that $\mathrm{Ind}\varphi_1 \& \ldots \& \mathrm{Ind}\varphi_n \vdash T$. Since all the functions $F_i(t)$ are provably total in $I^-\Sigma_1$, T proves that F_{n+2} is total. By our theorem, $F_{n+2} < F_{n+1}$ almost everywhere, contradiction.

References

[A] Z. Adamowicz, *Algebraic approach to $\exists_1$ induction*, Seminarbericht Nr. 70, Proceedings of the Third Easter Conference on Model Theory, Groß Köris, April 8–13, 1985, pp. 5–15

[G] A. Grzegorczyk, *Some classes of recursive functions*, Rozprawy Matematyczne 4 (1953).

[W] S. Wainer, *Ordinal recursion and a refinement of the extended Grzegorczyk hierarchy*, J. Symbolic Logic 37 (1972), 281–292.

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Received 6 December 1986;
in revised form 4 January 1988

An isomorphism theorem of Hurewicz type in the proper homotopy category

by

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Abstract. Numerous mathematicians have proved theorems of Hurewicz type in different contexts shape theory, pro-categories, coherent categories. In this paper we obtain a Hurewicz Theorem in the proper homotopy category. In particular, we prove:

THEOREM. *Let (X, A) be a proper pair such that $\pi_0(X)$, $\pi_0(A)$, $\tau_0(X)$, $\tau_0(A)$ are trivial. Suppose that for $n \geqslant 2$ (X, A) is (π) n-connected and $(\tau)(n-1)$-connected. Then for each proper ray α in A, $\varrho_\tau\colon \tau_n(X, A, \alpha)/(\Omega^n_\tau(X, A, \alpha) \to J_{n+1}(X, A)$ is an isomorphism. In the case where (X, A) is (τ) n-simple, for example if $\pi_1(A, \alpha) = 0$, then $\varrho_\tau\colon \tau_n(X, A, \alpha) \to J_{n+1}(X, A)$ is an isomorphism.*

1. Introduction. A natural relationship between singular homology groups and Hurewicz homotopy groups is displayed by Hurewicz's Theorem. This theorem was established in terms of simplicial homology and absolute homotopy groups by Hurewicz [11] in 1935 for simply connected polyhedra. In 1944, Eilenberg proved that the fundamental group modulo the commutator subgroup is the first singular homology group. Blakers [2], the proposer of the concept of relative homology groups, proved in 1948 the Hurewicz Theorem in the relative case given the kernel of a homomorphism.

There are more Hurewicz type theorems in other homotopy theories. For example, in 1969 Artin and Mazur [1] proved a Hurewicz Theorem in the category pro-$\mathscr{R}_0$, where $\mathscr{R}_0$ is the pointed homotopy category of connected pointed CW-complexes, and pro-$\mathscr{R}_0$ is the category of inverse systems of objects of $\mathscr{R}_0$. Relative Hurewicz type theorems for pro-$\mathscr{R}_0^2$ and Sh^2 were proved by Mardešić and Ungar [15] and independently by Morita [16]. Raussen [17] proved a Hurewicz type theorem in pro-Ho(Top_*), where Ho(Top_*) is the homotopy category of pointed topological spaces. In 1972, Kuperberg [13] proved another Hurewicz type Theorem between the homotopy groups defined by Borsuk and the Vietoris-Čech homology groups. In 1979, Kodama and Koyama [12] proved a Hurewicz type theorem between the Quigley approaching groups and the Steenrod homology groups. In a recent paper, Koyama proved a Hurewicz Theorem in the coherent homotopy category of inverse systems of spaces CPHTop.

The present paper is devoted to the study of Hurewicz theorems in the proper homotopy category of pairs with base ray and proper maps. First we introduce certain proper invariants: In 1980 Čerin [4] defined the groups $\pi_n(X, \alpha)$ associated with the space X and the proper ray $\alpha\colon [0, \infty) \to X$. In his definition Čerin was considering proper homotopy classes of proper maps of the form $f\colon S^n \times [0, \infty) \to X$. In 1984 the second author of this paper defined the proper groups $\tau_n(X, \alpha)$ taking proper maps of type $S^n \times [0, \infty)/S^n \times 0 \to X$, see [7]. Independently, these groups have been considered by Brin and Thickstun [3]. An alternative definition was given in [8] in terms of the model category defined by Edwards and Hasting in [5]. We recall these notions with more details in Section 3. In the same preprint [7], Hernandez also defined the proper end homology E_* and the singular proper homology J_*. We recall the definition of these proper homologies in Section 4.

In Section 5, first we show that the Hurewicz Theorem for the transformation $\varrho_\pi\colon \pi_n(X, A, \alpha) \to E_{n+1}(X, A)$ can be reduced to the "classical" Hurewicz Theorem. This, however, is not the case with the transformation $\varrho_\tau\colon \tau_n(X, A, \alpha) \to J_{n+1}(X, A)$. The main result of this paper is a Hurewicz isomorphism theorem for the relative case and the natural transformation ϱ_τ (see Theorem 5.14). Notice that in this theorem to the usual conditions

$$\tau_i(X, A, \alpha) = 0 \quad \text{for } i < n$$

we must add other conditions on the Hurewicz groups:

$$\pi_i(X, \alpha(0)) = 0, \quad i \leqslant n.$$

We have also included in Section 6 an example illustrating how all these proper invariants can be computed by applying exact sequences and the Hurewicz Theorems for proper groups.

2. Notation and preliminaries. We use the following notation. I denotes the closed interval $[0, 1]$ and $I^n = I \times \overset{(n)}{\ldots} \times I$. J denotes the semiopen interval $[0, \infty)$ and $J^n = J \times \overset{(n)}{\ldots} \times J$. $\boldsymbol{R}$ denotes the set of real numbers. $\boldsymbol{R}^n = \boldsymbol{R} \times \overset{(n)}{\ldots} \times \boldsymbol{R}$, $\boldsymbol{Z}$ denotes the set of integers. D^n the n-disc, and S^n the n-sphere.

Definition 2.1. Let X, Y be topological spaces. A continuous map $f\colon X \to Y$ is said to be *proper* if $f^{-1}(K)$ is compact whenever K is a closed compact subset of Y.

Two given proper maps $f, g\colon X \to Y$ are said to be *properly homotopic* if there is a homotopy from f to g, which is proper. A subspace A of X is said to be *proper* if the inclusion map of A into X is proper. In this case, we say that (X, A) is a *proper pair*. In a natural way, one can define *proper maps* between proper pairs and proper homotopies between proper maps of this type. Similarly, one can define proper triplets. A ray in X is a proper map $\alpha\colon J \to X$. A *proper map between two spaces with base ray* $f\colon (X, \alpha) \to (Y, \beta)$ is a proper map $f\colon X \to Y$ satisfying $f \circ \alpha = \beta$. Similarly, one can define proper maps between proper pairs with base ray and the corresponding proper homotopies.

A subspace of $\boldsymbol{R}^n$ of the form $K_1 \times \ldots \times K_n$, where either $K_i = I$ or $K_i = J$ for each $i = 1, \ldots, n$, will be called a *proper n-cube*. Let $\partial(K_1 \times \ldots \times K_n)$ denote the subspace $\bigcup_{i,l} K_1 \times \ldots \times l \times \ldots \times K_n$ where $i = 1, \ldots, n$, and $l = 0, 1$ if $K_i = I$ and $l = 0$ if $K_i = J$. We say that $\partial(K_1 \times \ldots \times K_n)$ is the *boundary of* $K_1 \times \ldots \times K_n$.

One can define the *absolute proper homotopy extension property* (APHEP) in the same way as this is done with usual homotopy.

We shall use the following:

Proposition 2.2. *The boundary of a proper n-cube has the* APHEP.

Proof. If $K_1 \times \ldots \times K_n$ is compact, see [10. I. 9]. Otherwise, define a proper retraction

$$r\colon K_1 \times \ldots \times K_n \times I \to K_1 \times \ldots \times K_n \times 0 \cup \partial(K_1 \times \ldots \times K_n) \times I$$

by projecting from the point $(1/2, \ldots, 1/2, 2) \in \boldsymbol{R}^{n+1}$.

Now, suppose that $f\colon K_1 \times \ldots \times K_n \to Y$ is a proper map and $H\colon \partial(K_1 \times \ldots \times K_n) \times I \to Y$ is a partial proper homotopy of f. Consider the proper map G defined by $G|_{K_1 \times \ldots \times K_n \times 0} = f$, $G|_{\partial(k_1 \times \ldots \times K_n) \times I} = H$. Then $F = Gr$ is a proper extension of G to $K_1 \times \ldots \times K_n \times I$.

3. Proper homotopy groups. In this section we recall the definition of proper homotopy groups and some of their properties.

Let (X, A, α) a proper pair with base ray. In [4] Čerin defined $\pi_n(X, \alpha)$ as the set of proper homotopy classes of proper maps of type $f\colon (S^n \times J, * \times J) \to (X, \alpha)$, $* \in S^n$, $f(*, t) = \alpha(t)$, under the proper homotopy relation relative to $* \times J$. One can also define $\pi_n(X, A, \alpha)$ by considering proper maps of type $f\colon (D^n \times J, S^{n-1} \times J, * \times J) \to (X, A, \alpha)$. In the present paper we choose the approach applied in [18] for the description of these groups and their properties stated there.

