y]$ such that $\operatorname{dist}(z,y)$ is minimal and then apply the Lipschitz condition for couples (x,z) and (z,y). Besides it is convenient to assume that we start with f sufficiently close to g. Similarly to that, in the previous part it does not cause any loss of generality and allows us to facilitate the formulation of steps.

We denote a branch of the inverse of f by $f^{-1,+}$ when the preimage falls on the right-hand side of 0, and by $f^{-1,-}$ otherwise.

For f^{2^m} we restrict the mapping to the "basis" interval $[f^{2^m-1}(1), -f^{2^m-1}(1)]$, i.e. the one on which f^{2^m} is linearly equivalent to $T^m f$.

From now on we mean $f^{2^m}|_{[f^{2^m-1}(1),-f^{2^m-1}(1)]}$ while writting f^{2^m}.

We denote its inverse branches by $(f^{-2^m})^-$, $(f^{-2^m})^+$ according to the same rule as for f for even m and vice-versa for m odd.

We shall define a class of families of intervals denoted $z_i(f^{2^j}, x)$, or in other way by $a_r(f^{2^q})$, $b_p(f^{2^q})$, $c_p(f^{2^q})$.

At first we set:

$$a_0(f)=z_0(f,f^3(1))=(f^3(1),f^2(1))=z_0(f,f^2(1))=b_0(f),$$
$$c_0(f)=z_0(f,f(1))=[-1,f(1)],$$
$$a_1(f)=z_1(f,f^3(1))=[f^3(1),-f(1)],$$
$$b_1(f)=z_1(f,f^2(1))=(f^{-1,+})(a_1(f)),$$
$$c_1(f)=z_1(f,f(1))=(f^{-1,-})(b_1(f)).$$

The equality $T^n f=\dfrac{1}{f^{2^n-1}(1)}\cdot f^{2^n}(f^{2^n-1}(1)\cdot x)$ provides a linear equivalence of $T^n f$ to f^{2^n}.

Set $R_n(f)(x)=f^{2^n-1}(1)\cdot x$.

Consider $a_1(Tf)$, $b_1(Tf)$, $c_1(Tf)$. We have

$$f(1)=R_1(f)(1),$$
$$f^3(1)=R_1(f)(Tf(1)),$$
$$f^5(1)=R_1(f)((Tf)^2(1)),$$
$$f^7(1)=R_1(f)((Tf)^3(1)).$$

Then we define (see Figure 3)

$$a_0(f^2)=z_0(f^2,f^7(1))=R_1(f)(a_0(Tf))=R_1(f)(b_0(Tf))=z_0(f^2,f^5(1))=b_0(f^2),$$
$$c_0(f^2)=z_0(f^2,f^3(1))=R_1(f)(c_0(Tf)),$$

$$a_1(f^2) = z_1(f^2, f^7(1)) = R_1(f)(a_1(Tf)),$$
$$b_1(f^2) = z_1(f^2, f^5(1)) = R_1(f)(b_1(Tf)),$$
$$c_1(f^2) = z_1(f^2, f^3(1)) = R_1(f)(c_1(Tf)).$$

As $z_1(f, f^3(1))$ was defined before we consecutively denote

$$a_2(f) = z_2(f, f^3(1)) = c_1(f^2),$$
$$b_2(f) = z_2(f, f^2(1)) = (f^{-1,+})(a_2(f)),$$
$$c_2(f) = z_2(f, f(1)) = (f^{-1,-})(b_2(f)).$$

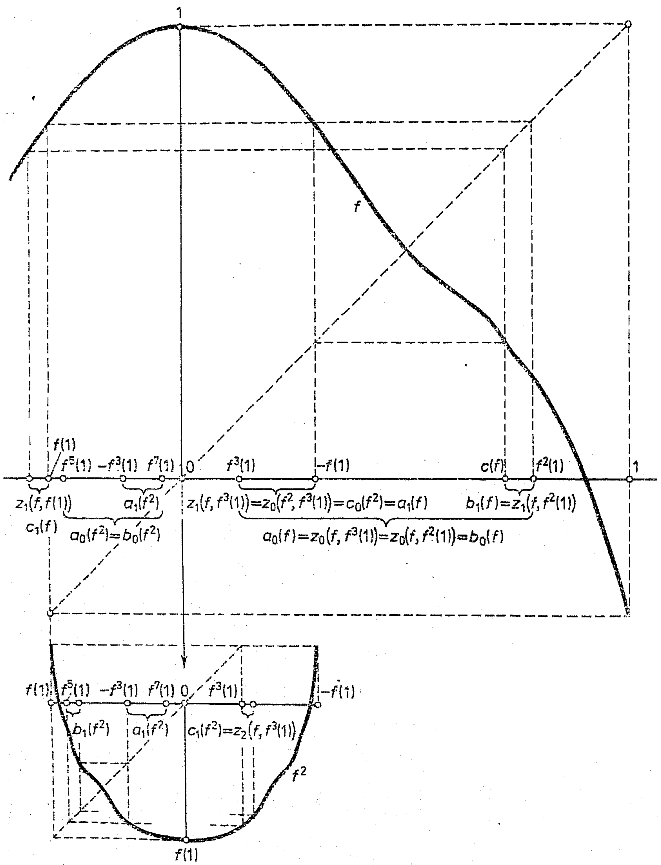

Fig. 3

In this way we can inductively define

$$a_n(f^2) = z_n(f^2, f^7(1)) = R_1(f)(a_n(Tf)),$$
$$b_n(f^2) = z_n(f^2, f^5(1)) = R_1(f)(b_n(Tf)),$$
$$c_n(f^2) = z_n(f^2, f^3(1)) = R_1(f)(c_n(Tf)),$$

$$a_{n+1}(f) = z_{n+1}(f, f^3(1)) = c_n(f^2),$$
$$b_{n+1}(f) = z_{n+1}(f, f^2(1)) = (f^{-1,+})(a_{n+1}(f)),$$
$$c_{n+1}(f) = z_{n+1}(f, f(1)) = (f^{-1,-})(b_{n+1}(f))$$

and furthermore

$$a_n(f^{2^m}) = z_n(f^{2^m}, f^{2^{m+2}-1}(1)) = R_m(f)(a_n(T^m f)),$$
$$b_n(f^{2^m}) = z_n(f^{2^m}, f^{3\cdot 2^m-1}(1)) = R_m(f)(b_n(T^m f)),$$
$$c_n(f^{2^m}) = z_n(f^{2^m}, f^{2^{m+1}-1}(1)) = R_m(f)(c_n(T^m f)).$$

Of course, we always have

$$b_n(f^{2^m}) = (f^{-2^m})^+(a_n(f^{2^m})),$$
$$c_n(f^{2^m}) = (f^{-2^m})^-(b_n(f^{2^m})),$$
$$a_{n+1}(f^{2^{m-1}}) = c_n(f^{2^m}).$$

Therefore we are able to introduce the following notation:

$$z_n(f^{2^{m+2}-1}(1)) \quad \text{for } a_n(f^{2^m}),$$
$$z_n(f^{3\cdot 2^m-1}(1)) \quad \text{for } b_n(f^{2^m}),$$
$$z_n(f(1)) = z_n(f, f(1)).$$

Let now l be any positive integer, greater than 3. There exists $m \geqslant 1$ such that $2^{m+1}-1 < l \leqslant 2^{m+2}-1$ so

$$2^{m+1} \leqslant l \leqslant 3\cdot 2^m - 1 \quad \text{or} \quad 3\cdot 2^m \leqslant l \leqslant 2^{m+2}-1.$$

Put $\bar{l}$ for $(3\cdot 2^m - 1)$ in the first case and $\bar{l} = 2^{m+2}-1$ in the other.

Then $l = \bar{l} - \varepsilon_0\cdot 2^0 - \varepsilon_1\cdot 2^1 - \ldots - \varepsilon_{m-1}\cdot 2^{m-1}$ where $\varepsilon_{m-1}\varepsilon_{m-2}\ldots\varepsilon_0$ is the binary representation of the number $\bar{l}-l$ (each term ε_i, $0 \leqslant i \leqslant m-1$, is either 0 or 1). Therefore, to every l, $2^{m+1} \leqslant l \leqslant 2^{m+2}-1$, there corresponds a unique binary number of m figures $\varepsilon_{m-1}\ldots\varepsilon_0$, so $(f^{2^{m-1}\cdot\varepsilon_{m-1}})\circ\ldots\circ(f^{2^0\cdot\varepsilon_0})(f^l(1))$ equals to $f^{3\cdot 2^m-1}(1)$ or to $f^{2^{m+2}-1}(1)$ respectively. Thus

$$f^l(1) = (f^{-2^0\cdot\varepsilon_0})^+\circ\ldots\circ(f^{-2^{m-1}\cdot\varepsilon_{m-1}})^+(f^{\bar{l}}(1)). \tag{2.1}$$

We emphasize that we need to act by the "positive" branches of inverse maps. Finally, we define

$$z_k(f^l(1)) = (f^{-2^0\cdot\varepsilon_0})^+\circ\ldots\circ(f^{-2^{m-1}\cdot\varepsilon_{m-1}})^+\big(z_k(f^{\bar{l}}(1))\big). \tag{2.2}$$

By the definition each family $\{z_1(f^l(1)), z_2(f^l(1)), z_3(f^l(1)), \ldots\}$ forms a descending sequence of intervals tending to $f^l(1)$, and for each $i \geqslant 1$ we have

$$z_i(f^l(1)) \subset (f^l(1), f^{l+2^m}(1)) \quad \text{if } l = 3\cdot 2^m - 1 \quad \text{or}$$
$$z_i(f^l(1)) \subset (f^l(1), f^{l-2^m}(1)) \quad \text{if } l = 2^{m+2} - 1 .$$

Obviously, $(f^l(1), f^{l+2^m}(1))$, or $(f^l(1), f^{l-2^m}(1))$, respectively, is an element of the family of open intervals forming $[-1, 1] \setminus J(f)$, as was mentioned before.

