References

[1] M. Artin and B. Mazur, *Etale Homotopy*, Lecture Notes in Math. Springer; 1969.
[2] A. L. Blakers, *Some relations between homology and homotopy groups*, Ann. of Maths. (2) 49 (1948) 428–461.
[3] M. G. Brin and T. L. Thikstun, *On the proper Steenrod homotopy groups and proper embeddings of planes into* 3-*manifolds*, Trans. Amer. Math. Soc. 289 (1985), 737–755.
[4] Z. Čerin, *On various relative proper homotopy groups*, Tsukuba J. Math. (2) 4 (1980), 177–202.
[5] D. Edwars and H. Hasting, *Čech and Steenrod homotopy theories with applications to Geometric Topology*, Lecture Notes Math. 542, Springer (1976).
[6] J. I. Extremiana, L. J. Hernández and M. T. Rivas, *Una* (*Co*)*Homologia propia*, Actas *X* Jornadas Hispano-Lusas de Matemáticas, Murcia 1985.
[7] L. J. Hernández, *A note on proper invariants*, Publ. Sem. Mat. Garcia de Galdeano. Serie II, sección 1 no 12 (1984).
[8] L. J. Hernández and T. Porter, *Proper pointed maps from* R^{n+1} *to a* σ-*compact space*. Math. Proc. Camb. Phil. Soc. 103 (1988), 457–462.
[9] S. T. Hu, *Algebraic local invariants of topological spaces*. Compositio Math. 13 (1958), 173–218,
[10] — *Homotopy theory*, Academic Press 1959.
[11] W. Hurewicz, *Beitrage zur Topologie der Deformationen*, I–IV, Neder. Akad. Wetensch Proc. Ser. A 38, 112–119, (1935) 521–528, 39, 117–126, (1936), 215–224.
[12] Y. Kodama and A. Koyama, *Hurewicz isomorphism theorem for Steenrod homology*. Amer. Math. Soc. (1979), 363–367.
[13] K. Kuperberg, *An isomorphism theorem of the Hurewicz type in Borsuk theory of shape*, Fund. Math. 77 (1972), 21–32.
[14] W. S. Massey, *Singular homology theory*, G.T.M. 70, Springer, 1980.
[15] S. Mardešić and S. Ungar, *The relative Hurewicz theorem in shape theory*, Glasnik Math. 9 (1974), 317–327.
[16] K. Morita, *The Hurewicz and Whitehead theorems in shape theory*, Sci. Rep. Tokyo Daigaku Sec. A (12) (1974), 246–258.
[17] M. Raussen, *Hurewicz isomorphism and Whitehead theorems in pro-categories*, Arch. Math. (Basel) 30 (1978), 153–164.
[18] M. T. Rivas, *Sobre invariantes de homotopia propia y sus relaciones*, Tesis. Publ. Sem. Mat. Garcia de Galdeano. Serie II, sección 2, 17 (1987).
[19] N. E. Steenrod, *The topology of fibre bundles*, P.M.S. 14. Princeton University Press, 1954.
[20] G. W. Whitehead, *Homotopy theory*, Massachusetts Institute of Technology, 1966.
[21] —, *Elements of homotopy theory*, G.T.M. 61, Springer, 1978.

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Jumps of entropy in one dimension

by

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Abstract. For continuous piecewise monotone maps of an interval we estimate possible jumps (discontinuities) of topological entropy under perturbations which preserve the number of pieces of monotonicity. We prove that for unimodal maps the topological entropy as a function of a map is continuons at all maps for which it is positive.

0. Introduction. This paper deals with the continuity properties of the topological entropy as a function of a map. For the discussion of this problem in the case of continuous maps of arbitrary compact spaces, we send the reader to the book [4], and in the case of smooth maps to the paper [11]. Here we shall concentrate on the case of continuous maps of the interval (it does not make any difference if we replace the interval by the circle, so these results apply also to the case of the maps of the circle).

It was proved in [10] and [8] that in this case the topological entropy is lower semi-continuous. Hence, what remains to investigate, is the problem how far it is from the upper semi-continuity. Clearly, one can modify any map by creating an invariant subinterval with arbitrarily large entropy, and this modification can be made small in the C^0-topology. However, the natural demand that we do not enlarge the number of turning points, excludes most of these examples.

In such a way we are left with the following problem: how high can the entropy jump up if we start with the piecewise monotone map and make arbitrarily small C^0 perturbations which do not enlarge the number of turning points? The answer (Theorem 1) is the following:

We look at all periodic orbits. On each of them we count the number of turning points and divide by the period. Then we take the maximum of these numbers over all periodic orbits and multiply by $\log 2$. *This is the maximal level to which the entropy can jump. If it is already above this level, then it is continuous at this map.*

Some problems are created if the map is not piecewise strictly monotone, i.e. if it can have "flat" pieces, and in particular turning intervals instead of turning points. Since the exact formulation of Theorem 1 needs some preliminary definitions, we leave it until the end of Section 1.