For each $n \geqslant 0$, $\tau_n(X, \alpha)$ denotes the set of classes of proper maps $f\colon (I^n \times J, \partial I^n \times J, I^n \times 0) \to (X, \alpha, \alpha(0))$.

$$f(x, t) = \alpha(t) \quad \text{for every } (x, t) \in \partial I^n \times J,$$

under the following relation: f, g are related if and only if there is a proper homotopy

$$H\colon (I^n \times J \times I, \partial I^n \times J \times I, I^n \times 0 \times I) \to (X, \alpha, \alpha(0))$$

relative to $(\partial I^n \times J, I^n \times 0)$ and such that

$$H(x, t, 0) = f(x, t), \quad (x, t) \in I^n \times J,$$

$$H(x, t, 1) = g(x, t), \quad (x, t) \in I^n \times J.$$

In the relative case, in order to define $\tau_n(X, A, \alpha)$ we consider proper maps of type

$$f\colon (I^n\times J, I^{n-1}\times J, T^{n-1}\times J, I^n\times 0)\to (X, A, \alpha, \alpha(0)),$$
$$f(x, t) = \alpha(t), \quad \text{for every } (x, t)\in T^{n-1}\times J,$$

where I^{n-1} stands for $I^{n-1}\times 0$, and T^{n-1} denotes the union of the other $(n-1)$-faces of I^n. The relations are given by proper homotopies

$$H\colon (I^n\times J\times I, I^{n-1}\times J\times I, T^{n-1}\times J\times I, I^n\times 0\times I)\to (X, A, \alpha, \alpha(0))$$

relative to $(T^{n-1}\times J, I^n\times 0)$.

In the same way we define $\pi_n(X, \alpha)$, $\pi_n(X, A, \alpha)$ taking proper maps

$$f\colon (I^n\times J, \partial I^n\times J, \partial I^n\times 0)\to (X, \alpha, \alpha(0)),$$
$$f\colon (I^n\times J, I^{n-1}\times J, T^{n-1}\times J, T^{n-1}\times 0)\to (X, A, \alpha, \alpha(0)), \text{ respectively.}$$

Notice that $\pi_0(X, \alpha)$ is the set of proper homotopy classes from J to X. We say that $\pi_0(X, \alpha)$ is the *set of proper ends* of X.

Given two proper maps $f, g\colon I^n\times J\to X$, $n>0$, such that

$$f(1, x_2, \ldots, x_n, t) = g(0, x_2, \ldots, x_n, t),$$

define $f+g\colon I^n\times J\to X$ by

$$f+g(x_1, x_2, \ldots, x_n, t) = \begin{cases} f(2x_1, x_2, \ldots, x_n, t), & \text{if } 0\leqslant x_1\leqslant 1/2,\\ g(2x_1-1, x_2, \ldots, x_n, t) & \text{if } 1/2\leqslant x_1\leqslant 1.\end{cases}$$

This operation gives group structure to $\tau_n(X, \alpha)$, $\pi_n(X, \alpha)$ for $n\geqslant 1$; the resulting groups are abelian for $n\geqslant 2$. In the relative case we get group structure on $\tau_n(X, A, \alpha)$, $\pi_n(X, A, \alpha)$ for $n\geqslant 2$, abelian for $n\geqslant 3$.

Definition 3.1. A space X is said to be (τ)*n-connected* if $\tau_q(X, \alpha) = 0$ for every $q\leqslant n$ and for every ray α in X. A proper pair (X, A) is said to be (τ)*n-connected* if $\tau_q(X, A, \alpha) = 0$ for every $q\leqslant n$ and for every ray α in A ($\tau_0(X, A, \alpha) = 0$ means that $\tau_0(A, \alpha)\to\tau_0(X, \alpha)$ is a surjection). In the same way define (π)*n-connectedness* for spaces and proper pairs.

For each $n\geqslant 1$ ($n\geqslant 2$), π_n, τ_n are covariant functors from the category of spaces (proper pairs) with base ray and proper maps to the category of groups and homomorphisms. These functors are invariants of the proper homotopy type; moreover, there is an exact sequence associated with a based proper pair (X, A, α) for each one of these functors. There are similar exact sequences for based proper triplets.

Now we are going to inspect relationships between the proper homotopy groups above and the Hurewicz homotopy groups.

Given a space with base ray (X, α), there is a natural map (which is homomorphism for $n\geqslant 1$)

(1) $$\varphi_\alpha\colon \pi_{n+1}(X, \alpha(0))\to\tau_n(X, \alpha)$$

defined as follows: Let $f\colon (I^{n+1}, \partial I^{n+1})\to (X, \alpha(0))$ be a representative map of an element ξ of $\pi_{n+1}(X, \alpha(0))$. Define

$$G\colon I^n\times I\times 0\cup I^n\times 0\times J\cup\partial I^n\times I\times J\to X \quad \text{by}$$
$$G(x, t, 0) = f(x, t) \quad \text{if } (x, t)\in I^n\times I,$$
$$G(x, 0, s) = \alpha(s) \quad \text{if } (x, s)\in I^n\times J,$$
$$G(y, t, s) = \alpha(s) \quad \text{if } (y, t, s)\in\partial I^n\times I\times J.$$

By the APHEP, see Proposition 2.2, the proper map G extends to a proper homotopy $F\colon I^n\times I\times J\to X$. Now we define $\varphi_\alpha(\xi)$ as the element of $\tau_n(X, \alpha)$ represented by F_1, where $F_1(x, s) = F(x, 1, s)$.

We also have another natural map,

(2) $$\Psi\colon \tau_n(X, \alpha)\to\pi_n(X, \alpha),$$

which sends the element of $\tau_n(X, \alpha)$ represented by a proper map f to the element of $\pi_n(X, \alpha)$ represented by the same f.

Finally, there is a natural boundary map

(3) $$\delta\colon \pi_n(X, \alpha)\to\pi_n(X, \alpha(0))$$

defined as follows: if f represents an element $\eta\in\pi_n(X, \alpha)$, then $\delta(\eta)$ is the element of $\pi_n(X, \alpha(0))$ represented by $\delta_0 f$, where $\delta_0 f(x) = f(x, 0)$ for each $x\in I^n$.

Using these transformations we can state:

Proposition 3.2. *For each $n\geqslant 0$, the following sequence is exact*

$$\ldots\to\pi_{n+1}(X, \alpha(0))\xrightarrow{\varphi_\alpha}\tau_n(X, \alpha)\xrightarrow{\psi}\pi_n(X, \alpha)\xrightarrow{\delta}\pi_n(X, \alpha(0))\to\ldots$$

A detailed proof of this Proposition can be found in [18], and another one (using pro-category methods) in [8].

An analogous exact sequence is obtained in the relative case, and we have the following commutative diagram:

$$\begin{array}{ccccccccc}
\ldots\to & \pi_{n+1}(X, A, \alpha(0)) & \xrightarrow{\varphi_\alpha} & \tau_n(X, A, \alpha) & \xrightarrow{\psi} & \pi_n(X, A, \alpha) & \xrightarrow{\delta} & \pi_n(X, A, \alpha(0)) & \to\ldots\\
& \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & \\
\ldots\to & \pi_{n+1}(A, \alpha(0)) & \xrightarrow{\varphi_\alpha} & \tau_n(A, \alpha) & \xrightarrow{\psi} & \pi_n(A, \alpha) & \xrightarrow{\delta} & \pi_n(A, \alpha(0)) & \to\ldots
\end{array}$$

where ∂ is the boundary operator of the corresponding sequences associated with (X, A, α).

We shall also need a certain notion of proper simplicity. There are various types of actions on the groups above, and we are going to have a closer look at some of them.

Definition 3.3. Let α, β be rays in X. A proper map $\mu\colon I\times J\to X$, such that $\mu(0, t) = \alpha(t)$, $\mu(1, t) = \beta(t)$ for every $t\in J$, will be called a *path from α to β*. Let γ be another ray in X, and ϱ a path from β to γ. Define the *composed path* $\mu\circ\varrho$ from α to γ by

$$\mu \circ \varrho(s,t) = \begin{cases} \mu(2s,t) & \text{if } 0 \leqslant s \leqslant 1/2, \\ \varrho(2s-1,t) & \text{if } 1/2 \leqslant s \leqslant 1. \end{cases}$$

For each $n \geqslant 0$, every path μ from α to β induces the natural map $\mu_n\colon \tau_n(X,\beta) \to \tau_n(X,\alpha)$ defined as follows:

If f is a representative proper map of $\xi \in \tau_n(X,\beta)$, consider the partial proper homotopy of f, $\varphi\colon \partial(I^n \times J) \times I \to X$, defined by $\varphi(x,t,s) = \mu(1-s,t)$ for every $(x,t,s) \in \partial(I^n \times J) \times I$. By the APHEP, φ extends to a proper map $\Phi\colon I^n \times J \times I \to X$. Then we define $\mu_n(\xi)$ as the element of $\tau_n(X,\alpha)$ represented by Φ_1, where $\Phi_1(x,t)$ $\Phi(x,t,1)$ for each $(x,t) \in I^n \times J$.

Proposition 3.4. *The map μ_n is an isomorphism (for $n \geqslant 1$ in the category of groups), which depends only on the proper homotopy class of μ relative to $0 \times J \cup 1 \times J$. Moreover, it has analogous properties to those of paths between base points; that is, $(\mu \circ \varrho)_n = \mu_n \circ \varrho_n$, $(\theta_\alpha)_n = \mathrm{id}$, where θ_α is constant path $\theta_\alpha(s,t) = \alpha(t)$.*

Observe that if X has only one proper end, then all groups τ_n with different proper rays are isomorphic.