To be in unity with the previous notation, from now on we shall write

$$z_0(f^l(1)) = z_0(f^{l+2^m}(1)) = (f^l(1), f^{l+2^m}(1)) \quad \text{for}$$
$$2^{m+1} \leqslant l \leqslant 3\cdot 2^m - 1, \quad m = 0, 1, 2, \ldots$$

Our next goal is to estimate the cross-ratio

$$\text{(2.3)} \qquad \frac{|z_{k+1}((T^n f)^l(1))|}{|z_k((T^n f)^l(1))|} \div \frac{|z_{k+1}(g^l(1))|}{|z_k(g^l(1))|}$$

depending neither on l nor on n.

We are going to start with $T^n f(1)$, $(T^n f)^2(1)$, $(T^n f)^3(1)$ and $g(1)$, $g^2(1)$, $g^3(1)$. By definition,

$$\text{(2.4)} \qquad z_k(T^n f(1)) = (T^n f)^{-1,-} \circ (T^n f)^{-1,+} \circ \ldots \circ ((T^n f)^{-2^{k-1}})^- \circ ((T^n f)^{-2^{k-1}})^+ (z_0((T^n f)^{2^k}, (T^n f)^{2^{k+1}-1}(1))),$$
$$z_{k+1}(T^n f(1)) = (T^n f)^{-1,-} \circ (T^n f)^{-1,+} \circ \ldots \circ ((T^n f^{-2^{k-1}})^- \circ ((T^n f)^{-2^{k-1}})^+ (z_1((T^n f)^{2^k}, (T^n f)^{2^{k+1}-1}(1)))$$

for $k \geqslant 1$ and

$$z_k((T^n f)^2(1)) = T^n f(z_k(T^n f(1))),$$
$$z_{k+1}((T^n f)^2(1)) = T^n f(z_{k+1}(T^n f(1))),$$
$$z_k((T^n f)^3(1)) = (T^n f) \circ (T^n f)(z_k(T^n f(1))),$$
$$z_{k+1}((T^n f)^3(1)) = (T^n f) \circ (T^n f)(z_{k+1}(T^n f(1))).$$

We also have

$$\text{(2.5)} \qquad z_k(T^n f(1)) \subset [-1, T^n f(1)],$$
$$T^n f(z_k(T^n f(1))) \subset [-(T^n f)(1), (T^n f)^2(1)],$$
$$(T^n f) \circ (T^n f)(z_k(T^n f(1))) \subset R_1(T^n f)[-1, T(T^n f)(1)] \quad \text{and generally}$$
$$(T^n f)^{2^t} \circ (T^n f)^{2^t} \circ \ldots \circ (T^n f) \circ (T^n f)(z_k(T^n f(1))) \subset R_j(T^n f)[-1, T^j(T^n f)(1)]$$
$$(T^n f)^{2^{j+1}} \circ (T^n f)^{2^j} \circ (T^n f)^{2^j} \circ \ldots \circ (T^n f) \circ (T^n f)(z_k(T^n f(1)))$$
$$\subset R_j(T^n f)[-T^j(T^n f)(1), (T^j(T^n f))^2(1)].$$

It is known (see [4]) that the Feigenbaum function g has the property

$$\text{(2.6)} \qquad (|g'|)|_{[-g(1),1]} > \xi^{-1} \quad \text{with some } 0 < \xi < 1$$

so for f sufficiently close to g also have

$$(|T^i f)'|)|_{[-(T^i f)(1), 1]} > \xi^{-1}, \quad i = 0, 1, 2, \ldots$$

Now we need the following lemma:

LEMMA 2.1. *There exists a positive integer u such that the inequality* (2.7)

$$\text{(2.7)} \qquad \frac{|z_{i+1}((T^{(u-1)i}(T^n f))(1))|}{|z_i((T^{(u-1)i}(T^n f))(1))|} \div \frac{|z_{i+1}(g(1))|}{|z_i(g(1))|} \leqslant 1 + \frac{\sigma}{2^i}$$

holds for all positive integers i and some constant $\sigma > 0$, independently on n.

Proof of the lemma. The basic idea of the proof is just like that for Lemma 1.1. Unfortunately, details are a bit more involved.

We use terms similar to (2.4). For $i = 0, 1, 2, \ldots$ and for any $u \geqslant 1$ we have

$$\text{(2.8)} \qquad z_i((T^{(u-1)i}(T^n f))(1))$$
$$= (T^{(u-1)i}(T^n f))^{-1,-} \circ (T^{(u-1)i}(T^n f))^{-1,+} \circ \ldots \circ ((T^{(u-1)i}(T^n f))^{-2^{i-1}})^-$$
$$\circ ((T^{(u-1)i}(T^n f))^{-2^{i-1}})^+ [z_0((T^{(u-1)i}(T^n f))^{2^i}, (T^{(u-1)i}(T^n f))^{2^{i+1}-1}(1))]$$
$$= \ldots = (T^{(u-1)i}(T^n f))^{-1,-}$$
$$\circ (T^{(u-1)i}(T^n))^{-1,+} (R_1(T^{(u-1)i}(T^n f)) ((T^{(u-1)i+1}(T^n f))^{-1,-}$$
$$\circ (T^{(u-1)i+1}(T^n f))^{-1,+} (R_1(T^{(u-1)i+1}(T^n f))) (T^{(u-1)i+2}(T^n f))^{-1,-}$$
$$\circ (T^{(u-1(i+2}(T^n f))^{-1,+} (R_1(T^{(u-1)i+2}(T^n f))$$
$$(\ldots R_1(T^{(u-1)i+(i-2)}(T^n f)) ((T^{(u-1)i+(i-1)}(T^n f))^{-1,-}$$
$$\circ (T^{(u-1)i+(i-1)}(T^n f))^{-1,+} R_1(T^{(u-1)+(i-1)}(T^n f)) [z_0(T^{u\cdot i}(T^n f)(1))])) \ldots))))))$$

For $z_{i+1}(T^{(u-1)i}(T^n f))$ we have the same formula with $z_1(T^{u\cdot i}(T^n f)(1))$ instead of z_0. Denote the left endpoint of $z_i f(1))$ by $\bar{z}_i(f(1))$

$$z_0(T^{ui}(T^n f)(1)) = [\bar{z}_0(T^{ui}(T^n f)(1)), T^{ui}(T^n f)(1)] = [-1, T^{ui}(T^n f)(1)],$$
$$z_1(T^{ui}(T^n f)(1)) = [\bar{z}_1(T^{ui}(T^n f)(1)), T^{ui}(T^n f)(1)]$$
$$= [(T^{ui}(T^n f))^{-1,-} \circ (T^{ui}(T^n f))^{-1,+}(-(T^{ui}(T^n f)(1))), T^{ui}(T^n f)(1)],$$
$$R_1(T^{ui-1}(T^n f))(\bar{z}_0(T^{ui}(T^n f)(1))) = (T^{ui-1}(T^n f)(1))\cdot(-1),$$
$$R_1(T^{ui-1}(T^n f))(\bar{z}_1(T^{ui}(T^n f)(1)))$$
$$= (T^{ui-1}(T^n f)(1))\cdot((T^{ui}(T^n f))^{-1,-} \circ (T^{ui}(T^n f))^{-1,+}(-(T^{ui}(T^n f)(1)))),$$
$$R_1(g)(\bar{z}_0(g(1))) = g(1)\cdot(-1),$$
$$R_1(g)(\bar{z}_1(g(1))) = g(1)\cdot(g^{-1,-} \circ g^{-1,+}(-g(1))).$$

We introduce a notation similar to that from § 1. Choose now B and δ so as to have fulfilled (1.3) and (2.9)

$$\|g-T^j f\|_{\sup} \leqslant B\cdot\delta^j . \tag{2.9}$$

Put $b_j = [\max(g^2(1), (T^j f)^2(1)), \max(g(1), T^j f(1))]$.

