Functions provably total in $I^\Sigma_1$

by

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Abstract. We estimate the rapidity of the growth of recursive functions which are provably total in a finite fragment of $\Sigma_0$ parameter-free induction subject to the size of the fragment.

The aim of this paper is to bound the rapidity of the growth of recursive ($\Sigma_0$ definable) functions which are provably total in $I^\Sigma_1$ induction (for parameter free $\Sigma_0$ formulas). We show that if in the proof of the totality of a recursive function $f$ from $N$ to $N$, $\Sigma_0$ induction is applied $n$ times then the function can be bounded by the $n+1$'s function in the Wainer hierarchy (see [W]).

The result is proved by means of a proof-theoretic analysis of proofs of sentences of the form $(\forall t)\phi(t)$ in $I^\Sigma_1$, (an analogous analysis for $\exists_1$ formulas and $I^\exists_1$ can be found in [A]). We consider here $\Sigma_0$ formulas $\phi$ without parameters.

Here $PA^-$ denotes the theory of discretely ordered rings. If $\phi$ is a formula then $Ind(\phi)$ denotes the following sentence:

$$PA^- \land [(0)(\forall t)\phi(t) \Rightarrow \phi(t+1)] \Rightarrow (\forall t)\phi(t)$$. 

To simplify the notation we will assume that for every formula of the form $\phi(y, \bar{x})$ the sentence $\phi(-1, 0, \ldots, 0)$ is true. Formally, this can be assumed since we can replace $\phi$ by the formula $\phi^*$ defined as $(y \geq 0 \land \phi(y, x)) \lor y < 0$. Then $\phi$ is equivalent to $\phi^*$ for all non-negative $y$'s we are interested in. Without causing confusion we identify $\phi$ and $\phi^*$.

**Definition 1.** Let the formulas $\phi_1, \ldots, \phi_n$ be of the form

$$\phi_i(t) = (\exists i)(\forall t)\phi_i(t, \bar{x})$$

where $\phi_i(t, \bar{x}) \in D_0$, $i = 1, \ldots, n$.

We assume that the quantifiers in the formulas $\phi_i$ are bounded by the free variable or by one of the variables of $\bar{x}$.

Let $M \models PA^-$, $n_1, \ldots, n_k \in D_0$. Assume that we have a fixed enumeration of polynomials. Let $K \subseteq N$. A set $H \subseteq M$ is called a $K$-closure of $\{n_1, \ldots, n_k\}$ with respect to $\{\phi_1, \ldots, \phi_n\}$ if there exist sets $H_0, \ldots, H_k$ such that

1. $H = H_0 \cup H_1 \cup \ldots \cup H_k$, and $\{n_1, \ldots, n_k\} = H_0$.
2. If $x \in H_j$ for a certain $j < k$ then for every $i \in \{1, \ldots, n\}$ there is an $x_i \in H_{i+1}$ such that $M \models \phi_i(x, x_i)$.
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$M \vdash \varphi(u_j)$ for $j \in \{j_1, ..., j_m\}$ since $M \vdash \varphi(u_{j_1}, j_1)$ and $\varphi_j$ is absolute w.r.t. $M'$ being $\mathcal{A}_1$. $M' \vdash \neg \varphi_j(u_j + 1)$ for $j \in \{j_1, ..., j_m\}$-immediate. $M' \vdash \neg \varphi_\omega$. Finally, $M' \vdash (\forall \varphi_\omega) \varphi_\omega(t_i)$ for $i \in \{i_1, ..., i_n\}$. It follows that $M' \vdash \text{Ind } \varphi_\omega, ..., \text{Ind } \varphi_\omega, \neg \varphi_\omega$ which contradicts (1).

(2) $\Rightarrow$ (1). Let $M$ be a model of $\text{Ind } \varphi_\omega, ..., \text{Ind } \varphi_\omega$. Assume that $\{i_1, ..., i_n\}$ is the set of those $i$'s from $\{1, ..., n\}$ for which $M' \vdash (\forall \varphi_\omega) \varphi_\omega(t_i)$. Let $\{j_1, ..., j_m\} = \{i_1, ..., i_n\}$. Then there are $u_{j_1}, ..., u_{j_m}$ in $M$ such that $M \vdash \varphi_\omega(u_{j_1})$ and $\neg \varphi_\omega(u_{j_1} + 1)$ for $j \in \{j_1, ..., j_m\}$.