As a consequence of the last proposition we obtain that $\pi_1(X,\alpha)$ acts on $\tau_n(X,\alpha)$ as an operator group. Define the action by $u * \xi = \mu_n(\xi)$ where $\xi \in \tau_n(X,\alpha)$ and μ is a representative map of u. Notice also that if $\pi_1(X,\alpha(0)) = 0$, then $\pi_1(X,\alpha)$ acts on $\tau_1(X,\alpha)$ by "conjugation".

Definition 3.5. A space X is said to be $(\tau)n$-*simple* if $\pi_1(X,\alpha)$ acts trivially on $\tau_n(X,\alpha)$ for every ray α in X.

For $n \geqslant 1$, $\Omega_\tau^n(X,\alpha)$ will denote the subgroup of $\tau_n(X,\alpha)$ generated by the elements of the form $\xi - u * \xi$ where $\xi \in \tau_n(X,\alpha)$ and $u \in \pi_1(X,\alpha)$. This subgroup is normal. In particular, $\Omega_\tau^1(X,\alpha)$ contains the commutator subgroup of $\tau_1(X,\alpha)$. Then the quotient group $\tau_n(X,\alpha)/\Omega_\tau^n(X,\alpha)$ is abelian for $n \geqslant 1$.

We also have analogous results to Proposition 3.4 in the relative case considering paths between rays in the subspace. For any based proper pair (X,A,α), on the one hand, we have actions of $\pi_1(X,\alpha)$ on the groups $\pi_n(X,\alpha)$, $\tau_n(X,\alpha)$, and on the other we have actions of $\pi_1(A,\alpha)$ on the groups $\pi_n(X,A,\alpha)$, $\tau_n(X,A,\alpha)$. Furthermore, there are compatible actions of $\pi_1(A,\alpha)$ on all the objects of the exact sequences of the functors π_n, τ_n associated with the based proper pair (X,A,α). We shall denote by $\Omega_\tau^n(X,A,\alpha)$, $\Omega_\pi^n(X,\alpha)$, $\Omega_\pi^n(X,A,\alpha)$ the normal subgroups determined by the corresponding actions. Notice that in the relative case the quotient groups are abelian for $n \geqslant 2$.

Finally, let us remark that one can define an action of π_1 on π_n through the action of the subgroup $\delta\pi_1$ of π_1 on π_n. In this way, we have compatible actions of π_1 on the sequence

$$\ldots \to \pi_{n+1} \to \tau_n \to \pi_n \to \pi_n \to \ldots$$

4. Proper homology groups. Now we are going to work out a proper homology theory inspired by the singular homology theory H_* developed by Massey in [14] with use of singular n-cubes (continuous maps from I^n to X) instead of singular n-simplexes. As in our case we have chosen to use proper singular n-cubes; see [6], [18].

Definition 4.1. A proper map $T\colon K_1 \times \ldots \times K_n \to X$ will be called a *proper singular n-cube*. If there is some i such that $T(x_1,\ldots,x_i,\ldots,x_n) = T(x_1,\ldots,x_i',\ldots,x_n)$ for every $x_i, x_i' \in K_i$ $(K_i = I)$, T is said to be *degenerated*.

Let $C_n(X)$ denote the free abelian group generated by all proper singular n-cubes modulo degenerated n-cubes. Define the boundary operator by $\partial T = \sum_{i=1}^{n}(-1)^i\big((\alpha_i^0)^*T - (\alpha_i^1)^*T\big)$; here $(\alpha_i^l)^*$ is the homomorphism induced by the injection α_i^l defined by

$$\alpha_i^l(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) = (x_1,\ldots,x_{i-1},l,x_{i+1},\ldots,x_n)$$

where $l = 0, 1$ if $K_i = I$; if $K_i = J$ then $l = 0$; in this case take $(\alpha_i^1)^* = 0$.

The resulting chain complex is denoted by $C_*(X)$. The nth homology group of this complex $H_n(C_*(X))$ will be denoted by $J_n(X)$ and will be called the *n-th proper singular homology group of X*.

Let $S_*(X)$ denote the chain complex of singular cubes of X modulo degenerated cubes. Consider the quotient complex $C_*(X)/S_*(X)$. The nth homology group $H_n(C_*(X)/S_*(X))$ will be denoted by $E_n(X)$ and called the nth *proper end homology group of X*.

In the usual way we define proper end and singular homology groups of a proper pair (X,A) and, for a given abelian group G, proper end and singular (co)homology groups with coefficients in G.

Let us review some properties of these proper homology groups: J_*, E_* are covariant functors from the proper homotopy category of proper pairs and proper maps to the category of abelian groups. There is an exact sequence

$$\ldots \to H_q(X,A) \to J_q(X,A) \to E_q(X,A) \to H_{q-1}(X,A) \to \ldots$$

which relates proper homology groups with singular homology groups. For each proper theory J_*, E_* there is an exact sequence associated with a proper pair. Similary to the singular theory, there are coefficient theorems for J_*, E_*. The homology theory J_* agrees with H_* on compact spaces, and satisfies the following condition: Let $F\colon J \times X \to Y$ be a proper map; define $f(x) = F(0,x)$ for every $x \in X$; then the induced homomorphism $f_*\colon J_q(X) \to J_q(Y)$ is zero for every integer q.

These proper theories satisfy weak excision and Mayer–Vietoris Theorems. These theorems are weak because we require some additional conditions on the set of Freudenthal ends $\mathscr{F}(X)$ of a space X. Let us recall the notion of a Freudenthal end: Consider the set $\{K\}$ of closed compact subsets of X directed by inclusion. Define $\mathscr{F}(X) = \lim \pi_0(X-K)$. Let $E \subset X$, and $e = \{U_K\} \in \mathscr{F}(X)$ Write $e < E$ if there is a closed compact subset K such that $U_K \subset E$. Denote: $E^{\mathscr{F}} = \{e \in \mathscr{F}(X) \mid e < E\}$. With these notation we have the following weak excision property:

Let (X, A) be a proper pair and $U \subset A$ such that $\operatorname{cl} U \subset \operatorname{int} A$, $\mathscr{F}(X) = A^{\mathscr{F}} \cup (X-U)^{\mathscr{F}}$; then $J_q(X-U, A-U) \to J_q(X, A)$ is an isomorphism for every integer q.

Using the properties of these homologies, one can prove that $J_q(R^n) = 0$ if $q \neq n$, and $J_n(R^n) \cong Z$; $J_q(R^n_+, R^{n-1}) = 0$ if $q \neq n$, and $J_n(R^n_+, R^{n-1}) \cong Z$, where R^n_+ is the upper-half euclidean n-space; $E_q(D^{n-1} \times J, S^{n-2} \times J) = 0$ if $q \neq n$, and $E_n(D^{n-1} \times J, S^{n-2} \times J) \cong Z$.

A suitable category for studying these proper homologies is the category FPCC of finite proper cubic complexes, which is described below.

Definition 4.2. Let $a^n = K_1 \times \ldots \times K_n$ be a proper n-cube. Any subspace of the form $t \circ g(a^n)$, where t is a translation and g is a linear isomorphism of R^q, $q \geqslant n$, will also be called a proper n-cube in R^q. In the natural way we define the faces of a proper n-cube. A finite proper cubic complex consists of a subspace $X \subset R^m$ together a finite family $\mathscr{S} = \{\sigma_i | \ i = 1, \ldots, p\}$ satisfying:

(i) σ_i is a proper n-cube for some n, $0 \leqslant n \leqslant m$,

(ii) $X = \bigcup_{i=1}^{p} \sigma_i$,

(iii) If σ_i is a face of $\sigma_j \in \mathscr{S}$, then $\sigma_i \in \mathscr{S}$,

(iv) If $\sigma_i, \sigma_j \in \mathscr{S}$, then either $\sigma_i \cap \sigma_j = \varnothing$ or $\sigma_i \cap \sigma_j$ is a common face or σ_i and σ_j.

One defines in the natural way subcomplexes of X and the r-skeleton X^r, $r \geqslant 0$. We say that (X, A) is a proper pair of finite proper cubic complexes if A is a subcomplex of X.

In the category of proper pairs of finite proper cubic complexes we have proposed a computing algorithm, see [6], [7], for the homologies H_*, J_*, E_*, similar to the "classical" algorithm for the singular homology in the category of simplicial complexes. In the case of singular homology, consider compact proper oriented cubes; for J_*, consider all proper oriented cubes; and for E_*, take only noncompact proper oriented cubes. When you take the oriented boundary, in the end homology, consider only noncompact faces.

Fortunately, in the category FPCC we have strong excision and Mayer–Vietoris Theorems: Let X_1, X_2 be subcomplexes of a finite proper cubic complex X such that $X = X_1 \cup X_2$. Then

$$J_q(X_1 \cup X_2, X_2) \cong J_q(X_1, X_1 \cap X_2)$$

for every q, and there is a connexion homomorphism Δ, such that the sequence

$$\ldots \to J_n(X_1 \cap X_2) \to J_n(X_1) \oplus J_n(X_2) \to J_n(X) \xrightarrow{\Delta} J_{n-1}(X_1 \cap X_2) \to \ldots$$

is exact for each integer n.

Remark 4.3. The homologies defined in this section can also be redefined by using only proper n-cubes of the form either I^n or $I^n \times J$, see [7]. In this case we use the same symbols C_*, C_*/S_*.