Then, by inequalities (2.6) and (2.9), similarly as in (1.4) we have

$$\|(T^j f)^{-1}|_{b_j} - g^{-1}|_{b_j}\|_{\sup} \leqslant B\cdot\delta^j \quad \text{on each branch.}$$

Thus, below we use inequalities

$$\|(T^j f)^{-1,+} - g^{-1,+}\|_{\sup} \leqslant B\cdot\delta^j \quad \text{and}$$
$$\|(T^j f)^{-1,-} - g^{-1,-}\|_{\sup} \leqslant B\cdot\delta^j$$

tacitly assuming that appropriate domains were slightly changed if necessary. Of course,

$$|\bar z_0(T^{ui}(T^n f)(1)) - \bar z_0(g(1))| = 0,$$
$$|R_1(T^{ui-1}(T^n f))(\bar z_0(T^{ui}(T^n f)(1))) - R_1(g)(\bar z_0(g(1)))| \leqslant B\cdot\delta^{ui-1+n},$$
$$|\bar z_1(T^{ui}(T^n f)(1)) - \bar z_1(g(1))| \leqslant 3B\cdot\delta^{ui+n}$$

due to (2.6) and to the argument from the proof of inequality (1.5) applied to $(T^{ui}(T^n f))^{-1}$ and g^{-1} instead of $\Sigma_f^{-1}, \Sigma_g^{-1}$. We also have

$$|R_1(T^{ui-1}(T^n f))(\bar z_1(T^{ui}(T^n f)(1))) - R_1(g)(\bar z_1(g(1)))|$$
$$\leqslant |(g(1)+B\cdot\delta^{ui-1+n})\cdot(\bar z_1(g(1))+3B\cdot\delta^{ui+n}) - \dot g(1)\cdot\bar z_1(g(1))| < 7B\delta^{ui-1+n}.$$

The second step gives

$$\Big|R_1(T^{ui-2}(T^n f))\Big((T^{ui-1}(T^n f))^{-1,-}$$
$$\circ (T^{ui-1}(T^n f))^{-1,+}\Big(R_1(T^{ui-1}(T^n f))(\bar z_0(T^{ui}(T^n f)(1)))\Big)\Big) -$$
$$- R_1(g)\Big(g^{-1,-}\circ g^{-1,+}\Big(R_1(g)(z_0(g(1)))\Big)\Big)\Big|$$
$$\leqslant |(g(1)+B\cdot\delta^{ui-2+n})\cdot\Big(g^{-1,-}\circ g^{-1,+}\Big(R_1(g)(\bar z_0(g(1)))\Big)+3B\delta^{ui-1+n}\Big) -$$
$$- g(1)\Big(g^{-1,-}\circ g^{-1,+}\Big(R_1(\dot g)(\bar z_0(g(1)))\Big)\Big)\Big| < 7B\delta^{ui-2+n},$$

$$\Big|R_1(T^{ui-2}(T^n f))(T^{ui-1}(T^n f))^{-1,-}$$
$$\circ (T^{ui-1}(T^n f))^{-1,+}\Big(R_1(T^{ui-1}(T^n f))(\bar z_1(T^{ui}(T^n f)(1)))\Big)\Big) -$$
$$- R_1(g)\Big(g^{-1,-}\circ g^{-1,+}\Big(R_1(g)(\bar z_1(g(1)))\Big)\Big)\Big|$$
$$\leqslant \Big|(g(1)+B\delta^{ui-2+n})\cdot\Big(g^{-1,-}\circ g^{-1,+}\Big(R_1(g)(\bar z_1(g(1)))\Big)+9B\delta^{ui-1+n}\Big) -$$
$$- g(1)\cdot\Big(g^{-1,-}\circ g^{-1,+}\Big(R_1(g)(\bar z_1(g(1)))\Big)\Big)\Big| < 19B\delta^{ui-2+n}.$$

After i steps we finally obtain an estimate

$$|\bar z_i(T^{(u-1)i}(T^n f)(1)) - \bar z_i(g(1))| \leqslant B\delta^{(u-1)i+n} + B\delta^{(u-1)i+n} + r_i\cdot B\delta^{(u-1)i+n}$$
$$\leqslant 3^{i+2}\cdot B\delta^{(u-1)i+n} \leqslant 9B\cdot 3^i\cdot\delta^{(u-1)i}$$

where $r_i = 2(r_{i-1}+2)+1$, $r_1 = 1$, and obviously

$$|\bar z_{i+1}(T^{(u-1)i}(T^n f)(1)) - \bar z_{i+1}(g(1))| \leqslant 27B\cdot 3^i\cdot\delta^{(u-1)i}.$$

Thus

$$|z_i(T^{(u-1)i}(T^n f)(1))| \geqslant |z_i(g(1))| - 10B\cdot 3^i\cdot\delta(u-1)i \quad \text{and}$$
$$|z_{i+1}(T^{(u-1)i}(T^n f)(1))| \leqslant |z_{i+1}(g(1))| + 28B\cdot 3^i\cdot\delta^{(u-1)i}$$

since

$$|T^{(u-1)i}(T^n f)(1) - g(1)| \leqslant B\delta^{(u-1)i}.$$

Because

$$\sup_{j\geqslant 0}\big(\max(\|R_1(T^j f)\|_{C_1}, \|T^j f|_{[-T^j f(1),1]}\|_{C_1})\big) = \theta < +\infty$$

so $|z_i(T^i f(1))| > \dfrac{1}{2\theta^{3^i}}$, as $|z_0(T^j f(1))| > \dfrac{1}{2}$.

Let σ be a real number such that

$$\max_{j\geqslant 0}\frac{|z_1(T^j f(1))|}{|z_0(T^j f(1))|} \div \frac{|z_1(g(1))|}{|z_0(g(1))|} < 1+\sigma .$$

Then it is enough to take u so as to have

$$0 < \frac{1+28B\cdot 3^2\cdot\delta^{(u-1)\cdot 2}\cdot 2\theta^6}{1-10B\cdot 3\cdot\delta^{u-1}\cdot 2\theta^3} < 1+\frac{\sigma}{2} \quad \text{and} \quad 3\delta^{u-1}\cdot\theta^3 < \frac{1}{4}$$

to obtain the statement of Lemma 2.1 by immediate induction, exactly in the same way as it was done for Lemma 1.1. ∎

Now we shall estimate $\dfrac{|z_{k+1}(T^n f(1))|}{|z_k(T^n f(1))|}$. Let $i = \left[\dfrac{k}{u}\right]$, $k = ui+r$, $0\leqslant r<u$. Transforming (2.8) we get:

$$z_k(T^n f(1)) = (T^n f)^{-1,-}\circ(T^n f)^{-1,+}\Big(R_1(T^n f)\Big((T(T^n f))^{-1,-} \tag{2.10}$$
$$\circ (T(T^n f))^{-1,+}\Big(R_1(T)((T^n f))\Big(\ldots (T^{r-1}(T^n f))^{-1,-}$$
$$\circ (T^{r-1}(T^n f))^{-1,+}\Big(R_1(T^{r-1}(T^n f))\Big((T^{(r-1)+1}(T^n f))^{-1,-}$$
$$\circ (T^{(r-1)+1}(T^n f))^{-1,+}\Big(R_1(T^{(r-1)+1}(T^n f))\Big(\ldots (T^{(r-1)+ui-i}(T^n f))^{-1,-}$$
$$\circ (T^{(r-1)+ui-i}(T^n f))^{-1,+}\big(R_1(T^{(r-1)+ui-i}(T^n f))\ \times$$
$$\times [z_1(T^{r+ui-i}(T^n f)(1))]\big)\ldots\Big)\Big)\Big)\Big)\ldots\Big)\Big)\Big)\Big)$$

and for $z_{k+1}(T^n f(1))$ we have the same formula with z_i replaced by z_{i+1}.

The number $\lambda = \max_{j\geqslant 0}\{|T^j f(1)|\}$ is less than 1.

Also the length of any interval $z_0(T^jf(1))$ is less than 1. By (2.6), acting by appropriate inverse mappings on intervals which occur in (2.10) makes them shorter with coefficient at least as ξ at each step, so

$$(2.11)\qquad \begin{aligned} &|z_i(T^{r+ui-i}(T^nf))| \leqslant \lambda^i\cdot\xi^{2i},\\ &\left|R_1(T^{(r-1)+ui-1}(T^nf))z_i(T^{r+ui-i}(T^nf)(1))\right| \leqslant \lambda^{i+1}\cdot\xi^{2i}. \end{aligned}$$

Set $\mathcal{M} = \max_j \|T^jf|_{(T^jf(1),1)}\|_{C_2}$.

Then by inductive application of Lemma 1.2 we get

$$\frac{|z_{k+1}(T^nf(1))|}{|z_k(T^nf(1))|} \leqslant \frac{|z_{i+1}(T^{r+ui-i}(T^nf)(1))|}{|z_i(T^{r+ui-i}(T^nf)(1))|}\times$$
$$\times e^{(\mathcal{M}\cdot\theta)(\lambda^{i+1}\xi^{2i}+\lambda^{i+1}\xi^{2i+1}+\lambda^{i+2}\xi^{2i+1}+\lambda^{i+2}\xi^{2i+2}+\ldots+\lambda^k\xi^{2(k-1)+1}+\lambda^k\xi^{2k})}$$
$$< \frac{|z_{i+1}(T^{r+ui-i}(T^nf)(1))|}{|z_i(T^{r+ui-i}(T^nf)(1))|}\cdot e^{B_1\cdot\lambda^{i+1}}, \quad \text{where } B_1 = \frac{2\mathcal{M}\theta}{1-\lambda}.$$

Analogously

$$\frac{|z_{k+1}(g(1))|}{|z_k(g(1))|} \geqslant \frac{|z_{i+1}(g(1))|}{|z_i(g(1))|}\cdot e^{-B_1\lambda^{i+1}}$$

and from Lemma 2.1 we have

$$(2.12)\qquad \frac{|z_{k+1}(T^nf(1))|}{|z_k(T^nf(1))|} \div \frac{|z_{k+1}(g(1))|}{|z_k(g(1))|} \leqslant \left(1+\frac{1}{2^i}\right)\cdot e^{2B_1\lambda^{i+1}}.$$

For similar cross-ratios taken at points $(T^nf)^2(1)$ and $g^2(1)$ or $(T^nf)^3(1)$ and $g^3(1)$ estimate (2.12) holds as well, since we apply the inverse mappings less times (one time or two times less).