We can draw some conclusions from Theorem 1. In the case of the unimodal maps, it turns out that the entropy is continuous at all maps for which it is positive (Theorem 2). Then we can deduce some properties of the behaviour of the kneading invariant as a function of a map.

The research which resulted in this paper started in order to answer some questions raised by Ch. Gillot in his talk given during the Semester on Dynamical Systems at the Stefan Banach Centre (Warsaw, 1986). Section 4 (Corollary 1) answers the question asked by W. Szczechla.

1. Notations and definitions. We adopt some notions from [10]. Let $f\colon I\to I$ be a continuous map of a closed interval into itself. A cover $\mathscr{A}$ of I is called f*-mono* if it is finite, its elements are intervals (possibly degenerated to points) and f is monotone on each element $A\in\mathscr{A}$. If such $\mathscr{A}$ exists then f is called *piecewise monotone.* We denote

$$c_n(f)=\min\{\operatorname{Card}\mathscr{A}\colon \mathscr{A}\text{ is an } f^n\text{-mono cover}\}.$$

Theorem 1 of [10] says that

$$h(f)=\lim_{n\to\infty}\frac{1}{n}\log c_n(f) \tag{1}$$

and

$$h(f)\leqslant\frac{1}{n}\log c_n(f)\quad\text{ for any } n. \tag{2}$$

It also follows from [10] (see (1.4) and Corollary 1 of [10]) that

(3) If $\mathscr{A}$ is an f-mono cover then $h(f)=h(f,\mathscr{A})$.

Another result useful to us is Lemma 5 of [10]:

(4) If $\sigma_n,\tau_n\geqslant 0$ for $n=0,1,2,\ldots$ then

$$\limsup_{n\to\infty}\frac{1}{n}\log\sum_{l=0}^{n-1}\sigma_l\tau_{n-l}\leqslant\max\left(\limsup_{n\to\infty}\frac{1}{n}\log\sigma_n,\ \limsup_{n\to\infty}\frac{1}{n}\log\tau_n\right).$$

Denote by $\mathscr{U}(f)$ the space of all continuous piecewise monotone maps $g\colon I\to I$ for which $c_1(g)\leqslant c_1(f)$. We endow this space with the C^0 topology.

Let f be continuous piecewise monotone. Let $\mathscr{C}(f)$ be the class of all maximal closed intervals (possibly degenerated to points) on which f is constant.

Let $J=[a,b]\in\mathscr{C}(f)$. By the maximality of J, there are 3 possibilities:

1. a is the left endpoint of I.
2. There exists $\varepsilon>0$ such that $f(x)<f(a)$ for all $x\in(a-\varepsilon,a)$.
3. There exists $\varepsilon>0$ such that $f(x)>f(a)$ for all $x\in(a-\varepsilon,a)$.

We set

$$l(J)=\begin{cases}0 & \text{in the first case},\\ -1 & \text{in the second case},\\ +1 & \text{in the third case}.\end{cases}$$

Analogously, we define $r(J)$ depending on what happens to the right of b (we replace a by b and $(a-\varepsilon,a)$ by $(b,b+\varepsilon)$). Then we set

$$T(f)=\{J\in\mathscr{C}(f)\colon |l(J)+r(J)|=2\},\qquad S(f)=\bigcup_{J\in T(f)}J.$$

In other words, $T(f)$ is the set of the turning intervals (points) of f, and $S(f)$ is the union of all of them. Clearly,

$$c_1(f)=\operatorname{Card}(T(f))+1.$$

Now we define

$$\alpha(f)=\limsup_{\substack{g\to f\\ g\in\mathscr{U}(f)}}h(g),$$

$$\beta(f)=\max\left\{\frac{p}{q}\log 2\colon \text{there exists a periodic orbit of } f \text{ of period } q \text{ with } p \text{ elements in } S(f)\right\}$$

(if there is no such orbit, we set $\beta(f)=0$).

Remark 0. *There are two different definitions of* lim sup: *one can take into account the value of the function at the limit point or not. Since $h(\cdot)$ is lower semi-continuous, the definition of $\alpha(f)$ does not depend on which definition we use.*

The main result of the paper is:

THEOREM 1. *If $f\colon I\to I$ is a continuous piecewise monotone map then $\alpha(f)=\max(h(f),\beta(f))$.*

2. Inequality. In this section we prove the inequality

$$\alpha(f)\leqslant\max(h(f),\beta(f)). \tag{5}$$

We take a continuous piecewise monotone map $f\colon I\to I$ and fix it for the rest of this section

LEMMA 1. *Let ζ be a map from $T(f)$ to the family of all open (in I) subintervals of I such that, for each $J\in T(f)$, $J\subset\zeta(J)$. Then there exists a neighbourhood V of f in $\mathscr{U}(f)$ such that for every $g\in V$ and $J\in T(f)$ there exists $K\in T(g)$ such that $K\subset\zeta(J)$.*