Finally, for any unexplained notions or facts concerning these homologies refer to [18], [6].

5. Theorem of Hurewicz type on proper homotopy. Given a based proper pair (X, A, α), we define $\varrho_\pi\colon \pi_n(X, A, \alpha) \to E_{n+1}(X, A)$ for $n \geqslant 1$ by $\varrho_\pi(\xi) = f_*([\mathrm{id}_{I^n \times J}])$, where f is a representative proper map of ξ and $f_*\colon E_{n+1}(I^n \times J, \partial I^n \times J) \to E_{n+1}(X, A)$ is the homomorphism induced by f. In the absolute case, we define $\varrho_\pi\colon \pi_n(X, \alpha) \to E_{n+1}(X)$ for $n \geqslant 1$, in a similar way.

We have the following relation between the groups just mentioned and the homotopy and homology local Hu's groups: On the one hand, let us consider the nth homotopy local Hu's group $\lambda_n(\hat{X}, \hat{A}, \infty; p_\alpha) = \pi_n(T(\hat{X}), T(\hat{A}), p_\alpha)$ (see [4], [9]), where $\hat{X}$ $(\hat{A})$ denotes the Alexandroff compactification of X (A) by the point ∞,

$$T(\hat{X}) = \{\sigma\colon (I, 1) \to (\hat{X}, \infty) | \ \sigma^{-1}(\infty) = 1\},$$

similarly $T(\hat{A})$, and p_α is defined by

$$p_\alpha(t) = \alpha(t/1-t) \quad \text{if } 0 \leqslant t < 1 \text{ and } p_\alpha(1) = \infty.$$

Now one can define an isomorphism

$$\Psi_1\colon \pi_n(T(\hat{X}), T(\hat{A}), p_\alpha) \to \pi_n(X, A, \alpha)$$

as follows: if g represents $\eta \in \pi_n(T(\hat{X}), T(\hat{A}), p_\alpha)$, then $\Psi_1(\eta)$ is the element of $\pi_n(X, A, \alpha)$ represented by the proper map $\bar{g}$ defined by $\bar{g}(x, t) = g(x)(t/1+t)$ for every $(x, t) \in I^n \times J$.

On the other hand, there is an isomorphism Ψ_2 from the nth homology local Hu's group $L_n(\hat{X}, \hat{A}, \infty) = H_n(T(\hat{X}), T(\hat{A}))$ to the proper end homology group $E_{n+1}(X, A)$ defined as follows: For a singular n-cube of $T(\hat{X})$, $v\colon I^n \to T(\hat{X})$, we put $\Psi_2(v) = \bar{v}$, where $\bar{v}(x, t) = v(x)(t/1+t)$. Notice that if $v \in S_*(T(\hat{A}))$, then $\bar{v} \in (C_*/S_*)(A)$. To check that Ψ_2 is an isomorphism, it suffices to take into account Remark 4.3, which is a consequence of the fact that an $(n+1)$-proper cube can be subdivided into a new proper simplicial complex in which all simplexes are homeomorphic to I^{n+1} or to $I^n \times J$.

Let ϱ_π denote the natural Hurewicz homomorphism from $\pi_n(X, A)$ to $H_n(X, A)$: see [21]. It is easy to verify that $\varrho_\pi \circ \Psi_1 = \Psi_2 \circ \varrho_\pi$. Consequently, we obtain the following:

Theorem 5.1. *If X, A have one proper end, and (X, A) is $(\pi)(n-1)$-connected $(n \geqslant 2)$, then for each ray α in A the map $\varrho_\pi\colon \pi_n(X, A, \alpha) \to E_{n+1}(X, A)$ is an epimorphism whose kernel is the subgroup $\Omega^n_\pi(X, A, \alpha)$ determined by the action of $\pi_1(A, \alpha)$ in $\pi_n(X, A, \alpha)$.*

Similarly, in the absolute case we have:

Theorem 5.2. *Suppose that X is $(\pi)(n-1)$-connected $(n \geqslant 1)$. Then for each ray α in X the map $\varrho_\pi\colon \pi_n(X, \alpha) \to E_{n+1}(X)$ is an isomorphism for $n \geqslant 2$, and an epimorphism for $n = 1$ with kernel $\Omega^1_\pi(X, \alpha)$.*

The homomorphisms ϱ_π commute (up to sign when dimension changes) with the exact sequence associated with a based proper pair, and are functorial with respect to proper maps between spaces (proper pairs) with base ray.

Now we are going to study the relation between τ_* and J_*.

For $n \geq 1$ we define $\varrho_\tau\colon \tau_n(X, A, \alpha) \to J_{n+1}(X, A)$ by

$$\varrho_\tau(\xi) = f_*([\mathrm{id}_{I^n \times J}]),$$

where f is a representative proper map of ξ and

$$f_*\colon J_{n+1}(I^n \times J, \partial(I^n \times J)) \to J_{n+1}(X, A)$$

is the homomorphism induced by f.

Similarly ϱ_τ is defined in the absolute case and $n \geq 1$.

It is easy to verify that ϱ_τ is a homomorphism for $n \geq 2$ ($n \geq 1$) in the relative (absolute) case using the Mayer–Vietoris exact sequence associated with the finite proper cubic complex

$$K^n = \partial((I^n \times J)_-) \cup \partial((I^n \times J)_+)$$

and the subcomplexes $\partial(I^n \times J)_-$, $\partial(I^n \times J)_+$, where

$$(I^n \times J)_- = [0, 1/2] \times I^{n-1} \times J, \quad (I^n \times J)_+ = [1/2, 1] \times I^{n-1} \times J;$$

see the end of Section 4.

The homomorphisms ϱ_τ "commute" (up to sign) with the (τ) exact sequence associated with a proper pair with base ray. The following diagram is also "commutative" (there is a similar diagram for the absolute case):

$$\begin{array}{ccccccccc}
\ldots \to & \pi_{n+1}(X, A, \alpha(0)) & \to & \tau_n(X, A, \alpha) & \to & \pi_n(X, A, \alpha) & \to & \pi_n(X, A, \alpha(0)) & \to \ldots \\
& \downarrow \varrho_\pi & & \downarrow \varrho_\tau & & \downarrow \varrho_\pi & & \downarrow \varrho_\pi & \\
\ldots \to & H_{n+1}(X, A) & \to & J_{n+1}(X, A) & \to & E_{n+1}(X, A) & \to & H_n(X, A) & \to \ldots
\end{array}$$

Then, applying the Hurewicz Theorems for ϱ_π and ϱ_π and the fourth Lemma, we obtain:

PROPOSITION 5.3. *If X is $(\pi)n$-connected and $(\tau)(n-1)$-connected, then for each ray α in X the map $\varrho_\tau\colon \tau_n(X, \alpha) \to J_{n+1}(X)$ is an isomorphism if $n \geq 2$ and an epimorphism if $n = 1$.*

PROPOSITION 5.4. *Suppose that (X, A) is $(\pi)n$-connected and $(\tau)(n-1)$-connected ($n \geq 2$). Assume also that $\pi_0(X)$, $\pi_0(A)$, $\tau_0(X)$, $\tau_0(A)$ are trivial. Then, for each ray α in A, $\varrho_\tau\colon \tau_n(X, A, \alpha) \to J_{n+1}(X, A)$ is an epimorphism.*

Notice that $\Omega_\tau^n(X, A, \alpha) \subset \operatorname{Ker} \varrho_\tau^n$ (the same holds for $n = 1$ in the absolute case). Later we shall prove hat $\operatorname{Ker} \varrho_\tau^n$ is equal to $\Omega_\tau^n(X, A, \alpha)$.

Let $\tau_n^*(X, A)$ denote the set of classes of proper maps of type

$$(I^n \times J, \partial(I^n \times J)) \to (X, A),$$

under the relation of proper homotopy of proper maps. When A has only one proper end, for each ray α in A there is a natural bijection

$$\tau_n(X, A, \alpha)/\Omega_\tau^n(X, A, \alpha) \to \tau_n^*(X, A),$$

which gives an abelian group structure to $\tau_n^*(X, A)$. Provided that (X, A) is $(\tau)n$-simple a geometric interpretation, which is independent of the chosen base ray, is obtained for the group $\tau_n(X, A, \alpha)$.

Similarly, let $\pi_{n+1}^*(X, A)$ denote the set of classes of maps of type

$$(I^{n+1}, \partial I^{n+1}) \to (X, A)$$

under the relation of "free" homotopy. Recall (see [20.1, 2.2]) that if A is path-connected, then for each point $x_0 \in A$ there is a natural bijection

$$\pi_{n+1}(X, A, x_0)/\Omega_\pi^{n+1}(X, A, x_0) \to \pi_{n+1}^*(X, A),$$

which gives an abelian group structure to $\pi_{n+1}^*(X, A)$. Recall that $\Omega_\pi^{n+1}(X, A, x_0)$ denotes the subgroup determined by the action of $\pi_1(A, x_0)$ on $\pi_{n+1}(X, A, x_0)$.

Provided that A is path-connected and has one proper end, the homomorphism

$$\varphi_\alpha\colon \pi_{n+1}(X, A, \alpha(0)) \to \tau_n(X, A, \alpha),$$

where α is a base ray in A, induces a homomorphism $\varphi^*\colon \pi_{n+1}^*(X, A) \to \tau_n^*(X, A)$, which does not depend on the base ray.