Since

$$z_k(f^{\tilde{l}}(1)) = R_m(f)\big(z_k((T^mf)^2(1))\big) \quad \text{if } \tilde{l} = 3\cdot 2^m-1 \quad \text{and}$$
$$z_k(f^{\tilde{l}}(1)) = R_m(f)\big(z_k((T^mf)^3(1))\big) \quad \text{if } \tilde{l} = 2^{m+2}-1,$$

so

$$\frac{|z_{k+1}(f^{\tilde{l}}(1))|}{|z_k(f^{\tilde{l}}(1))|} \div \frac{|z_{k+1}(g^{\tilde{l}}(1))|}{|z_k(g^{\tilde{l}}(1))|}$$

equals to

$$\frac{|z_{k+1}((T^mf)^2(1))|}{|z_k((T^mf)^2(1))|} \div \frac{|z_{k+1}(g^2(1))|}{|z_k(g^2(1))|}$$

or to

$$\frac{|z_{k+1}((T^mf)^3(1))|}{|z_k((T^mf)(1))|} \div \frac{|z_{k+1}(g^3(1))|}{|z_k(g^3(1))|}.$$

By the above, at points of the form $f^{\tilde{l}}(1)$ estimate (2.5) gives

$$\frac{|z_{k+1}(f^{\tilde{l}}(1))|}{|z_k(f^{\tilde{l}}(1))|} \div \frac{|z_{k+1}(g^{\tilde{l}}(1))|}{|z_k(g^{\tilde{l}}(1))|} \leqslant \left(1+\frac{\sigma}{2^{[\frac{k}{u}]}}\right)e^{2B_1\lambda^{[\frac{k}{u}]+1}}.$$

Now we consider an arbitrarily chosen point of the orbit of 1, namely $f^l(1)$, $2^{m+1}-1 < l \leqslant 2^{m+2}-1$. By the definition of $z_k(f^l(1))$ we have:

$$(2.13)\qquad z_k(f^l(1)) = (f^{-1,+})^{\varepsilon_0}\circ\big((R_1(f)\circ(Tf))^{-1,+}\big)^{\varepsilon_1}\circ\ldots$$
$$\ldots\circ\big((R_{m-1}(f)\circ(T^{m-1}f))\big)^{\varepsilon_{m-1}}\quad \big(z_k(f^{\tilde{l}}(1))\big).$$

Here a function in 0-th power means the identity and in 1-st power means the same function itself.

We also have:

$$z_k(f^l(1)) \subset [-f(1), 1] \qquad \text{if } \varepsilon_0 = 1$$
$$(f^{2^0\cdot\varepsilon_0})\big(z_k(f^l(1))\big) \subset R_1(f)[-(Tf)(1), 1] \quad \text{if } \varepsilon_1 = 1.$$

......

$$(R_{m-2}(f)\circ(T^{m-2}f))^{\varepsilon_{m-2}}\circ\ldots\circ(R_1(f)\circ(Tf))^{\varepsilon_1}\circ f^{\varepsilon_0}\big(z_k(f^l(1))\big)$$
$$= (f^{2^{m-2}\cdot\varepsilon_{m-2}})\circ\ldots\circ(f^{2^0\cdot\varepsilon_0})\big(z_k(f^l(1))\big)$$
$$\subset R_{m-1}(f)[-(T^{m-1}f)(1), 1] \quad \text{if } \varepsilon_{m-2} = 1.$$

Thus, by (2.13), (2.6) and inductive using of Lemma 1.2 we obtain

$$(2.14)\qquad \frac{|z_{k+1}(f^l(1))|}{|z_k(f^l(1))|} \leqslant \frac{|z_{k+1}(f^{\tilde{l}}(1))|}{|z_k(f^{\tilde{l}}(1))|}\cdot e^{\mathcal{M}\theta\xi^{2(k-1)}(\lambda^k+\ldots+\lambda^{k+m-1})}$$
$$< \frac{|z_{k+1}(f^{\tilde{l}}(1))|}{|z_k(f^{\tilde{l}}(1))|}\cdot e^{\mathcal{M}\theta\frac{1}{1-\lambda}\cdot\lambda^k},$$

and

$$\frac{|z_{k+1}(g^l(1))|}{|z_k(g^l(1))|} \geqslant \frac{z_{k+1}(g^{\tilde{l}}(1))|}{|z_k(g^{\tilde{l}}(1))|}\cdot e^{-\mathcal{M}\theta\frac{1}{1-\lambda}\cdot\lambda^k}$$

since, of course,

$$|z_k(f^{\tilde{l}}(1))| \leqslant \sup_{j\geqslant 0}\big(\max(|z_k(T^jf)^2(1))|, |z_k((T^jf)^3(1))|)\big) \leqslant \lambda^k\cdot\xi^{2(k-1)}.$$

Consequently,

$$(2.15)\qquad \frac{|z_{k+1}(f^l(1))|}{|z_k(f^l(1))|} \div \frac{|z_{k+1}(g^l(1))|}{|z_k(g^l(1))|} \leqslant \left(1+\frac{\sigma}{2^{[\frac{k}{u}]}}\right)e^{B_2\lambda^k}$$

with some positive constant B_2.

The same method provides also the estimate

$$\frac{|z_{k+1}(f^l(1))|}{|z_k(f^l(1))|} \div \frac{|_{k+1}(g^l(1))|}{|z_k(g^l(1))|} \geqslant \left(1-\frac{\sigma}{2^{\left[\frac{k}{u}\right]}}\right) e^{-B_2\lambda^k} \tag{2.16}$$

increasing σ and B_2 if necessary.

So we obtained an upper estimate for the cross-ratio (2.3) depending exponentially on k.

Finally, let us observe that

$$\begin{aligned}\frac{|z_{k+1}(f^l(1))|}{|z_k(f^l(1))|} &\geqslant \frac{|z_{k+1}(g^l(1))|}{|z_k(g^l(1))|}\cdot e^{-B_2\lambda^k}\left(1-\frac{\sigma}{2^i}\right)\\ &\geqslant \frac{|z_{k+1}(g^{\tilde{l}}(1))|}{|z_k(g^{\tilde{l}}(1))|}\cdot e^{-2B_2\lambda^k}\left(1-\frac{\sigma}{2^i}\right)\\ &= \frac{|z_{k+1}(g^{((\tilde{l}+1)/2^m}(1))|}{|z_k(g^{((\tilde{l}+1)/2^m)-1}(1))|}\cdot e^{-2B_2\lambda^k}\left(1-\frac{\sigma}{2^i}\right)\\ &\geqslant \frac{|z_{k+1}(g(1))|}{|z_k(g(1))|}\, e^{-2B_2\lambda^k}\left(1-\frac{\sigma}{2^i}\right) e^{-\mathcal{M}\theta(\lambda^k\xi^{2(k-1)}+\lambda^k\xi^{2k-1})}\end{aligned}$$

by (2.11). But

$$\frac{|z_{k+1}(g(1))|}{|z_k(g(1))|} \geqslant \frac{|z_1(g(1))|}{|z_0(g(1))|}\cdot e^{-\mathcal{M}\theta(\lambda+\lambda^2\xi^2+\ldots+\lambda^k\xi^{2(k-1)})}$$

so $\dfrac{|z_{k+1}(f^l(1))|}{|z_k(f^l(1))|} \geqslant B_3 > 0$ independently on k, l.

Now consider any point $x = f^l(1)$ for some l and $y \in (f^l(1), f^{l+2^m}(1))$, $2^{m+1} \leqslant l \leqslant 3\cdot 2^m - 1$. There exists k such that $y \in z_k(x)$ and $y \notin z_{k+1}(x)$. Then $h(y) \in z_k(h(x))$ and $h(y) \notin z_{k+1}(h(x))$. Thus $\operatorname{dist}(h(x), h(y)) \geqslant |z_{k+1}(h(x))|$ and $\operatorname{dist}(x, y) \leqslant |z_k(x)|$, so

$$\frac{\operatorname{dist}(x,y)}{\operatorname{dist}(h(x),h(y))} \leqslant \frac{|z_k(x)|}{|z_{k+1}(h(x))|} = \frac{\dfrac{|z_k(x)|}{|z_{k-1}(x)|}\cdots\dfrac{|z_1(x)|}{|z_0(x)|}\cdot|z_0(x)|}{\dfrac{|z_{k+1}(h(x))|}{|z_k(h(x))|}\cdots\dfrac{|z_1(h(x))|}{|z_0(h(x))|}\cdot|z_0(h(x))|}$$

$$\leqslant \frac{1}{B_3}\cdot e^{B_2(1+\lambda+\ldots\lambda^k)} \underbrace{\left(1+\frac{\sigma}{2^0}\right)\ldots\left(1+\frac{\sigma}{2^0}\right)}_{u\text{-times}} \cdot \underbrace{\left(1+\frac{\sigma}{2^1}\right)\ldots\left(1+\frac{\sigma}{2^1}\right)}_{u\text{-times}} \ldots$$

$$\times \underbrace{\left(1+\frac{\sigma}{2^i}\right)\ldots\left(1+\frac{\sigma}{2^i}\right)}_{r\text{-times}} \cdot 2 \leqslant B_4 < +\infty .$$

The estimation $\operatorname{dist}(h(x), h(y)) \leqslant B_5\cdot\operatorname{dist}(x, y)$ may be handled in the same way, and the proof of Theorem 2 is complete.