Proof. If we take 3 points in $\zeta(J)$: x to the left of J, y in J, z to the right of J, then either $f(x)<f(y)$ and $f(z)<f(y)$ or $f(x)>f(y)$ and $f(z)>f(y)$. For some neighbourhood V_J of f, if $g\in V_J$ then the above inequalities hold with f replaced by g. Therefore if $g \in V_J$ then there is $K \in T(g)$ with $K \subset (x, z) \subset \zeta(J)$. We set $V = \bigcap_{J \in T(f)} V_J$. ∎

Remark 1. *If in the above situation* $\{\zeta(J)\}_{J \in T(f)}$ *are pairwise disjoint, then, since* $g \in \mathcal{U}(f)$, *there is exactly one* $K \in T(g)$ *in each* $\zeta(J)$ *and there are no more elements of* $T(g)$.

We define a partition $\mathscr{A}$ of I by taking all elements of $T(f)$ and all connected components of $I \setminus S(f)$. Notice that the elements of $T(f)$ are closed and the elements of $\mathscr{A} \setminus T(f)$ are open (in I). Clearly, the cover $\mathscr{A}$ is f-mono. For $\mathscr{J} = (J_0, J_1, \ldots, J_{n-1})$, where $J_0, J_1, \ldots, J_{n-1} \in \mathscr{A}$, we denote $A(\mathscr{J}) = \bigcap_{i=0}^{n-1} f^{-i}(J_i)$. Then we set $Z = \{\mathscr{J}\colon A(\mathscr{J}) \neq \emptyset\}$.

Lemma 2. *For each* $n \geqslant 1$ *there exists a neighbourhood* V_n *of* f *in* $\mathcal{U}(f)$ *and a map* ξ *from* $\mathscr{A}$ *to the family of all open (in* I*) subintervals of* I *such that if we denote* $D_g(\mathscr{J}) = \bigcap_{i=0}^{n-1} g^{-i}(\xi(J_i))$ *for* $\mathscr{J} = (J_0, J_1, \ldots, J_{n-1})$ *then for all* $g \in V_n$ *we have:*

(a) *if* $J \in T(f)$ *then* $\xi(J)$ *can be divided into two subintervals on which* g *is monotone,*

(b) *if* $J \in \mathscr{A} \setminus T(f)$ *then* g *is monotone on* $\xi(J)$,

(c) $\{D_g(\mathscr{J})\colon \mathscr{J} \in Z\}$ *is a cover of* I.

Proof. We fix $n \geqslant 1$. For each $J \in T(f)$ we take an open interval $\varphi(J)$ containing J in such a way that for $J_1 \neq J_2$ the intervals $\varphi(J_1)$ and $\varphi(J_2)$ are disjoint. For each $J \in \mathscr{A} \setminus T(f)$, we set $\varphi(J) = J$. Then $\{\varphi(J)\colon J \in \mathscr{A}\}$ is an open cover of I.

For $\mathscr{J} = (J_0, J_1, \ldots, J_{n-1}) \in Z$ set $B(\mathscr{J}) = \bigcap_{i=0}^{n-1} f^{-i}(\varphi(J_i))$. Since for all $J \in \mathscr{A}$ we have $J \subset \varphi(J)$ and $\{A(\mathscr{J})\colon \mathscr{J} \in Z\}$ is a partition of I, we obtain that $\{B(\mathscr{J})\colon \mathscr{J} \in Z\}$ is an open cover of I. Therefore we can find a compact cover $\{C(\mathscr{J})\colon \mathscr{J} \in Z\}$ of I such that $C(\mathscr{J}) \subset B(\mathscr{J})$ for all $\mathscr{J} \in Z$.

For each $J \in \mathscr{A}$ we define $\psi(J)$ as the union of the sets $f^i(C(\mathscr{J}))$ over all $\mathscr{J} = (J_0, J_1, \ldots, J_{n-1}) \in Z$ for which $J_i = J$ (and over all $i = 0, 1, \ldots, n-1$). Then $\psi(J)$ is a compact subset of $\varphi(J)$. Now we choose an open interval $\xi(J)$ such that

$$\psi(J) \subset \xi(J) \subset \overline{\xi(J)} \subset \varphi(J).$$

If $\mathscr{J} = (J_0, J_1, \ldots, J_{n-1}) \in Z$ and $0 \leqslant i \leqslant n-1$ then $f^i(C(\mathscr{J}))$ is a compact subset of $\xi(J_i)$. Therefore, if g is sufficiently close to f then also $g^i(C(\mathscr{J})) \subset \xi(J_i)$. Hence, if g is sufficiently close to f then $C(\mathscr{J}) \subset D_g(\mathscr{J})$ for all $\mathscr{J} \in Z$ and consequently (c) holds.