In order to prove a Hurewicz Theorem in the relative case we need an addition theorem, which we now formulate. The main difference between this theorem and the classical one is that here we are summing elements represented by singular non-compact cubes together with elements represented by singular compact cubes. This is done with the aid of the homomorphism φ^*. We follow the procedure applied by Whitehead in [20].

THEOREM 5.5 (proper homotopy addition theorem). *Let (X, A) be a proper pair with $\pi_0(A) = \tau_0(A) = 0$. Suppose that for each $n \geq 2$ the proper map $f\colon I^{n+1} \times J \to X$ carries every n-face of $I^{n+1} \times J$ into the subspace A. For each i satisfying $1 \leq i \leq n+1$ and $l \in \{0, 1\}$, let γ_i^l be the element of $\tau_n^*(X, A)$ represented by the proper map*

$$f \circ \alpha_i^l\colon (I^n \times J, \partial(I^n \times J)) \to (X, A).$$

Let $\gamma = \varphi^*(\bar{\gamma}) \in \tau_n^*(X, A)$, where $\bar{\gamma}$ is the element of $\pi_{n+1}^*(X, A)$ represented by $f \circ \alpha_{n+2}^0\colon (I^{n+1}, \partial I^{n+1}) \to (X, A)$. Then the following relation holds in the abelian group $\tau_n^*(X, A)$:

$$\Big(\sum_{i=1}^{n+1} \sum_{l=0,1} (-1)^{i+l} \gamma_i^l\Big) - (-1)^{n+1} \gamma = 0.$$

(A similar theorem is true for the absolute case for $n \geq 1$).

To prove this theorem we need some auxiliary results.

LEMMA 5.6. *Let* $f, g\colon (I^n\times J, \partial(I^n\times J)) \to (X, A)$ *be proper maps such that* $f(1, t_2, \ldots, t_n; t) = g(0, t_2, \ldots, t_n, t)$ *for every* $(t_2, \ldots, t_n, t) \in I^{n-1}\times J$. *Let* $h\colon (I^n\times J, K^n) \to (X, A)$ *be the proper map defined by*

$$h(t_1, t_2, \ldots, t_n, t) = \begin{cases} f(2t_1, t_2, \ldots, t_n, t) & \text{if } 0 \leqslant t_1 \leqslant 1/2, \\ g(2t-1, t_2, \ldots, t_n, t) & \text{if } 1/2 \leqslant t_1 \leqslant 1. \end{cases}$$

Let a, b, c be the elements of $\tau_n^*(X, A)$ *represented by f, g, h, respectively. Then* $c = a+b$.

The proof of this Lemma is analogous to that of the similar "classical" Lemma; see [20].

LEMMA 5.7. *Let* $f\colon (I^{n+1}, \partial I^{n+1}) \to (X, A)$,

$$g\colon (I^n\times J, \partial(I^n\times J)) \to (X, A)$$

be proper maps such that $f(t_1, t_2, \ldots, t_n, 1) = g(t_1, t_2, \ldots, t_n, 0)$ *for every* $(t_1, t_2, \ldots, t_n) \in I^n$. *Let a, b be the elements of* $\pi_{n+1}^*(X, A)$, $\tau_n^*(X, A)$ *represented by f, g, respectively. Then the element c of* $\tau_n^*(X, A)$ *represented by the proper map*

$$h\colon (I^n\times J, \partial(I^n\times J)) \to (X, A)$$

defined by

$$h(t_1, t_2, \ldots, t_n, t) = \begin{cases} f(t_1, t_2, \ldots, t_n, 2t) & \text{if } 0 \leqslant t \leqslant 1/2, \\ g(t_1, t_2, \ldots, t_n, 2t-1) & \text{if } 1/2 \leqslant t, \end{cases}$$

satisfies the equality $c = b - \varphi^*(a)$.

Proof. Let P^n be the union of all n-faces of I^{n+1} different from $I^{n-1}\times 0\times I$, and let $v = (0, 0, \ldots, 0, 1) \in I^{n+1}$. Since, the triplet $(P^n, I^n\times 1, \{v\})$ has the same homotopy type as $(\{v\}, \{v\}, \{v\})$, there is a deformation $\Delta\colon P^n\times I \to P^n$ such that

$$\Delta(x, 0) = x \quad \text{if } x\in P^n,$$
$$\Delta(x, 1) = v \quad \text{if } x\in P^n,$$
$$\Delta(y, 1, s) = (1-s)(y, 1)+s(0, 1) \quad \text{where } y, 0\in I^n.$$

Define

$$F|_{P^n\times I}\colon (P^n\times I, \partial(I^{n-1}\times 0\times I)\times I) \to (X, A)$$

by

$$F(x, s) = f(\Delta(x, s)).$$

Now, applying twice the homotopy extension property we can extend $F|_{P^n\times I}$ to a homotopy

$$F\colon (I^n\times I\times I, \partial(I^n\times I)\times I) \to (X, A)$$

such that

$$F(x, 0) = f(x) \quad \text{if } x\in I^n\times I,$$
$$F(x, 1) = f(v) \quad \text{if } x\in \dot{P}^n.$$

Notice that $F_1(x) = F(x, 1)$, $x\in I^{n+1}$, represents in $\pi_{n+1}^*(X, A)$ the same element that f does.

On the other hand, we also have a proper deformation

$$\Delta'\colon (I^n\times 0\cup T^{n-1}\times J)\times I \to (I^n\times 0\cup T^{n-1}\times J)$$

such that

$$\Delta'(x, 0) = x \quad \text{if } x\in I^n\times 0\cup T^{n-1}\times J,$$
$$\Delta'(r, t, 1) = (0, \ldots, 0, t)\in I^n\times J \quad \text{if } (r, t)\in I^n\times 0\cup T^{n-1}\times J,$$
$$\Delta'(y, 0, s) = (1-s)(y, 0)+s(0, 0) \quad \text{if } y, 0\in I^n.$$

Define $G|_{(I^n\times 0\cup T^{n-1}\times J)\times I}$ by $G(x, s) = g(\Delta'(x, s))$ for every

$$(x, s)\in (I^n\times 0\cup T^{n-1}\times J)\times I.$$

Now, using consecutively the proper homotopy extension we obtain a proper homotopy $G\colon (I^n\times J\times I, \partial(I^n\times J)\times I) \to (X, A)$ such that $G(x, 0) = g(x)$ if $x\in I^n\times J$, and $G(x, 0, s) = F(x, 1, s)$ for each $(x, s)\in I^n\times I$. Note that $G_1(x) = G(x, 1)$, for $x\in I^n\times J$, represents in $\tau_n^*(X, A)$ the same element that g does.

Define $H\colon I^n\times J\times I \to X$ by

$$H(t_1, t_2, \ldots, t_n, t, s) = \begin{cases} F(t_1, t_2, \ldots, t_n, 2t, s) & \text{if } 0\leqslant t\leqslant 1/2, \\ G(t_1, t_2, \ldots, t_n, 2t-1, s) & \text{if } 1/2\leqslant t. \end{cases}$$

H is a proper homotopy and $H(x, 0) = h(x)$ for $x\in I^n\times J$. Since $H(\partial(I^n\times J)\times I)\subset A$, we see that H_1 represents the same element as h in $\tau_n^*(X, A)$.

Consider the proper map $R\colon I^n\times J\times I \to X$ given by

$$R(t_1, t_2, \ldots, t_n, t, s) =$$

$$\begin{array}{lll} 0\leqslant t\leqslant 1/2 & \begin{cases} F_1((2/2-s)t_1, t_2, \ldots, t_n, 2t) \\ F_1(1, t_2, \ldots, t_n, 2t) \end{cases} & \begin{array}{l}\text{if } 0\leqslant t_1\leqslant 1-s/2, \\ \text{if } 1-s/2\leqslant t_1\leqslant 1,\end{array} \\ 1/2\leqslant t & \begin{cases} G_1(0, t_2, \ldots, t_n, 2t-1) \\ G_1((1/2-s)(2t_1-s), t_2, \ldots, t_n, 2t-1) \end{cases} & \begin{array}{l}\text{if } 0\leqslant t_1\leqslant s/2, \\ \text{if } s/2\leqslant t_1\leqslant 1\end{array} \end{array}$$

We have $R_0 = H_1$ and $R(\partial(I^n\times J)\times I)\subset A$. Hence, R_1 represents the element $c\in\tau_n^*(X, A)$. Define $K, K'\colon I^n\times J \to X$ by

$$K(t_1, t_2, \ldots, t_n, t) = R_1(t_1/2, t_2, \ldots, t_n, t),$$
$$K'(t_1, t_2, \ldots, t_n, t) = R_1((t_1+1)/2, t_2, \ldots, t_n, t),$$

for each $(t_1, t_2, \ldots, t_n, t)\in I^n\times J$. It is easy to verify that K represents in $\tau_n^*(X, A)$ the image by φ^* of the opposite element of $\pi_{n+1}^*(X, A)$ represented by F_1, and K' represents the same element as G_1. In view of the equality

$$R_1(t_1, t_2, \ldots, t_n, t) = \begin{cases} K(2t_1, t_2, \ldots, t_n, t) & \text{if } 0\leqslant t_1\leqslant 1/2, \\ K'(2t_1-1, t_2, \ldots, t_n, t) & \text{if } 1/2\leqslant t_1\leqslant 1, \end{cases}$$

we conclude from Lemma 5.6 that R_1 represents $b-\varphi^*(a)$.

We shall also need the following result; a detailed proof can be found in [18].