§ 3. The Lipschitz condition on the whole interval $[-1, 1]$. Now we intend to examine when h and h^{-1} can simultaneously be Lipschitz continuous mappings on the whole interval $[-1, 1]$. We start with the observation that typically it is not the case. Below we give a necessary condition (C) for h, h^{-1} to be Lipschitz; (C) is fulfilled only on a submanifold of infinite codimension.

Set $a = g'(x_0) > 1$, $g(x_0) = x_0$.

$$\begin{aligned} &f'(a_1) = a, && \text{where } f(a_1) = a_1,\\ &(Tf)'(a_2) = a && \text{where } (Tf)(a_2) = a_2,\\ &(T^2 f)'(a_3) = a && \text{where } (T^2 f)(a_3) = a_3,\\ &\ldots \end{aligned} \tag{C}$$

For a given function f from $W^s(g)$ denote $\check{\alpha}_n(f) = \underbrace{(f^{-1,+})\circ\ldots\circ(f^{-1,+})}_{n\text{-times}}(f^2(1))$ $n = 0, 1, 2, \ldots$ Consider a family of points

$$\{1, -1, \check{\alpha}_0(f), -\check{\alpha}_0(f), \check{\alpha}_1(f), -\check{\alpha}_1(f), \check{\alpha}_2(f), -\check{\alpha}_2(f) \ldots\}.$$

They form a partition of the interval $[-1, 1]$, symmetrical with respect to 0, concentrating around the fixed point α_1 and around $-\alpha_1$. We shall call it $1(f)$-*partition.* Clearly, $\check{\alpha}_n(g) = h(\check{\alpha}_n(f))$.

Set $f'(a_1) = \mu \geqslant 1$ and assume that $\mu \neq a$, $|\mu - \alpha| = \varepsilon$, say $\mu > \alpha$. In a small neighborhoods of fixed points a_1, x_0 functions f, g are "almost" linear. Thus, for sufficiently small neighborhoods $\mathcal{U} \ni a_1$, $\mathcal{V} \ni x_0$ we have

$$\frac{\operatorname{dist}(a_1, \check{\alpha}_{n+1}(f))}{\operatorname{dist}(a_1, \check{\alpha}_n(f))} \leqslant \frac{1}{\mu - \frac{\varepsilon}{3}}, \quad \frac{\operatorname{dist}(x_0, \check{\alpha}_{n+1}(g))}{\operatorname{dist}(x_0, \check{\alpha}_n(g))} \geqslant \frac{1}{a + \frac{\varepsilon}{3}}.$$

So when $n \to \infty$

$$\frac{\operatorname{dist}(x_0, \check{\alpha}_n(g))}{\operatorname{dist}(a_1, \check{\alpha}_n(f))} \to \infty, \quad \text{but } h(a_1) = x_0,\ h(\check{\alpha}_n(f)) = \check{\alpha}_n(g)$$

hence the conjugacy cannot be Lipschitz continuous at the point a_1 when $f'(a_1) \neq g'(x_0)$.

Since

$$(Tf)'(a_2) = f'(f(1)\cdot a_2)\cdot f'(f(f(1)\cdot a_2)), \quad \text{and} \quad f^2(f(1)\cdot a_2) = f(1)\cdot a_2,$$

we can get in the same way another necessary condition for h to be Lipschitz continuous. Actually, considering f^{2^n}, g^{2^n}, we get that the product of derivatives being taken over all points of a given periodic orbit for f must equal to the one taken over the conjugate orbit for g. So we deliver infinitely many independent conditions; their conjunction is what we called condition (C). Now we state the main result of this part:

THEOREM 3. *Conjugacies h and h^{-1} are simultaneously Lipschitz continuous mappings if and only if condition* (C) *is fulfilled.*

Proof. What is left is to prove sufficiency of the condition. Similarly to that in §§ 1, 2 we can deal with the case of f close to g. Since h is an odd function, it is enough to examine it on the interval $[0, 1]$. Let us take two points $x, y \in [0, 1]$ and $\bar{x}, \bar{y}$, $\bar{x} = h(x)$, $\bar{y} = h(y)$. By Theorem 2 there is no loss of generality in assumption that $x, y \notin J(f)$. We call the points x, y *1(f)-separated* if they belong to two different intervals of $1(f)$-partition; otherwise we say that x, y are *not 1(f)-separated.*

For $1(T^p f)$-partition we take its linear image on the interval

$$[f^{2^{p-1}}(1), -f^{2^{p-1}}(1)]$$

and call it $(p+1)(f)$-partition. If x, y are not 1-separated then both belong to the same interval of $1(f)$-partition; take its image under iterated acting of f (i_1-times) until the image falls into the interval $[f(1), -f(1)]$ for the first time. If the obtained points $f^{i_1}(x), f^{i_1}(y)$ occur to belong to two different intervals of $2(f)$-partition we say that x, y are 2-separated. If not, consider the interval of $2(f)$-partition including $[f^{i_1}(x), f^{i_1}(y)]$ and take its consecutive images under acting of f^2 (i_2-times) until it falls for the first time into the interval $[-f^3(1), f^3(1)]$. If points $f^{i_1+2i_2}(x), f^{i_1+2i_2}(y)$ belong to two different intervals of $3(f)$-partition we say that x, y are 3-separated. If not i.e. $f^{i_1+2i_2}(x), f^{i_1+2i_2}(y)$ belong to the same interval of $3(f)$-partition, we continue with the consecutive images of this interval under acting of f^4 until it falls into $[f^7(1), -f^7(1)]$ for the first time, and so on. So, having defined what it means that x, y are not n-separated we can proceed as described above to say if they are $(n+1)$-separated or not. Since f has no homtervals, every $x \neq y$ have to occur n-separated for some $n \geq 1$. Let now $\mathscr{A}, M, \delta < 1$ be real numbers such that (1.3) is fulfilled with $B = \mathscr{A}$ and also a stronger version of (2.9), namely (3.1):

$$\|T^n f - g\|_{C_1} < \mathscr{A}\delta^n \quad \text{holds} \tag{3.1}$$

and $M = \sup_{n\geq 0}\big(\sup_{z\in(-1,1)}(|(T^n f)''(z)|)\big)$.
For given n consider arbitrarily chosen n-separated x, y and their images $f^s(x), f^s(y)$ which belong to two different intervals of $n(f)$-partition, $s = i_1 + 2_{i_2} + \ldots + 2^{n-1} i_n$, each term of this sequence is a non-negative integer.

LEMMA 3.1.

$$\frac{1}{A}\cdot \mathrm{dist}\big(f^s(x), f^s(y)\big) \leq \mathrm{dist}(g^s\bar{x}, g^s\bar{y}) \leq A\cdot \mathrm{dist}\big(f^s(x), f^s(y)\big).$$

Here A is some constant which does not depend on x, y, n.

Proof of the Lemma. Due to Theorem 1

$$\frac{|g^{2^n-1}(1), -g^{2^n-1}(1)|}{|f^{2^n-1}(1), -f^{2^n-1}(1)|} < \mathscr{K}$$

for some Lipschitz constant $\mathscr{K}$ which may be given not depending on n.

The partitions $n(f)$ and $n(g)$ are linear images of $1(T^{n-1}f)$, $1(g)$, so by Theorem 1 it is enough to show that $\frac{1}{\tilde{A}}\ \mathrm{dist}(x, y) \leq \mathrm{dist}(x, y) \leq \tilde{A}\cdot\mathrm{dist}(x, y)$ for $x, y - 1(T^{n-1}f)$-separated, with $\tilde{A}$ not depending on x, y, n. Consider the intervals

$$D = [\check{\alpha}_0(T^{n-1}f), \gamma], \quad \tilde{D} = [\check{\alpha}_0(g), \bar{\gamma}]$$

where $\gamma \in [\check{\alpha}_0(T^{n-1}f), \check{\alpha}_2(T^{n-1}f)]$, $\bar{\gamma}$ conjugate to γ by h_n: $g \circ h_n = h_n \circ T^{n-1}f$. By Theorem 1 and Theorem 2, $\frac{|\tilde{D}|}{|D|} < \varkappa$ for some Lipschitz constant $\varkappa$, depending neither on n nor on γ.