For each $J \in T(f)$ we can take an open interval $\zeta(J)$ containing J and disjoint from $\xi(K)$ for all $K \in \mathscr{A} \setminus \{J\}$. By Lemma 1 and Remark 1, there exists a neighbourhood V_n of f in $\mathcal{U}(f)$ such that if $g \in V_n$ then for each $J \in T(f)$ there is only one element of $T(g)$ inside $\zeta(J)$ and these are all elements of $T(g)$. Therefore (a) and (b) hold. ∎

Using the above lemma, we can easily prove (5). Let $\mathscr{J} = (J_0, J_1, \ldots, J_{n-1}) \in Z$ and $g \in V_n$. Denote by $k(\mathscr{J})$ the number of $i \in \{0, 1, \ldots, n-1\}$ for which $J_i \in T(f)$. Then by Lemma 2(a) and (b), $D_g(\mathscr{J})$ can be divided into $2^{k(\mathscr{J})}$ intervals on which g^n is monotone. In view of Lemma 2(c), it follows that

$$c_n(g) \leqslant \sum_{\mathscr{J} \in Z} 2^{k(\mathscr{J})}. \tag{6}$$

For $J \in T(f)$ and $0 \leqslant l \leqslant n-1$ we set

$$Y(J, l) = \{\mathscr{J} = (J_0, J_1, \ldots, J_{n-1}) \in Z\colon J_i \notin T(f) \text{ for } i < l \text{ and } J_l = J\}.$$

Notice that if $J \in T(f)$ then $f^i(J)$ consists of one point for all $i \geqslant 1$, and hence if $(J_0, J_1, \ldots, J_{n-1}) \in Z$ and $J_l = J$ then $J_{l+1}, \ldots, J_{n-1}$ are uniquely determined. Consequently $\operatorname{Card}(Y(J, l)) \leqslant \operatorname{Card}(\mathscr{A}^l)$, where $\mathscr{A}^l = \bigvee_{i=0}^{l-1} f^{-i}(\mathscr{A})$ (we count non-empty sets only). Clearly, if $\mathscr{J} \in Y(J, l)$ then $k(\mathscr{J}) = s(n-l, J)$, where

$$s(m, J) = \operatorname{Card}\{i\colon 0 \leqslant i \leqslant m-1 \text{ and } f^i(J) \subset S(f)\},$$

and if

$$\mathscr{J} \in X = Z \setminus \bigcup_{l=0}^{n-1} \bigcup_{J \in T(f)} Y(J, l)$$

then $k(J) = 0$. Then, looking at the decomposition of Z:

$$Z = X \cup \bigcup_{l=0}^{n-1} \bigcup_{J \in T(f)} Y(J, l),$$

we get

$$\sum_{\mathscr{J} \in Z} 2^{k(\mathscr{J})} \leqslant \operatorname{Card}(\mathscr{A}^n) + \sum_{l=0}^{n-1} \operatorname{Card}(\mathscr{A}^l) \sum_{J \in T(f)} 2^{s(n-l, J)}. \tag{7}$$

From (2), (6) and (7), we get

$$\sup_{g \in V_n} h(g) \leqslant \frac{1}{n} \log\Big(\operatorname{Card}(\mathscr{A}^n) + \sum_{l=0}^{n-1} \operatorname{Card}(\mathscr{A}^l) \sum_{J \in T(f)} 2^{s(n-l, J)}\Big). \tag{8}$$

Clearly,

$$\alpha(f) \leqslant \limsup_{n \to \infty} \big(\sup_{g \in V_n} h(g)\big).$$

From this, (8) and (4), we get

$$\alpha(f) \leqslant \max\Big(\limsup_{n \to \infty} \frac{1}{n} \log \operatorname{Card}(\mathscr{A}^n),\ \limsup_{n \to \infty} \frac{1}{n} \log \sum_{J \in T(f)} 2^{s(n, J)}\Big). \tag{9}$$

Since $\mathscr{A}$ is an f-mono partition, we have in view of (3)

$$\lim_{n\to\infty}\frac{1}{n}\log\operatorname{Card}(\mathscr{A}^n) = h(f,\mathscr{A}) = h(f). \tag{10}$$

Clearly,

$$\limsup_{n\to\infty}\frac{1}{n}\log\sum_{J\in T(f)}2^{s(n,J)} = \max_{J\in T(f)}\left(\limsup_{n\to\infty}\frac{1}{n}s(n,J)\right)\log 2. \tag{11}$$

If $J\in T(f)$ then:

if for some $i>0$, $f^i(J)=\{x\}$ with x belonging to a periodic orbit of period q with p elements in $S(f)$, then

$$\lim_{n\to\infty}\frac{1}{n}s(n,J)=\frac{p}{q};$$

otherwise $\lim\limits_{n\to\infty}\frac{1}{n}s(n,J)=0$.

From this, (9), (10) and (11) there follows (5).

3. Construction of jumps of entropy. In view of (5) and Remark 0, to complete the proof of Theorem 1, it is sufficient to prove

$$\alpha(f)\geqslant\beta(f). \tag{12}$$

We may assume that f is not constant (globally).