LEMMA 5.8. *Let σ be a permutation of $\{1, 2, \ldots, n\}$ and let $\tilde{\sigma}$: $(I^n\times J, \partial(I^n\times J)) \to (I^n\times J, \partial(I^n\times J))$ be given by $\tilde{\sigma}(t_1, t_2, \ldots, t_n, t) = (t_{\sigma(1)}, t_{\sigma(2)}, \ldots, t_{\sigma(n)}, t)$. Then, the induced map*

$$\tilde{\sigma}_*\colon J_{n+1}(I^n\times J, \partial(I^n\times J)) \to J_{n+1}(I^n\times J, \partial(I^n\times J))$$

satisfies $\tilde{\sigma}_*([\mathrm{id}_{I^n\times J}]) = \operatorname{sign}\sigma\cdot[\mathrm{id}_{I^n\times J}]$.

Consider the proper maps $\varphi_i^l\colon I^n\times J\to I^{n+1}\times J$ defined as follows. For $1<i\leqslant n+1$, $l=0$,

$$\varphi_i^l(t_1, t_2, \ldots, t_n, t) = \begin{cases}(2t_{i-1}, t_1, \ldots, t_{i-2}, 0, t_i, \ldots, t_n, t) & \text{if } 0\leqslant t_{i-1}\leqslant 1/2,\\ (1, t_1, \ldots, t_{i-2}, 2t_{i-1}-1, t_i, \ldots, t_n, t) & \text{if } 1/2\leqslant t_{i-1}\leqslant 1.\end{cases}$$

For $1<i\leqslant n+1$, $l=1$,

$$\varphi_i^l(t_1, t_2, \ldots, t_n, t) = \begin{cases}(1, t_1, \ldots, t_{i-2}, 2t_{i-1}, t_i, \ldots, t_n, t) & \text{if } 0\leqslant t_{i-1}\leqslant 1/2,\\ (2-2t_{i-1}, t_1, \ldots, t_{i-2}, 1, t_i, \ldots, t_n, t) & \text{if } 1/2\leqslant t_{i-1}\leqslant 1.\end{cases}$$

For $i=n+2$, $l=0$,

$$\varphi_{n+2}^0(t_1, t_2, \ldots, t_n, t) = \begin{cases}(2t, t_1, \ldots, t_{n-1}, t_n, 0) & \text{if } 0\leqslant t\leqslant 1/2,\\ (1, t_1, \ldots, t_{n-1}, t_n, 2t-1) & \text{if } 1/2\leqslant t.\end{cases}$$

With these notations we have:

LEMMA 5.9. *For each i satisfying $1<i\leqslant n+1$ and for $l\in\{0,1\}$, $f\circ\varphi_i^l$ represents the element $\gamma_1^1+(-1)^{i+l}\gamma_i^l$. The proper map $f\circ\varphi_{n+2}^0$ represents the element $\gamma_1^1-(-1)^{n+2}\gamma$.*

Proof. For i satisfying $1<i\leqslant n+1$ consider the permutation $(1, 2, \ldots, t-1)$ of $\{1, \ldots, n\}$, for $i=n+2$ consider the permutation $(1, 2, \ldots, n+1)$ of $\{1, \ldots, n+1\}$. Now, the desired result follows from Lemmas 5.6, 5.7, 5.8, and the fact that $\tau_n^*(X, A)$ is abelian for $n\geqslant 2$.

Let us denote by $(I^{n+1}\times J)_{j,\eta}$ the $(n+1)$-face of $I^{n+1}\times J$ which is equal to $\operatorname{Im}\alpha_j^\eta$; see the beginning of Section 4. Using formulas analogous to those of [20] and the proper homotopy extension property, it is easy to obtain:

LEMMA 5.10. *For each (i, l), where $1<i\leqslant n+1$, $l\in\{0,1\}$, or $i=n+2$, $l=0$, there is a proper homotopy $\Phi\colon I^{n+1}\times J\times I\to X$ such that*

(1) *$\Phi(u, 0) = f(u)$ for every $u\in I^{n+1}\times J$,*
(2) *$\Phi(u, s)\in A$ if $(u, s)\in\partial((I^{n+1}\times J)_{j,\eta})\times I$ and $(1, 1)\neq(j, \eta)\neq(i, l)$,*
(3) *$\Phi(u, 1)\in A$ if $u\in(I^{n+1}\times J)_{i,l}$,*
(4) *$\Phi(u, s)\in A$ if $(u, s)\in(I^{n+1}\times J)_{j,\eta}\times I$ and $f((I^{n+1}\times J)_{j,\eta})\subset A$,*
(5) *$\Phi(\alpha_1^1(x), 1) = f\circ\varphi_i^l(x)$ if $x\in I^n\times J$.*

Now we are in position to give:

Proof of the proper homotopy addition theorem. We proceed by induction on the number m of pairs (i, l) with $i>0$ such that $f((I^{n+1}\times J)_{i,l})\not\subset A$.

For $m=0$, the result is obvious. Assume that the result holds for $m\geqslant 0$. Let f be a proper map such that for $(m+1)$ pairs (i, l) with $i>1$, $f((I^{n+1}\times J)_{i,l})\not\subset A$. Choose one such pair (i, l). Consider the proper homotopy Φ of Lemma 5.10. Define $f'(u) = \Phi(u, 1)$. Then f' satisfies Theorem 5.5 and there are m pairs (j, η) with $j>1$ such that $f'((I^{n+1}\times J)_{j,\eta})\not\subset A$. Now, by the induction hypothesis and Lemma 5.9, the theorem holds for $m+1$. This completes the proof.

Now suppose that $\pi_0(X) = \tau_0(X) = 0$. We are going to introduce new chain complexes, whose homology groups will be intermediate between τ_n and J_{n+1}. These groups will play in the proof of our Hurewicz type Theorem a similar role to that played by the Eilenberg–Blakers homology groups in the case of singular homology: see [20].

For a ray α in A, define $C_*^{(n)}(X, A)$ as the subcomplex of $C_*(X)$ generated by all proper singular cubes $T\colon\sigma\to X$ (where $\sigma = I^{q-1}\times J$ or $\sigma = I^q$, see Remark 4.3) satisfying: (i) T sends all vertices of σ to $\alpha(0)$, (ii) T sends the n-skeleton of σ into A, (iii) If σ is not compact, T maps the non-compact 1-faces of σ in the same way as α. In case of confusion we shall denote $C_*^{(n)}(X, A)$ by $C_*^{(n)}(X, A, \alpha)$.

THEOREM 5.11. *If (X, A) is $(\pi)n$-connected and $(\tau)(n-1)$-connected, then the inclusion map*

$$i\colon (C_*^{(n)}(X, A), C_*^{(0)}(A, \alpha)) \to (C_*(X), C_*(A))$$

is a homotopy equivalence between pairs of chain complexes.

Proof. We are going to construct a sequence of homomorphisms $F_q\colon C_q(X)\to C_{q+1}(X)$ with the properties:

(1) If T is a compact (non-compact) proper singular q-cube, then F_qT is a compact (non-compact) proper singular $(q+1)$-cube,
(2) $(F_qT)\circ\alpha_1^0 = T$,
(3) $(F_qT)\circ\alpha_1^1\in C_q^{(n)}(X, A)$,
(4) $F_q(T\circ\alpha_i^l) = (F_qT)\circ(1\times\alpha_i^l)$ for every $i\geqslant 1$,
(5) F_qT is "stationary" if $T\in C_q^{(n)}(X, A)$,
(6) If $T\in C_q(A)$, $F_qT\in C_{q+1}(A)$.

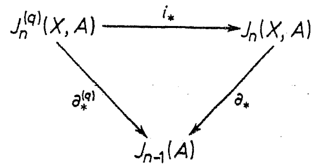
To define F_0 we use the fact that X, A are path-connected. Suppose we have already constructed $F_0, \ldots, F_q$ satisfying (1)–(6). To define F_{q+1}, we distinguish three cases: for $q=0$, we use the fact that $\tau_0(X, \alpha) = \tau_0(A, \alpha) = \pi_1(X, A, \alpha(0)) = 0$; for $0\leqslant q\leqslant n-1$ we use the fact that $\pi_{q+1}(X, A, \alpha(0))$, $\tau_q(X, A, \alpha)$ are trivial; finally, for $q\geqslant n$ we apply the proper homotopy extension property. Now we define $r\colon C_*(X)\to C_*^{(n)}(X, A)$ by $rT = FT|_{1\times\sigma}$ for each singular proper n-cube $T\colon\sigma\to X$. It follows that $r\circ i = \mathrm{id}_{C_*^{(n)}(X,A)}$, $i\circ r\simeq\mathrm{id}_{C_*(X)}$. The theorem is thus proved.

DEFINITION 5.12. The nth *proper homology group of Eilenberg–Blakers type* of a based proper pair (X, A, α) is defined as the relative homology group $H_n(C_*^{(q)}(X, A), C_*^{(q)}(A, \alpha))$ and it is denoted by $J_n^{(q)}(X, A)$ (or by $J_*^{(q)}(X, A, \alpha)$ to avoid confusion). In the absolute case this notion is defined for (X, α) by $J_n^{(q)}(X) = H_n(C_*^{(q)}(X, \alpha), C_*^{(q)}(\alpha))$.