Set now $S(k) = \sup_n\big(\mathrm{dist}(\check{\alpha}_k(T^{n-1}f), a_n)\big)$ and choose the least k_0 so as to have $S(k_0-1)\cdot M < a$.

k_0 does not depend on n, so $\frac{\tilde{D}_{k_0}}{D_{k_0}} < \hat{\varkappa}$, where

$$D_j = \underbrace{(T^{n-1}f)^{-1,+} \circ \ldots \circ (T^{n-1}f)^{-1,+}}_{j\text{-times}}(D), \quad \tilde{D}_j = \underbrace{g^{-1,+} \circ \ldots \circ g^{-1,+}}_{j\text{-times}}(\tilde{D})$$

and

$$\hat{\varkappa} = \varkappa\cdot\left(\frac{\max_n(\sup_{z\in[-T^n f(1),1]}|(T^n f)'(z)|)}{\inf_{z\in[-g(1),1]}|g'(z)|}\right)^{k_0}.$$

Now for any k let us take k preimages of $D, \tilde{D}$ by $(T^{n-1}f)^{-1,+}$, $g^{-1,+}$ respectively. If $k \geq k_0$ we replace acting of $(k-k_0)$ appropriate inverse functions by linearization at points a_n, x_0 respectively. After k_0 steps we have

$$\sup_{z\in D_{k_0}}(\mathrm{dist}(z, a_n)) \leq S(k_0), \quad \sup_{z\in\tilde{D}_{k_0}}(\mathrm{dist}(z, x_0)) \leq S(k_0)$$

so by (2.6) for $k > k_0$ we obtain

$$\sup_{z\in D_k}(\mathrm{dist}(z, a_n)) \leq S(k_0-1)\cdot\xi^{k-k_0}, \quad \sup_{z\in\tilde{D}_k}(\mathrm{dist}(z, x_0)) \leq S(k_0-1)\cdot\xi^{k-k_0}.$$

Thus

$$\frac{|\tilde{D}_k|}{|D_k|} = \frac{|\overbrace{g^{-1,+}\circ\ldots\circ g^{-1,+}}^{(k-k_0)\text{-times}}(\tilde{D}_{k_0})|}{|\underbrace{(T^{n-1}f)^{-1,+}\circ\ldots\circ(T^{n-1}f)^{-1,+}}_{(k-k_0)\text{-times}}(D_{k_0})|} \quad \text{and by linearization}$$

$$\frac{|\tilde{D}_k|}{|D_k|} \leq \frac{\frac{1}{a-MS(k_0-1)\xi^{k-k_0+1}}\cdots\frac{1}{a-MS(k_0-1)\xi^0}\cdot|\tilde{D}_{k_0}|}{\frac{1}{a+MS(k_0-1)\xi^{k-k_0-1}}\cdots\frac{1}{a+MS(k_0-1)\xi^0}\cdot|D_{k_0}|}$$

$$\leq \hat{\varkappa}\cdot\prod_{n=1}^{\infty}\left(\frac{a+MS(k_0-1)\xi^n}{a-MS(k_0-1)\xi^n}\right) = \tilde{\varkappa} < +\infty.$$

Likewise

$$\frac{|D_k|}{|\tilde{D}_k|} < \tilde{\varkappa}, \quad \text{increasing } \tilde{\varkappa} \text{ if necessary.}$$

The same method provides an analogous estimate when choose intervals $(\check{\alpha}_2(T^{n-1}f), y)$ and $(\check{\alpha}_2(g), \bar{y})$ for D and $\tilde{D}$ respectively; details are left to the reader.

Finally, consider x, y. If they are $1(T^{n-1}f)$-separated there exists $m \geqslant 0$ such that $\check{\alpha}_m(T^{n-1}f)$ lies inside the interval (x, y). Assuming $x < y$ let us denote

$$\check{\alpha}_r(T^{n-1}f) = \min_i \{\check{\alpha}_i(T^{n-1}f) > x\}, \quad \check{\alpha}_s(T^{n-1}f) = \max_i \{\check{\alpha}_i(T^{n-1}f) < y\}$$

and

$$\check{\alpha}_r(g) = \min_i \{\check{\alpha}_i(g) > x\}, \quad \check{\alpha}_s(g) = \max_i \{\check{\alpha}_i(g) < y\}.$$

Then, by the above

$$\frac{1}{\tilde{\varkappa}} \leqslant \frac{\check{\alpha}_s(T^{n-1}f) - \check{\alpha}_r(T^{n-1}f)}{\check{\alpha}_s(g) - \check{\alpha}_r(g)} \leqslant \tilde{\varkappa}.$$

If $x \in [0, \check{\alpha}_1(T^{n-1}f)]$ or $y \in [(T^{n-1}f)^2(1), 1]$ then, by Theorem 2 it is clear that $\frac{1}{\varkappa} \leqslant \frac{\operatorname{dist}(\bar{x}, \bar{y})}{\operatorname{dist}(x, y)} \leqslant \varkappa$, so we may assume that $r \geqslant 3$, $s \geqslant 2$, and then

$x \in (\check{\alpha}_{r-2}(T^{n-1}f), \check{\alpha}_r(T^{n-1}f))$, for r odd $\quad y \in (\check{\alpha}_s(T^{m-1}f), \check{\alpha}_{s-2}(T^{n-1}f))$, for s even

$x \in (\check{\alpha}_r(T^{n-1}f), \check{\alpha}_{r+2}(T^{n-1}f))$, for r even $\quad y \in (\check{\alpha}_s(T^{n-1}f), \check{\alpha}_{s+2}(T^{n-1}f))$, for s odd.

We shall consider the case of s even and r odd; other cases are similar. In the case we have

$$(x, \check{\alpha}_r(T^{n-1}f)) = D_{r-2} \quad \text{and} \quad (\check{\alpha}_s(T^{n-1}f)) = D'_{s-2},$$

$D = (\gamma, \check{\alpha}_2(T^{n-1}f))$, $D' = (\check{\alpha}_2(T^{n-1}f), \gamma')$ for some $\gamma, \gamma' \in (\check{\alpha}_2(T^{n-1}f), \check{\alpha}_0(T^{n-1}f))$. Thus

$$\begin{aligned}\operatorname{dist}(x, y) &= \operatorname{dist}(x, \check{\alpha}_r(T^{n-1}f)) + \operatorname{dist}(\check{\alpha}_r(T^{n-1}f), \check{\alpha}_s(T^{n-1}f)) + \\ &\quad + \operatorname{dist}(\check{\alpha}_s(T^{n-1}), y) \leqslant \tilde{\varkappa} \cdot |\tilde{D}_{r-2}| + \tilde{\varkappa} \cdot \operatorname{dist}(\check{\alpha}_r(g), \check{\alpha}_s(g)) + \tilde{\varkappa} |\tilde{D}'_{s-2}| \\ &= \tilde{\varkappa} \cdot \operatorname{dist}(\bar{x}, \bar{y}).\end{aligned}$$

Similarly we obtain

$$\operatorname{dist}(\bar{x}, \bar{y}) \leqslant \tilde{\varkappa} \cdot \operatorname{dist}(x, y)$$

and the proof of Lemma 3.1 is complete. ■

Now we shall prove another important lemma.

LEMMA 3.2. *If $T^n f$ converges to g with exponential rate, then h_n: $g \circ h_n = h \circ T^n f$ converges to identity in the $\|\cdot\|_{\sup}$ norm; also h_n^{-1} converges exponentially to identity.*

Proof of the Lemma. By (1.3) and (2.9) due to the argument from the proof of inequality (1.5) we get $|\check{\alpha}_k(g) - \check{\alpha}_k(T^n f)| \leqslant k \cdot B \cdot \delta^n$. Hence, for any fixed positive integer $k(n)$ there holds $\|h_n|_{1(T^n f)} - \mathrm{id}\| \leqslant \xi^{k(n)} + k(n) \cdot B\delta^n$, since $\check{\alpha}_{k(n)}(T^n f)$, $\check{\alpha}_{k(n)+1}(T^n f)$, $\check{\alpha}_{k(n)+2}(T^n f)$, ... and $\check{\alpha}_{k(n)}(g)$, $\check{\alpha}_{k(n)+1}(g)$, $\check{\alpha}_{k(n)+2}(g)$, ... are in the "small" neighborhoods (of radius $\xi^{k(n)}$) of the points a_{n+1}, x_0 respectively.

If now r is chosen such that $2\delta^r < (\xi + \frac{1}{2}(1-\xi))$ then

$$\xi^{k(n)+1} + (k(n)+1) \cdot B\delta^{n+r} \leqslant \zeta(\xi^{k(n)} + k(n) \cdot B\delta^n) \quad \text{with } \zeta = \xi + \tfrac{1}{2}(1-\xi) < 1. \tag{3.2}$$

For $i = 1, 2, \ldots$ we also have

$$\|h_{n+i}|_{1(T^{n+i}f)} - \mathrm{id}\|_{\sup} \leqslant \xi^{k(n)} + k(n) \cdot B\delta^{n+1}. \tag{3.3}$$

Consider $T^{n+1}f$; if f is close enough to g (e.g. $\frac{g(1)}{T^n f(1)}$ does not differ from 1 very much), then for points from $2(T^n f)$-partition we also obtain

$$\sup_{i \geqslant 0} |R_1(T^n f)(\check{\alpha}_i(T^{n+1}f)) - R_1(g)(\check{\alpha}_i(g))| < k(n) \cdot B\delta^{n+1} + \xi^{k(n)}.$$

Thus, if q is a preimage of any point forming $2(T^n f)$-partition by $(T^n f)^{-1,+}$ we have

$$|q - h(q)| \leqslant (k(n) \cdot B\delta^{n+1} + \xi^{k(n)}) + k(n) \cdot B\delta^n + \xi^{k(n)}.$$