Let $(x_0,x_1,\ldots,x_{q-1})$ be a periodic orbit of f of period q such that $f(x_i)=x_{i+1}$ for $i=0,1,\ldots,q-1$ (in the whole section the addition of the subindices is mod q) and $x_i\neq x_j$ for $i\neq j$. Assume that p of the points $x_0,x_1,\ldots,x_{q-1}$ belong to $S(f)$. We shall show that

(13) *arbitrarily close to f there are maps in $\mathscr{U}(f)$ with entropy at least* $\frac{p}{q}\log 2$.

If J is an interval, then we will denote the set of its endpoints by ∂J. We will also use the notation introduced in Section 1.

If $\varepsilon>0$ is sufficiently small then there exist open (in I) intervals D_i, $i=0,1,\ldots,q-1$, such that

(i) $x_i\in D_i$,
(ii) $D_i\cap D_j=\emptyset$ for $i\neq j$,
(iii) the intersection $D_i\cap S(f)$ is empty if $x_i\notin S(f)$ and is contained in J if $x_i\in J\in T(f)$,
(iv) if $x_i\in J$ for some $J\in\mathscr{C}(f)$ with $J\cap\partial I\neq\emptyset$ then $J\subset D_i$,
(v) if $y\in\partial D_i\setminus\partial I$ then $|f(y)-x_{i+1}|=\varepsilon$.

We define a continuous map $g\colon I\to I$ by setting $g=f$ outside $\bigcup_{i=0}^{q-1}D_i$, then defining g at some points inside $\bigcup_{i=0}^{q-1}D_i$ and extending g linearly (speaking more precisely, affinely) in the remaining gaps. These points in D_i are x_i (not always), $x_i\pm\delta$ (if they belong to I), and occasionally the endpoints of I. We define δ in such a way that

(vi) $0<\delta<\varepsilon$,
(vii) $\delta<|x_i-y|$ for all i and $y\in\partial I$ for which $x_i\neq y$,
(viii) $[x_i-\delta,x_i+\delta]\cap I\subset D_i$ for all i.

We denote

$$x_i^-=\begin{cases}x_i-\delta & \text{if } x_i \text{ is not the left-hand endpoint of } I,\\ x_i & \text{if } x_i \text{ is the left-hand endpoint of } I,\end{cases}$$

$$x_i^+=\begin{cases}x_i+\delta & \text{if } x_i \text{ is not the right-hand endpoint of } I,\\ x_i & \text{if } x_i \text{ is the right-hand endpoint of } I.\end{cases}$$

If f is non-decreasing on D_i then we set

$$g(x_i^-)=x_{i+1}^-,\quad g(x_i^+)=x_{i+1}^+.$$

Analogously, if f is non-increasing on D_i then we set

$$g(x_i^-)=x_{i+1}^+,\quad g(x_i^+)=x_{i+1}^-.$$

Additionally, if $x_i\in J\in\mathscr{C}(f)$, $J\cap\partial I=\{y\}$ and $x_i\neq y$ then we set

$$g(y)=\begin{cases}g(x_i^-) & \text{if } y \text{ is the left-hand endpoint of } I,\\ g(x_i^+) & \text{if } y \text{ is the right-hand endpoint of } I.\end{cases}$$

In the remaining cases, $x_i\in J\in T(f)$. Then we set

$$g(x_i)=\begin{cases}x_{i+1}^- & \text{if } l(J)+r(J)=2,\\ x_{i+1}^+ & \text{if } l(J)+r(J)=-2,\end{cases}$$

$$g(x_i^-)=g(x_i^+)=\begin{cases}x_{i+1}^+ & \text{if } l(J)+r(J)=2,\\ x_{i+1}^- & \text{if } l(J)+r(J)=-2.\end{cases}$$

It is easy to see that $g\in\mathscr{U}(f)$ and

$$\sup_{y\in I}|g(y)-f(y)|<\varepsilon+2\delta<3\varepsilon.$$

For all i we have

$$g([x_i^-,x_i^+])=[x_{i+1}^-,x_{i+1}^+]$$

and if $x_i\in S(f)$ then g is 2-to-1 on $[x_i^-,x_i^+]$. Therefore $h(g)\geqslant\frac{p}{q}\log 2$. This proves (13).

Since our periodic orbit was chosen arbitrarily, (12) follows. Hence, the proof of Theorem 1 is complete. ■

Remark 2. *Clearly, as in* [10], *everything done above can be done also in the case of the circle instead of the interval.*

4. Dependence of $\beta(f)$ upon f. Since Theorem 1 shows that the behaviour of the entropy depends on $\beta(f)$, it may be interesting to investigate the behaviour of β itself.