$J_n^{(q)}$ are covariant functors, which are invariants of the proper homotopy type. We also have the following natural transformations:

$$0 = J_n^{(n)}(X, A) \to J_n^{(n-1)}(X, A) \to \ldots \to J_n^{(0)}(X, A) \to J_n(X, A).$$

Since the map $i_*^{(0)}\colon J_{n-1}^{(0)}(A) \to J_{n-1}(A)$, induced by the inclusion, is an isomorphism, $\partial_*^{(q)}$ is a natural transformation. Moreover, the diagram



is commutative.

THEOREM 5.13. *Let (X, A) be a proper pair such that $\pi_0(X)$, $\pi_0(A)$, $\tau_0(X)$, $\tau_0(A)$ are trivial. Then for each $n > 1$ and each ray α in A, $\tau_n(X, A, \alpha)/\Omega_\tau^n(X, A, \alpha)$ is isomorphic to $J_{n+1}^{(n)}(X, A)$. Similarly in the absolute case for $n \geq 1$.*

Proof. Consider the homomorphism

$$\bar{\varrho}_\tau\colon \tau_n(X, A, \alpha)/\Omega_\tau^n(X, A, \alpha) \to J_{n+1}^{(n)}(X, A)$$

defined as follows: Let

$$f\colon (I^n \times J, I^{n-1} \times J, T^{n-1} \times J, I^n \times 0) \to (X, A, \alpha, \alpha(0))$$

be a proper representative map of ξ. Then f is a proper singular $(n+1)$-cube, which belongs to $C_*^{(n)}(X, A, \alpha)$. Furthermore, f is a relative $(n+1)$-cycle (modulo $C_*^{(0)}(A, \alpha)$). Define $\bar{\varrho}_\tau(\xi)$ as the class of f in $J_{n+1}^{(n)}(X, A)$. On the other hand, if we consider the Hurewicz homomorphism

$$\varrho_\tau\colon \tau_n(X, A, \alpha)/\Omega_\tau^n(X, A, \alpha) \to J_{n+1}(X, A)$$

and the homomorphism

$$i_*\colon J_{n+1}^{(n)}(X, A) \to J_{n+1}(X, A)$$

induced by the inclusion, we see that $\varrho_\tau = i_* \circ \bar{\varrho}_\tau$.

In order to prove that $\bar{\varrho}_\tau$ is an isomorphism, consider the homomorphism η defined as follows: Let T be a non-compact singular proper $(n+1)$-cube of X which sends the n-faces into A, the "vertices" into $\alpha(0)$, and the non-compact 1-faces into α. Via the isomorphism $\tau_n^*(X, A) \to \tau_n(X, A, \alpha)/\Omega_\tau^n(X, A, \alpha)$, T represents an element ηT of the latter group. Now, let T be a compact singular proper $(n+1)$-cube of X; then, similarly, T represents an element T_π of $\pi_{n+1}(X, A, \alpha(0))/\Omega_\pi^{n+1}(X, A, \alpha(0))$. Define $\eta T = -\varphi^*(T_\pi)$. If $T \in C_{n+1}^{(0)}(A, \alpha)$, then $\eta T = 0$. From the "classical" homotopy addition theorem and from Theorem 5.5 (on the addition of proper homotopies) it follows that if T is an $(n+1)$-boundary of $C_*^{(n)}(X, A, \alpha)$ modulo $C_*^{(0)}(A, \alpha)$, then $\eta T = 0$. Therefore η factorizes into a new homomorphism $\eta\colon J_{n+1}^{(n)}(X, A) \to \tau_n(X, A, \alpha)/\Omega_\tau^n(X, A, \alpha)$, which is the desired inverse homomorphism of $\bar{\varrho}_\tau$.

Finally, as a consequence of Theorems 5.11, 5.13 we obtain the following theorems of Hurewicz type for τ:

THEOREM 5.14 *Let (X, A) be a proper pair such that $\pi_0(X)$, $\pi_0(A)$, $\tau_0(X)$, $\tau_0(A)$ are trivial. Suppose that for $n \geq 2$ (X, A) is $(\pi)n$-connected and $(\tau)(n-1)$-connected. Then for each ray α in A*

$$\varrho_\tau\colon \tau_n(X, A, \alpha)/\Omega_\tau^n(X, A, \alpha) \to J_{n+1}(X, A)$$

is an isomorphism. In the case where (X, A) is $(\tau)n$-simple, for example if $\pi_1(A, \alpha) = 0$, the map $\varrho_\tau\colon \tau_n(X, A, \alpha) \to J_{n+1}(X, A)$ is an isomorphism.

THEOREM 5.15 *For $n \geq 1$, suppose that X is $(\pi)n$-connected and $(\tau)(n-1)$-connected. Then for each ray α in X, $\varrho_\tau\colon \tau_n(X, \alpha) \to J_{n+1}(X)$ is an isomorphism if $n \geq 1$, and if $n = 1$, $\varrho_\tau\colon \tau_1(X, \alpha)/\Omega_\tau^1(X, \alpha) \to J_2(X)$ is an isomorphism. In the case where X is $(\tau)1$-simple, for example, if $\pi_1(X, \alpha) = 0$, the map $\varrho_\tau\colon \tau_1(X, \alpha) \to J_2(X)$ is an isomorphism.*

Remark 5.16. It is not difficult to verify that if X is $(\pi)(n-1)$-connected, then ϱ_π^{n+1} is an epimorphism. Similarly, if X is $(\pi)n$-connected and $(\tau)(n-1)$-connected then ϱ_τ^{n+1} is also an epimorphism $(n \geq 2)$.

6. Example. Let M be a separable compact Hausdorff differentiable n-manifold. Suppose also that M has no boundary, is orientable and m-connected for $m > 1$. Notice that n must be larger than m. Consider the tangent bundle of M: $\mathbf{R}^n \to TM \to M$. Since M admits a Riemannian metric, see [19; p. 58], one can consider the tangent sphere bundle: $S^{n-1} \to S^{n-1}M \to M$. Since M is compact, so is the tangent sphere bundle $S^{n-1}M$. It is not difficult to verify that TM has one Freudenthal end and there is a "neighbourhood" of this end homeomorphic to $S^{n-1}M \times J$. To prove this it is enough to consider the map $\theta\colon S^{n-1}M \times [1, \infty) \to TM$ given by $\theta(x, \lambda) = \lambda x$.

In this section we try to reduce the calculation of some proper homotopy and homology groups of the tangent bundle TM to the calculation of homotopy and homology groups of M.

Consider the fibration $\mathbf{R}^n \to TM \to M$. Since $\mathbf{R}^n$ is contractible, we can consider the homotopy exact sequence, and Serre's homology sequence (see [21]) to obtain

$$(1)\qquad \begin{aligned} \pi_q(TM) &\cong \pi_q(M), \quad q \geq 0, \\ H_q(TM) &\cong H_q(M), \quad q \geq 0. \end{aligned}$$

Now consider the tangent sphere bundle $S^{n-1} \to S^{n-1}M \to M$. Since S^{n-1} is $(n-2)$-connected, applying the homotopy exact sequence and the Serre's homology sequence we set

$$\begin{array}{ll} & \pi_q(S^{n-1}M) \cong \pi_q(M),\ 0 \leqslant q \leqslant n-2, \\ & \pi_{n-1}(S^{n-1}) \to \pi_{n-1}(S^{n-1}M) \to \pi_{n-1}(M) \to 0 \text{ is exact}, \\ (2) & H_q(S^{n-1}M) \cong H_q(M),\ 0 \leqslant q \leqslant n-2 \text{ or } n+1 \leqslant q \leqslant m+n-1 \\ & H_{n-1}(S^{n-1}M) \to H_{n-1}(M) \text{ is an epimorphism}, \\ & H_n(S^{n-1}M) \to H_n(M) \text{ is a monomorphism.} \end{array}$$

As we saw in Section 5, we have the commutative diagram

$$(2)\qquad \begin{array}{ccccccccc} \ldots \to & \pi_{q+1}(TM) & \to & \tau_q(TM) & \to & \pi_q(TM) & \to & \pi_q(TM) & \to \ldots \\ & \downarrow{\scriptstyle \varrho_\pi} & & \downarrow{\scriptstyle \varrho_\tau} & & \downarrow{\scriptstyle \varrho_\pi} & & \downarrow{\scriptstyle \varrho_\pi} & \\ \ldots \to & H_{q+1}(TM) & \to & J_{q+1}(TM) & \to & E_{q+1}(TM) & \to & H_q(TM) & \to \ldots \end{array}$$

M being m-connected, we obtain from diagram (3), in view of (1),

$$(4)\qquad \begin{array}{l} \tau_q(TM) \cong \pi_q(TM),\ q < m, \\ \tau_m(TM) \to \pi_m(TM) \text{ is an epimorphism}, \\ J_q(TM) \cong E_q(TM),\ 2 \leqslant q \leqslant m, \\ J_{m+1}(TM) \to E_{m+1}(TM) \text{ is an epimorphism}, \\ J_{n+1}(TM) \to E_{n+1}(TM) \text{ is a monomorphism}, \\ J_q(TM) \cong E_q(TM),\ n+1 < q, \\ J_1(TM) = 0,\ E_1(TM) \cong \mathbf{Z}, \\ J_0(TM) = E_0(TM) = 0. \end{array}$$

From the properties of the functor π_q, E_q (see Sections 3, 4) we set

$$(5)\qquad \begin{array}{ll} \pi_q(TM) \cong \pi_q(S^{n-1}M \times J) \cong \pi_q(S^{n-1}M), & 0 \leqslant q, \\ E_q(TM) \cong E_q(S^{n-1}M \times J) \cong H_{q-1}(S^{n-1}M), & 0 \leqslant q. \end{array}$$