Iterating this procedure $(r-1)$-times we obtain, that for any point z of the form

$$z = (T^n f)^{-1,+} \circ \ldots \circ (T^n f)^{-1,+} \circ \ldots \circ ((T^n f)^{-2^n})^+ \circ \ldots \circ ((T^n f)^{-2^n})^+(\tilde{z}),$$

where $\tilde{z}$ is one of the points forming $(s+1)(T^n f)$-partition, $s+1 \leqslant r-1$, the inequality

$$|z - h(z)| \leqslant r \cdot \xi^{k(n)} + k(n) \cdot B\delta^n + \ldots + k(n) \cdot B\delta^{n+r-1} < r(\xi^{k(n)} + k(n) \cdot B\delta^n) \quad \text{holds}.$$

Due to (3.2) and (3.3) following steps give that for a point $\tilde{v}$ from the family forming $(r+j)(T^n f)$ $j = 0, 1, 2, \ldots$, the inequality (3.4) is satisfied:

$$|\tilde{v} - h(\tilde{v})| \leqslant \zeta(\xi^{k(n)} + k(n) \cdot B\delta^n). \tag{3.4}$$

So, for v' of the form

$$v' = (T^n f)^{-1,+} \circ \ldots \circ (T^n f)^{-1,+} \circ \ldots \circ ((T^n f)^{-2^{r+j-1}})^+ \circ \ldots \circ ((T^n f)^{-2^{r+j-1}})^+(\tilde{v})$$

we have

$$|v' - h(v')| \leqslant (j+1) \cdot \zeta(\xi^{k(n)} + k(n) \cdot B\delta^n) + r \cdot (\xi^{k(n)} + k(n) \cdot B\delta^n)$$

and by obvious induction, when $\tilde{\chi}$ is any point of $m(T^n f)$, m-arbitrarily chosen positive integer, then for any χ of the form

$$\chi = (T^n f)^{-1,+} \circ \ldots \circ (T^n f)^{-1,+} \circ \ldots \circ ((T^n f)^{-2^{m-1}})^+ \circ \ldots \circ ((T^n f)^{-2^{m-1}})^+(\tilde{\chi})$$

we have

$$|\chi - h(\chi)| \leqslant r \cdot (\xi^{k(n)} + k(n) \cdot B\delta^n)(1 + \zeta + \ldots + \zeta^{[\frac{m}{r}]+1}) < \frac{r}{1-\zeta}(\xi^{k(n)} + k(n) \cdot B\delta^n).$$

But points of this form are a dense set in [0, 1], so

$$\|h_n-\mathrm{id}\|_{\sup}\leqslant \frac{r}{1-\zeta}\left(\xi^{k(n)}+k(n)\cdot B\delta^n\right).$$

But for h_{n+r} we have

$$\|h_{n+r}-\mathrm{id}\|_{\sup}\leqslant \frac{r}{1-\zeta}\left(\xi^{k(n)+1}+(k(n)+1)\cdot B\delta^{n+r}\right)\leqslant \frac{r}{1-\zeta}\cdot\zeta\left(\xi^{k(n)}+k(n)\cdot B\delta^{n+r}\right).$$

Thus, putting $k(0)=1$ we finally obtain

$$\|h_i-\mathrm{id}\|_{\sup}<(B+1)\cdot\frac{r}{1-\zeta}\cdot\zeta^{\left[\frac{i}{r}\right]-1}.$$

Similar considerations apply to h^{-1} and the lemma follows. ∎

Now we are in a position to make final estimates. The arguments for them are very similar to that from § 2, so some explanations are omitted. From now on v denotes $f^s(x)$, w denotes $f^s(y)$, $\bar v=h(v)$, $\bar w=h(w)$, where $s=i_1+2i_2+\ldots+2^{n-1}i_n$ is as given on page (250), x, y are some n-separated points. We have known that

$$\frac{1}{A}\leqslant\frac{\operatorname{dist}(\bar v,\bar w)}{\operatorname{dist}(v,w)}\leqslant A.$$

By the definition of n-separated points we have

$$x=(f^{-1,+})^{i_1}\circ((f^{-2})^+)^{i_2}\circ\ldots\circ((f^{-2^{n-1}})^+)^{i_n}(v),$$
$$y=(f^{-1,+})^{i_1}\circ((f^{-2})^+)^{i_2}\circ\ldots\circ((f^{-2^{n-1}})^+)^{i_n}(w)$$

and the same for $\bar x$, $\bar y$ with $\bar v$, $\bar w$ and g instead of v, w, f respectively. Let us denote $g_i(a,b)=\sup_{z\in(a,b)}|((g^{-2^i})^+)'(z)|$ and $f_i(a,b)=\inf_{z\in(a,b)}|((f^{-2^i})^+)'(z)|$ where $(a,b)\subset[-g^{2^{i+1}-1}(1),g^{2^{i+1}-1}(1)]$. We shall also write

$$f^{j,2^k}(v)=((f^{-2^k})^+)^{i_k-j}\circ((f^{-2^{k+1}})^+)^{i_{k+1}}\circ\ldots\circ((f^{-2^{n-1}})^+)^{i_n}(v)$$

and we introduce identical notation for w, $\bar v$, $\bar w$ in obvious way.
Then

$$\frac{\operatorname{dist}(\bar x,\bar y)}{\operatorname{dist}(x,y)}\leqslant\frac{\operatorname{dist}(\bar v,\bar w)}{\operatorname{dist}(v,w)}\times$$
$$\times\frac{g_{n-1}(\bar v,\bar w)}{f_{n-1}(v,w)}\cdot\frac{g_{n-1}(g^{i_n-1,2^{n-1}}(\bar v),g^{i_n-1,2^{n-1}}(\bar w))}{f_{n-1}(f^{i_n-1,2^{n-1}}(v),f^{i_n-1,2^{n-1}}(w))}\cdots\frac{g_{n-1}(g^{1,2^{n-1}}(\bar v),g^{1,2^{n-1}}(\bar w))}{f_{n-1}(f^{1,2^{n-1}}(v),f^{1,2^{n-1}}(w))}\times$$
$$\times\frac{g_{n-2}(g^{0,2^{n-1}}(\bar v),g^{0,2^{n-1}}(\bar w))}{f_{n-2}(f^{0,2^{n-1}}(v),f^{0,2^{n-1}}(w))}\cdot\frac{g_{n-2}(g^{i_{n-1}-1,2^{n-2}}(\bar v),g^{i_{n-1}-1,2^{n-2}}(\bar w))}{f_{n-2}(f^{i_{n-1}-1,2^{n-2}}(v),f^{i_{n-1}-1,2^{n-2}}(w))}\cdots$$
$$\cdots\frac{g_{n-2}(g^{1,2^{n-2}}(\bar v),g^{1,2^{n-2}}(\bar w))}{f_{n-2}(f^{1,2^{n-2}}(v),f^{1,2^{n-2}}(w))}\cdots\frac{g_0(g^{0,2^1}(\bar v),g^{0,2^1}(\bar w))}{f_0(f^{0,2^1}(v),f^{0,2^1}(w))}\times$$

$$\times\frac{g_0(g^{i_1-1,2^0}(\bar v),g^{i_1-1,2^0}(\bar w))}{f_0(f^{i_1-1,2^0}(v),f^{i_1-1,2^0}(w))}\cdots\frac{g^0(g^{1,2^0}(\bar v),g^{1,2^0}(\bar w))}{f_0(f^{1,2^0}(v),f^{1,2^0}(w))}$$
$$\leqslant A\cdot\prod_{j=i_n}^{-\infty}\frac{g_{n-1}(g^{j,2^{n-1}}(\bar v),g^{j,2^{n-1}}(\bar w))}{f_{n-1}(f^{j,2^{n-1}}(v),f^{j,2^{n-1}}(w))}\cdot\prod_{j=i_{n-1}}^{-\infty}\frac{g_{n-2}(g^{j,2^{n-2}}(\bar v),g^{j,2^{n-2}}(\bar w))}{f_{n-2}(f^{j,2^{n-2}}(v),f^{j,2^{n-2}}(w))}\cdots$$
$$\cdots\prod_{j=i_1}^{-\infty}\frac{g_0(g^{j,2^0}(\bar v),g^{j,2^0}(\bar w))}{f_0(f^{j,2^0}(v),f^{j,2^0}(w))}.$$

Now consider any term of the form

$$\prod_{j=i_k}^{-\infty}\frac{g_{k-1}(g^{j,2^{k-1}}(\bar v),g^{j,2^{k-1}}(\bar w))}{f_{k-1}(f^{j,2^{k-1}}(v),f^{j,2^{k-1}}(w))}.$$

Let $k=r\cdot m+p$, r is as in the proof of Lemma 3.2, $p=0,1,\ldots,r-1$. First notice that

(3.5)
$$\frac{|[f^{i_k,2^{k-1}}(v),f^{i_k,2^{k-1}}(w)]|}{|[0,f^{2^{k-1}}(1)]|}=\frac{|[(R_{k-1}(f))^{-1}(f^{i_k,2^{k-1}}(v)),(R_{k-1}(f))^{-1}(f^{i_k,2^{k-1}}(w))]|}{|[0,1]|}\leqslant 2\lambda^{n-k}$$

λ was defined in § 2. Thus

(3.6)
$$\frac{|[f^{j,2^{k-1}}(v),f^{j,2^{k-1}}(w)]|}{|[0,f^{2^{k-1}}(1)]|}\leqslant 2\lambda^{n-k}\cdot\xi^{i_k-j}.$$