PROPOSITION 1. (a) *If $\beta(f)>0$ then there is a neighbourhood U of f in $\mathcal{U}(f)$ such that for every $g\in U$, $\beta(g)\leqslant\beta(f)$.*

(b) *If $\beta(f)=0$ then for every $\varepsilon>0$ there is a neighbourhood U of f in $\mathcal{U}(f)$ such that for every $g\in U$, $\beta(g)<\varepsilon$.*

Proof. For a periodic orbit Q of $g\in\mathcal{U}(f)$ let us denote by β_Q the number of elements of Q belonging to $S(g)$ divided by the period (i.e. cardinality) of Q. Since the image of each $K\in T(g)$ consists of one point, if $x, y\in Q$ and $x\neq y$ then x and y cannot belong to the same $K\in T(g)$. Consequently,

$$\beta_Q\leqslant\frac{\operatorname{Card}(T(g))}{\operatorname{Card}(Q)}.$$

Since $g\in\mathcal{U}(f)$, $\operatorname{Card}(T(g))\leqslant\operatorname{Card}(T(f))$. Therefore

$$\beta_Q\leqslant\frac{\operatorname{Card}(T(f))}{\operatorname{Card}(Q)}. \tag{14}$$

Fix $n>1$. We can choose a map ζ as in Lemma 1 with $\{\zeta(J)\}_{J\in T(f)}$ pairwise disjoint and such that for each $0<i\leqslant n$ and $K\in T(f)$, either $f^i(K)\subset S(f)$ or $f^i(K)\cap\bigcup_{J\in T(f)}\zeta(J)=\emptyset$ (remember that each $f^i(K)$ consists of one point).

Let V be the neighbourhood of f from Lemma 1 and $g\in V$. By Remark 1, there is a bijection $\psi\colon T(f)\to T(g)$ such that $\psi(J)\subset\zeta(J)$ for each $J\in T(f)$. By the continuity argument, there is a neighbourhood U_n of f in $\mathcal{U}(f)$ such that if $g\in U_n$, $0<i\leqslant n$ and $K\in T(f)$ then either

$$f^i(K)\subset M \text{ for some } M\in T(f) \text{ and } g^i(\psi(K))\subset\zeta(M) \tag{15}$$

or

$$g^i(\psi(K))\cap S(g)=\emptyset. \tag{16}$$

Let Q be a periodic orbit of g. There are 3 cases possible:

1. $Q\cap S(g)=\emptyset$,
2. $Q\cap S(g)\neq\emptyset$ and $\operatorname{Card}(Q)>n$,
3. $Q\cap S(g)\neq\emptyset$ and $\operatorname{Card}(Q)\leqslant n$.

In the first case, $\beta_Q=0$. In the second case, by (14),

$$\beta_Q<\frac{\operatorname{Card}(T(f))}{n}.$$

The third case is the most complicated one. Let us take $x\in Q\cap S(g)$. Denote $m=\operatorname{Card}(Q)$. For some $K\in T(f)$ we have $x\in\psi(K)$. For each $0<i\leqslant m$ we have either (15) or (16). However, $g^i(\psi(K))=\{g^i(x)\}$, and thus (16) is equivalent to

$$g^i(x)\notin S(g). \tag{17}$$

Since $g^m(x)=x\in S(g)$, we have (15) for $i=m$. Since

$$g^m(\psi(K))=\{g^m(x)\}=\{x\}\subset\zeta(K)$$

and $\{\zeta(J)\}_{J\in T(f)}$ are pairwise disjoint, we obtain in this case $M=K$. Therefore $f^m(K)\subset K$ and consequently ther exists $y\in K$ such that $f^m(y)=y$. If $0<i\leqslant m$ and $f^i(y)\notin S(f)$ then (15) does not occur and consequently (17) does. Hence, $\beta_Q\leqslant\beta(f)$.

In all three cases we get

$$\beta_Q\leqslant\max\left(\beta(f),\frac{\operatorname{Card}(T(f))}{n}\right).$$

Since this applies to all periodic orbits of g, we obtain for all $g\in U_n$

$$\beta_Q(g)\leqslant\max\left(\beta(f),\frac{\operatorname{Card}(T(f))}{n}\right).$$

To prove (a), we take $n>\dfrac{\operatorname{Card}(T(f))}{\beta(f)}$ and $U=U_n$; to prove (b) we take $n>\dfrac{\operatorname{Card}(T(f))}{\varepsilon}$ and $U=U_n$. ■

COROLLARY 1. *$\beta(\cdot)$ as a function on $\mathcal{U}(f)$ is upper semi-continuous.*

5. Unimodal maps. Let us look closer at the case when $T(f)$ consists of one element. We shall call such maps *weakly unimodal* (cf [9]). Recall that f is *unimodal* if it is weakly unimodal and there are no proper intervals on which f is constant. Notice that essentially the whole kneading theory (see e.g. [7], [3]), as long as it does not use any smoothness, works as well for weakly unimodal maps. Of course we have to replace the turning point by $S(f)$. When we will use the kneading theory, we will keep to the notations of [3].