Now, from (4), (5), (2) we deduce that

$$(6)\qquad \begin{array}{ll} \tau_q(TM) \cong \pi_q(TM) \cong 0, & q < m, \\ J_q(TM) \cong E_q(TM) \cong 0, & 2 \leqslant q \leqslant m \text{ or } n+2 \leqslant q \leqslant m+n-1. \end{array}$$

We are now going to consider two cases:

(A) Assume $m \geqslant [n/2]$. In this case, using Poincaré duality and the (π) Hurewicz theorem, we may suppose that $m = n-1$. Taking the diagram (3) for $q = n-1$, and using (4), we obtain

$$\begin{array}{ccccccccccc} & \tau_n(TM) & \to & \pi_n(TM) & \to & \pi_n(TM) & \to & \tau_{n-1}(TM) & \to & \pi_{n-1}(TM) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & J_{n+1}(TM) & \to & E_{n+1}(TM) & \to & H_n(TM) & \to & J_n(TM) & \to & E_n(TM) & \to 0 \end{array}$$

From (5) and (2) we deduce that $E_n(TM) \cong \pi_{n-1}(S^{n-1}M)$ is a cyclic group. Suppose that $E_n(TM) \cong \mathbf{Z}/p$, $0 \leqslant p < \infty$. Since $H_n(TM) \cong \mathbf{Z}$, we obtain the exact sequence

$$(7)\qquad \mathbf{Z} \to J_n(TM) \to \mathbf{Z}/p \to 0\,.$$

From the exactness of sequences in the above diagram we also see that

$$(8)\qquad E_{n+1}(TM) \cong \begin{cases} 0, \\ \mathbf{Z}, \end{cases} \quad J_{n+1}(TM) \cong \begin{cases} 0, \\ \mathbf{Z}. \end{cases}$$

Finally, by Theorems 5.2, 5.15 we have

$$(9)\qquad \begin{array}{l} \tau_{n-1}(TM) \cong J_n(TM)\,, \\ \pi_{n-1}(TM) \cong E_n(TM) \cong \mathbf{Z}/p\,. \end{array}$$

(B) Suppose that $m < [n/2]$ and $n \geqslant 6$. We have the following commutative diagram:

$$\begin{array}{ccccccccc} \ldots \to & \pi_{m+1}(TM) & \to & \pi_{m+1}(TM) & \to & \tau_m(TM) & \to & \pi_m(TM) & \to 0 \\ & \downarrow & & \downarrow{\scriptstyle \varrho_\pi} & & \downarrow{\scriptstyle \varrho_\tau} & & \downarrow{\scriptstyle \varrho_\pi} & \\ \ldots \to & E_{m+2}(TM) & \to & H_{m+1}(TM) & \to & J_{m+1}(TM) & \to & E_{m+1}(TM) & \to 0 \end{array}$$

Applying the Hurewicz Theorem and Theorems 5.2, 5.15, we infer that ϱ_π, ϱ_τ, ϱ_π are isomorphisms.

Consider the following commutative diagrams:

$$\begin{array}{ccc} \pi_q(TM) & \longrightarrow & \pi_q(TM) \\ \wr\| & & \\ \pi_q(S^{n-1}M) & & \wr\| \\ \downarrow{\scriptstyle (p_*)^\pi} & & \downarrow \\ \pi_q(M) & \xrightarrow{\text{id}} & \pi_q(M) \end{array} \qquad \begin{array}{ccc} E_{q+1}(TM) & \longrightarrow & H_q(TM) \\ \wr\| & & \\ H_q(S^{n-1}M) & & \wr\| \\ \downarrow{\scriptstyle (p_*)^H} & & \downarrow \\ H_q(M) & \xrightarrow{\text{id}} & H_q(M) \end{array}$$

From (2) we deduce that:

$$(10)\qquad \begin{array}{ll} \tau_q(TM) = 0, & q \leqslant n-2, \\ \pi_q(TM) \cong \pi_q(M), & q \leqslant n-2, \\ \pi_{n-1}(TM) \to \pi_{n-1}(M) & \text{is an epimorphism}, \\ J_q(TM) = 0, & q < n-1, \\ E_q(TM) \cong H_{q-1}(M), & 1 \leqslant q \leqslant n-1, \\ E_n(TM) \to H_{n-1}(M) & \text{is an epimorphism}, \\ E_{n+1}(TM) \to H_n(M) & \text{is a monomorphism.} \end{array}$$

Acknowledgement. The authors wish to acknowledge financial aid from the I. E. R. (Instituto de Estudios Riojanos).

References

[1] M. Artin and B. Mazur, *Etale Homotopy*, Lecture Notes in Math. Springer; 1969.
[2] A. L. Blakers, *Some relations between homology and homotopy groups*, Ann. of Maths. (2) 49 (1948) 428–461.
[3] M. G. Brin and T. L. Thikstun, *On the proper Steenrod homotopy groups and proper embeddings of planes into 3-manifolds*, Trans. Amer. Math. Soc. 289 (1985), 737–755.
[4] Z. Čerin, *On various relative proper homotopy groups*, Tsukuba J. Math. (2) 4 (1980), 177–202.
[5] D. Edwars and H. Hasting, *Čech and Steenrod homotopy theories with applications to Geometric Topology*, Lecture Notes Math. 542, Springer (1976).
[6] J. I. Extremiana, L. J. Hernández and M. T. Rivas, *Una (Co)Homologia propia*, Actas X Jornadas Hispano-Lusas de Matemáticas, Murcia 1985.
[7] L. J. Hernández, *A note on proper invariants*, Publ. Sem. Mat. Garcia de Galdeano. Serie II, sección 1 no 12 (1984).
[8] L. J. Hernández and T. Porter, *Proper pointed maps from R^{n+1} to a σ-compact space.* Math. Proc. Camb. Phil. Soc. 103 (1988), 457–462.
[9] S. T. Hu, *Algebraic local invariants of topological spaces.* Compositio Math. 13 (1958), 173–218,
[10] — *Homotopy theory*, Academic Press 1959.
[11] W. Hurewicz, *Beitrage zur Topologie der Deformationen*, I–IV, Neder. Akad. Wetensch Proc. Ser. A 38, 112–119, (1935) 521–528, 39, 117–126, (1936), 215–224.
[12] Y. Kodama and A. Koyama, *Hurewicz isomorphism theorem for Steenrod homology.* Amer. Math. Soc. (1979), 363–367.
[13] K. Kuperberg, *An isomorphism theorem of the Hurewicz type in Borsuk theory of shape*, Fund. Math. 77 (1972), 21–32.
[14] W. S. Massey, *Singular homology theory*, G.T.M. 70, Springer, 1980.
[15] S. Mardešić and S. Ungar, *The relative Hurewicz theorem in shape theory*, Glasnik Math. 9 (1974), 317–327.
[16] K. Morita, *The Hurewicz and Whitehead theorems in shape theory*, Sci. Rep. Tokyo Daigaku Sec. A (12) (1974), 246–258.
[17] M. Raussen, *Hurewicz isomorphism and Whitehead theorems in pro-categories*, Arch. Math. (Basel) 30 (1978), 153–164.
[18] M. T. Rivas, *Sobre invariantes de homotopia propia y sus relaciones*, Tesis. Publ. Sem. Mat. Garcia de Galdeano. Serie II, sección 2, 17 (1987).
[19] N. E. Steenrod, *The topology of fibre bundles*, P.M.S. 14. Princeton University Press, 1954.
[20] G. W. Whitehead, *Homotopy theory*, Massachusetts Institute of Technology, 1966.
[21] —, *Elements of homotopy theory*, G.T.M. 61, Springer, 1978.

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Received 18 February 1987;
in revised form 28 September 1987

Jumps of entropy in one dimension

by

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Abstract. For continuous piecewise monotone maps of an interval we estimate possible jumps (discontinuities) of topological entropy under perturbations which preserve the number of pieces of monotonicity. We prove that for unimodal maps the topological entropy as a function of a map is continuons at all maps for which it is positive.

0. Introduction. This paper deals with the continuity properties of the topological entropy as a function of a map. For the discussion of this problem in the case of continuous maps of arbitrary compact spaces, we send the reader to the book [4], and in the case of smooth maps to the paper [11]. Here we shall concentrate on the case of continuous maps of the interval (it does not make any difference if we replace the interval by the circle, so these results apply also to the case of the maps of the circle).

It was proved in [10] and [8] that in this case the topological entropy is lower semi-continuous. Hence, what remains to investigate, is the problem how far it is from the upper semi-continuity. Clearly, one can modify any map by creating an invariant subinterval with arbitrarily large entropy, and this modification can be made small in the C^0-topology. However, the natural demand that we do not enlarge the number of turning points, excludes most of these examples.

In such a way we are left with the following problem: how high can the entropy jump up if we start with the piecewise monotone map and make arbitrarily small C^0 perturbations which do not enlarge the number of turning points? The answer (Theorem 1) is the following:

We look at all periodic orbits. On each of them we count the number of turning points and divide by the period. Then we take the maximum of these numbers over all periodic orbits and multiply by log2. *This is the maximal level to which the entropy can jump. If it is already above this level, then it is continuous at this map.*

Some problems are created if the map is not piecewise strictly monotone, i.e. if it can have "flat" pieces, and in particular turning intervals instead of turning