Likewise

(3.7)
$$\frac{|[g^{j,2^{k-1}}(\bar v),g^{j,2^{k-1}}(\bar w)]|}{|[0,g^{2^{k-1}}(1)]|}\leqslant 2\lambda^{n-k}\cdot\xi^{i_k-j}.$$

By (3.6) we have

(3.8)
$$f_{k-1}(f^{j,2^{k-1}}(v),f^{j,2^{k-1}}(w))=[\sup\{|(f^{2^{k-1}})'\}|_{(f^{j-1,2^{k-1}}(v),f^{j-1,2^{k-1}}(w))}\}]^{-1}\geqslant[|(f^{2^{k-1}})'(f^{j-1,2^{k-1}}(v))|+M\cdot 2\lambda^{n-k}\cdot\xi^{i_k-(j-1)}]^{-1}.$$

Similarly

(3.9)
$$g_{k-1}(g^{j,2^{k-1}}(\bar v),g^{j,2^{k-1}}(\bar w))=[\inf\{|(g^{2^{k-1}})'|_{(g^{j-1,2^{k-1}}(\bar v),g^{j-1,2^{k-1}}(\bar w))}\}]^{-1}\leqslant[|(g^{2^{k-1}})'(\bar v)|-2M\lambda^{n-k}\cdot\xi^{i_k-(j-1)}]^{-1}.$$

Since $g^{2^{k-1}}$, $f^{2^{k-1}}$ are linear images of g, $T^{k-1}f$ respectively, thus by (3.1) Lemma 3.2 we have:

(3.10)
$$|(g^{2^{k-1}})'(\bar v)-(f^{2^{k-1}})'(v)|\leqslant\mathscr{A}\cdot\delta^{k-1}+M(B+1)\frac{r}{1-\zeta}\cdot\zeta^{\left[\frac{k-1}{r}\right]-1}.$$

Hence we can rewrite inequality (3.9) in the form:

$$g_{k-1}(g^{j,2^{k-1}}(\bar v), g^{j,2^{k-1}}(\bar w)) \leqslant \Big[|(f^{2^{k-1}})'(f^{j-1,2^{k-1}}(v))| - \mathscr{A}\delta^{k-1} - M(B+1)\frac{\;}{1-\zeta}\cdot\zeta^{\left[\frac{k-1}{r}\right]-1} - 2M\lambda^{n-k}\cdot\xi^{i_k-j+1}\Big]^{-1} \tag{3.11}$$

For $|i_k-j| > \left[\frac{k}{r}\right] = m$ we also have

$$f_{k-1}(f^{j,2^{k-1}}(v), f^{j,2^{k-1}}(w)) \geqslant \frac{1}{a+M\xi^{i_k-j}} \quad \text{and} \quad g_{k-1}(g^{j,2^{k-1}}(\bar v), g^{j,2^{k-1}}(\bar w)) \leqslant \frac{1}{a-M\xi^{i_k-j}}. \tag{3.12}$$

Now we estimate our product using (3.8), (3.1) for $j \leqslant m$ and (3.12) for $j > m$. We also assume that all denominators in the formula given below are positive (it is fulfilled whenever $n_0 \leqslant k \leqslant n-n_0$ with some appropriate positive integer n_0).

$$\prod_{j=i_k}^{-\infty} \frac{g_{k-1}(g^{j,2^{k-1}}(\bar v), g^{j,2^{k-1}}(\bar w))}{f_{k-1}(f^{j,2^{k-1}}(v), f^{j,2^{k-1}}(w))} \tag{3.13}$$

$$\leqslant \prod_{j=i_k}^{j=i_k-m} \frac{|(f^{2^{k-1}})'(f^{j-1,2^{k-1}}(v))| + 2M\lambda^{n-k}\cdot\xi^{i_k-j+1}}{|(f^{2^{k-1}})'(f^{j-1,2^{k-1}}(v))| - 2M\lambda^{n-k}\cdot\xi^{i_k-j+1} - \mathscr{A}\delta^{k-1} - M(B+1)\frac{r}{1-\zeta}\cdot\zeta^{m-2}} \times$$

$$\times \prod_{j=i_k-m-1}^{-\infty} \frac{a+M\xi^{|i_k-j|}}{a-M\xi^{|i_k-j|}} \leqslant \frac{a+M_1\cdot\xi^{m+1}}{a-M_1\cdot\xi^{m+1}} \cdot \prod_{i=1}^{m} \frac{1+2M\lambda^{n-k}\xi^i}{1-M_2\vartheta^{m-2}-2M\lambda^{n-k}\cdot\xi^i}.$$

Here $\vartheta = \min(\xi, \delta)$, M_1, M_2-appropriate constants. Assuming now that the denominators in the fractions in the product are greater than $1-\frac{\log 2}{2}$, (which is fulfilled when $n_1 \leqslant k \leqslant n-n_1$ with some appropriate $n_1 \geqslant n_0$) then, using $e^{-2x} < 1-x$ for $0 < x < 1-\frac{\log 2}{2}$, we have

$$\prod_{j=i_k}^{-\infty} \frac{g_{k-1}(g^{j,2^{k-1}}(\bar v), g^{j,2^{k-1}}(\bar w))}{f_{k-1}(f^{j,2^{k-1}}(v), f^{j,2^{k-1}}(w))} \leqslant \frac{a+M_1\cdot\xi^{m+1}}{a-M_1\cdot\xi^{m+1}} \times \tag{3.14}$$

$$\times e^{2M\lambda n-k\left(\xi\cdot\frac{1-\xi^{m+1}}{1-\xi}\right)} \cdot e^{2mM_2\vartheta^{m-2}+4M\lambda^{n-k}\left(\xi\cdot\frac{1-\xi^{m+1}}{1-\xi}\right)} \leqslant \frac{a+M_1\xi^{m+1}}{a-M_1\xi^{m+1}} \cdot e^{M_3\lambda^{n-k}} \cdot e^{mM_4\cdot\vartheta^{m-2}}.$$

Finally, if $k \leqslant n_1$ or $k \geqslant n-n_1$ we have an obvious estimation

$$\prod_{j=i_k}^{-\infty} \frac{g_{k-1}(g^{j,2^{k-1}}(\bar v), g^{j,2^{k-1}}(\bar w))}{f_{k-1}(f^{j,2^{k-1}}(v), f^{j,2^{k-1}}(w))} \tag{3.15}$$

$$\leqslant \left(\frac{\max_{k\geqslant 0}(\sup T^k f)'|_{[-T^k f(1),1]})}{\inf(g'|_{[-g(1),1]})}\right)^{n_0} \cdot \prod_{i=n_0+1}^{\infty} \frac{a+M\xi^i}{a-M\xi^i} \leqslant \mathscr{P} < +\infty.$$

Hence, replacing the finite product of our terms by infinite multiplication we have

$$\frac{\operatorname{dist}(\bar x, \bar y)}{\operatorname{dist}(x, y)} \leqslant A\cdot\mathscr{P}^{2n_1} \cdot \prod_{k=1}^{\infty} \frac{a+M_1\cdot\xi^{\left[\frac{k}{r}\right]}}{a-M_1\cdot\xi^{\left[\frac{k}{r}\right]}} \cdot e^{M_3\lambda^k} \cdot e^{\left[\frac{k}{r}\right]M_4\cdot\vartheta^{\left[\frac{k}{r}\right]-2}} \leqslant \mathscr{R} < +\infty \tag{3.16}$$

for some $\mathscr{R}$.

The estimate

$$\frac{\operatorname{dist}(x, y)}{\operatorname{dist}(\bar x, \bar y)} \leqslant \tilde{\mathscr{R}}$$

for some $\tilde{\mathscr{R}} < +\infty$ can be handled in the same way and Theorem 3 is proved.

QUESTION. Assume condition (C). Does h have to be real analytic function in that case?

Remark 2. When we consider a family of functions of class C^∞ converging to g then using small smooth perturbation we can easily construct a function f such that $f \in C^\infty$, $T^n f \to g$, h: $g \circ h = h \circ f$ is Lipschitz continuous and h is not of class C^1.

References

[1] P. Collet, J.-P. Eckmann, *Iterated maps on the interval as dynamical systems*, Progress in Physics, Birkhauser, Boston 1980.

[2] M. Misiurewicz, *Structure of mappings of an interval with zero entropy*, Publ. Math. IHES, 53 (1981), 5–16.

[3] F. Ledrappier, M. Misiurewicz, *Dimension of invariant measures for maps with exponent zero*, Ergod. Th. and Dyn. Syst. 5 (1985), 595–610.

[4] H. Epstein, J. Lascoux, *Analyticity properties of the Feigenbaum function*. Comm. Math. Phys. 81 (1981), 437–453.

[5] M. Campanino, H. Epstein, *On the existence of Feigenbaum's fixed point*, Comm. Math. Phys. 79 (1981), 261–302.

[6] O. E. Lanford, *A computer-assisted proof of the Feigenbaum conjectures*, Bull. Amer. Math. Soc. (New Series) 6 (1982) 427–434.

[7] M. Shub, D. Sullivan, *Expanding endomorphisms of the circle revisited*, Ergod. Th. and Dyn. Syst. 5 (1985) 285–289.
[8] F. Przytycki, *On holomorphic perturbations of $z \to z^n$*, Bull. Pol. Acad. Sci. (Math.) 34 (1986), 127–132.

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Received 1 April 1987;
in revised form 17 December 1987