THEOREM 2. *The topological entropy, as a function on the set of all weakly unimodal maps, is continuous at all points at which it is positive.*

Proof. By [10], the topological entropy is lower semi-continuous. Hence, by Theorem 1, it is enough to prove that if f is weakly unimodal and $h(f)>0$ then $h(f)\geqslant\beta(f)$. It is known (see [3], [5], [6]) that if

$$\frac{1}{2^{n+1}}\log 2<h(f)\leqslant\frac{1}{2^n}\log 2$$

then there exists an interval J, containing $S(f)$, such that the interiors of the intervals $J, f(J), f^2(J), f^3(J), \ldots, f^{2^n-1}(J)$ are pairwise disjoint, $J^{2^n}(J)\subset J$ and $h(f) = \frac{1}{2^n}h(f^{2^n}|_J)$. Moreover, $f^{2^n}|_J$ is also weakly unimodal, and therefore (since its entropy is positive) it has no fixed point in $S(f^{2^n}|_J)$. Consequently, each periodic orbit containing an element of $S(f)$ has period $k2^n$ for some $k>1$. From this it follows that

$$\beta(f)\leqslant\frac{1}{k2^n}\log 2\leqslant\frac{1}{2^{n+1}}\log 2\leqslant h(f). \blacksquare$$

Remark 3. In one-parameter families of piecewise smooth (even piecewise linear) unimodal maps, the jumps of entropy from 0 to some positive value often do occur. One can easily see that if for example the one-sided derivatives of f at the critical point are 1 from the left and $-\alpha$ from the right, then

$$\lim_{\mu\searrow 0} h(f_\mu)\geqslant\log\gamma_{E(\alpha)}$$

where $f_\mu(x) = f(x)+\mu$, $E(\alpha)$ is the integer part of α, and γ_m is the largest zero of the polynomial $x^{m+1}-2x^m+1$ (this is due to the appearence of a certain periodic point in the piecewise linear situation; the polynomial can be easily computed by the methods of [1]). If $\alpha=\infty$ then we obtain $\lim_{\mu\searrow 0} h(f_\mu) = \log 2$ (this time we obtain a full 2-horseshoe). Similar considerations can be applied to some iterates of f and slightly different families.

For example, we can consider two families of piecewise linear maps:

1. $f_\lambda(0) = 0$, $f_\lambda(\frac{1}{2}) = 1$, $f_\lambda(1) = \lambda$; linear in between,
2. $f_\lambda(0) = \lambda$, $f_\lambda(\frac{1}{2}) = 1$, $f_\lambda(1) = 0$; linear in between.

The second family has been studied by Brodiscou, Gillot and Gillot (see [2]). For the first family we look at the second iterate and discover the jump of the entropy from 0 to $\frac{1}{2}\log\gamma_2$ at $\lambda=\frac{1}{2}$. For the second family we look at the fourth iterate and discover the jump of the entropy from 0 to $\frac{1}{4}\log\gamma_4$ at $\lambda=\frac{3}{4}$.

The next result is a kind of an intermediate value theorem for kneading invariants. For the smooth unimodal maps a theorem of this type has been proved already by Milnor and Thurston [7]. Here we deal with continuous maps, so necessarily our result has to be weaker. We shall formulate this result for weakly unimodal maps; however as we mentioned already, the kneading theory works in this case as well. The kneading invariant of f will be denoted by $K(f)$.

For any $1<\mu\leqslant 2$ we consider the unimodal map $g_\mu\colon [0,1]\to[0,1]$ with the constant slope μ:

$$g_\mu(x) = \frac{\mu}{2}-\mu\left|x-\frac{1}{2}\right|.$$

We have $h(g_\mu) = \log\mu$ (see [10]).

THEOREM 3. *Let (f_λ) be a continuous one-parameter family of weakly unimodal maps. Assume that $0<h(f_{\lambda_1})<\log\mu<h(f_{\lambda_2})$, and that $h(f_\lambda)>0$ for all λ between λ_1 and λ_2. Then there exists λ_0 between λ_1 and λ_2 such that $K(f_{\lambda_0}) = K(g_\mu)$.*

Proof. It is well known that if $K(g_\mu)$ is infinite then it is the only possible kneading sequence of a (weakly) unimodal map with the entropy $\log\mu$ (it has been essentially proved in [5]). Therefore, in this case, the existence of λ_0 with $K(f_{\lambda_0}) = K(g_\mu)$ follows immediately from Theorem 2.

Consider now the case of $K(g_\mu)$ finite. Then we can find μ_1 and μ_2 such that $K(g_{\mu_1})$ and $K(g_{\mu_2})$ are infinite,

$$h(f_{\lambda_1})<\log\mu_1<\log\mu<\log\mu_2<h(f_{\lambda_2})$$

and for all $\vartheta\in[\mu_1,\mu_2]$ if $\vartheta\neq\mu$ then $K(g_\vartheta)$ is longer than $K(g_\mu)$. By Theorem 2, there exist λ_3 and λ_4 between λ_1 and λ_2 such that

$$\{h(f_{\lambda_3}), h(f_{\lambda_4})\} = \{\log\mu_1, \log\mu_2\}$$

and $K(f_\lambda)$ is longer than $K(g_\mu)$ for all λ between λ_3 and λ_4, unless $K(f_\lambda) = K(g_\mu)$.

Assume that $K(f_\lambda)<K(g_\mu)$ and $K(f_\lambda)$ is longer than $K(g_\mu)$. If we take a neighbourhood U of $S(f_\lambda)$ then for all ϑ sufficiently close to λ we have $S(f_\vartheta)\subset U$. Hence, in this situation if ϑ is sufficiently close to λ then $K(f_\vartheta)<K(g_\mu)$. The same is true if we replace "$<$" by "$>$". Hence, if we assume that $K(f_\lambda)\neq K(g_\mu)$ for all λ between λ_3 and λ_4 then the interval with these endpoints can be divided into two open disjoint sets — a contradiction. This ends the proof in the case of $K(g_\mu)$ finite. $\blacksquare$

References

[1] L. Block, J. Guckenheimer, M. Misiurewicz and L.-S. Young, *Periodic points and topological entropy of one dimensional maps*, in *Global theory of dynamical systems*, Lecture Notes in Math. 819, Springer Verlag, Berlin 1980, 18–34.

[2] C. Broudiscou, Ch. Gillot and G. Gillot, *Variations d'entropie d'une famille de transformations de l'intervalle unite, unimodales et linéaires par morceaux*, C. R. Acad. Sci. Paris 300 (1985), ser. I, n. 16, 585–588.

[3] P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Progr. in Phys. 1, Birkhauser, Boston 1980.

[4] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic Theory on Compact Spaces*, Lecture Notes in Math. 527, Springer Verlag, Berlin 1976.

[5] J. Guckenheimer, *Sensitive dependence on initial conditions for one-dimensional maps*, Comm. Math. Phys. 70 (1979), 133–160.

[6] L. Jonker and D. Rand, *Bifurcations in one dimension*, I: *The nonwandering set*, Invent. Math. 62 (1981), 347–365.
[7] J. Milnor and W. Thurston, *On iterated maps of the interval*, preprint (1977).
[8] M. Misiurewicz, *Horseshoes for mappings of an interval*, Bull. Acad. Pol. Sci., Ser. sci. math., astron. and phys. 27 (1979), 167–169.
[9] M. Misiurewicz and J. Smital, *Smooth chaotic maps with zero topological entropy*, preprint (1987).
[10] M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. 67 (1980), 45–63.
[11] S. Newhouse, *Continuity properties of entropy*, preprint (1986).

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The Lipschitz condition for the conjugacies of Feigenbaum-like mappings

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Abstract. For a map f in the stable manifold $W^s(g)$ of the Feigenbaum function g the conjugacies h, h^{-1}: $h\circ f\circ h^{-1} = g$ are Lipschitz continuous maps at points of the Cantor set attractors. Moreover, h and h^{-1} occur to be Lipschitz continuous on the whole interval $[-1, 1]$ if and only if the products of derivatives of f as taken over periodic orbits are all equal.

§ 0. Introduction. In this paper, we study some properties of mappings topologically conjugate to Feigenbaum's fixed point, i.e. a concave analytic solution $g\colon [-1, 1]\to[-1, 1]$ of the functional equation $Tg = g$ with $Tf(x)$ defined as in Section 1.

We are interested in even analytic functions f conjugate to g and such that for inductively defined $T^n f = T(T^{n-1}f)$ we have $T^n f\to g$ with exponential rate.

For f chosen like above a conjugacy h: $g\circ h = h\circ f$ is uniquely given by the kneading invariant. Furthermore, there exists an f-invariant Cantor set attractor, such that $\lim_{n\to\infty}\operatorname{dist}(f^n(x), J(f)) = 0$ for every x which is not eventually periodic.

We show (§ 1) that h considered as a mapping with the domain restricted to $J(f)$ is a Lipschitz continuous function. Using this, we also prove (§ 2) that there exists a constant $\mathscr{K}$ such that h fulfils the Lipschitz condition with this constant at arbitrarily chosen point $x\in J(f)$ with respect to any point $y\in[-1, 1]$, when regarded h as a function from $[-1, 1]$ into itself. This leads us to deal with general question when h: $[-1, 1]\to[-1, 1]$ can be Lipschitz continuous on the whole interval. The answer as mentioned in the abstract is given in § 3.

The results of this paper are an expanded version of §§ 1, 2 of my Masters Thesis written in 1985 under the supervision of professor Michał Misiurewicz; I would like to thank him for calling my interest to the problem and encouragement.

After this paper was written I have learnt that D. Sullivan obtained the result covering the statement of Theorem 1.

Finally, in § 3 there is stated the question of analyticity of h, which seems to be an interesting direction of further work, by similarity to the known results for expanding mappings of the circle (cf. [7], [8]).