Let $u \in M$ be arbitrary $u \geq 0$. Since $M \vdash (\forall \varphi_\omega) \varphi_\omega(t_i)$ for $i \in \{i_1, ..., i_n\}$, there exists a $K$-closure of $\{u, u_{j_1}, ..., u_{j_m}, j_1, ..., j_m\}$ w.r.t. $\varphi_{j_1}, ..., \varphi_{j_m}$ in $M$, where $\varphi_j$ are such that $M \vdash \varphi_j(u_j, j)$ for $j \in \{j_1, ..., j_m\}$ and $K$ is taken from (2). In this closure there exists an $\bar{y}$ such that $M \vdash \varphi(u_\bar{y}, \bar{y})$, by (2). Since $u$ was arbitrary, $M' \vdash (\forall \varphi_\omega) \varphi_\omega(t_i)$.

Now we are going to formulate the main theorem. We define a hierarchy of functions based on the hierarchy defined by Grzegorczyk [G] and Wainer [W].

**Definition 2.** Let $F_i$ be functions from $N$ to $N$ defined as:

$$F_0(t) = 2^t, \quad F_{k+1}(t) = F_k(t)^{F_k(t)}.$$ 

**Theorem.** Let $n \in N$. Let $\varphi_1, ..., \varphi_n$ be as before. Define the function $f: N \to N$ as follows:

$$f_t(x) = w \iff (\exists \varphi(t, i) \land i \leq w \land (\forall \varphi) \neg \varphi(t, w)).$$

Assume that $\text{Ind } \varphi_1, ..., \text{Ind } \varphi_n \vdash (\forall \varphi_\omega) \varphi_\omega(t)$. Then $f_{\bar{t}}(\bar{x}) < f_t(\bar{x})$ for almost all $t \in N$.

**Proof.** We need the following definition:

**Definition 3.** Let $L, K \in N$. Let $a_1, ..., a_n \in N$. We say that a subset $H$ of $N$ is an $L$th $K$-closure of $\{a_1, ..., a_n\}$ with respect to $\emptyset$ if $H = H_{\emptyset} \cup ... \cup H_{\emptyset}$ where $H_{\emptyset}$ is a $K$-closure of $\{a_1, ..., a_n\}$ w.r.t. $\emptyset$ and $H_{\emptyset}$ is a $K$-closure of $H_{\emptyset} \cup H_{\emptyset}$ w.r.t. $\emptyset$ for $i = 2, ..., n$.

**Proof.** From the lemma it follows that in consecutive $K$-closures of the set $\{t, t - 1\}$ w.r.t. $\emptyset$ there appear new sequences $(\bar{m}, \bar{d})$ satisfying one of the formulas $\varphi_1, ..., \varphi_\omega$. We shall say that a sequence $(\bar{m}, \bar{d})$ satisfying $\varphi_1$ or $\varphi_\omega$ is the $m$th sequence for $\varphi_1$ or $\varphi_\omega$, respectively. Assume without loss of generality that for a given $m$ there is just one such sequence.

It cannot happen that there appear only consecutive sequences $(\emptyset, d), (1, 1), ...$ for one of the formulas $\varphi_1, ..., \varphi_\omega$, say for $\varphi_1$, and the $r$th sequence for $\varphi_\omega$ does not appear. Indeed, if it was so, then a $K$-closure of $\{t, t - 1\}$ w.r.t. $\varphi_1$ would exist. But in this $K$-closure there must be the $0$th sequence for one of the formulas $\varphi_1, ..., \varphi_\omega$, say for $\varphi_1$, or the $r$th sequence for $\varphi_\omega$ (by the lemma), contradiction.
Reasoning in the same way, we infer it cannot happen that building the consecutive K-closures of \([t, -1, 0)\) w.r.t. \(\emptyset\) only sequences for two of the formulas \(\varphi_1, \ldots, \varphi_n\) say for \(\varphi_1, \varphi_2\) are generated and the rth sequence for \(\varphi\) does not appear. Similarly for three, etc.

If so, then there is an \(L \in \mathbb{N}\) such that in the \(L\)th K-closure of the set \([t, -1, 0)\) w.r.t. \(\emptyset\) there are all sequences for \(\varphi_1, \ldots, \varphi_n\) which are needed to generate the rth sequence for \(\varphi\) (\(L \in \mathbb{N}\) since the theorem is formulated for functions from \(N\) to \(N\), in the case of a nonstandard model \(L\) can be nonstandard).

It either can happen that there are all the sequences for \(\varphi_1, \ldots, \varphi_n\) which are in the K-closure of \([t)\) w.r.t. \(\{\varphi_1, \ldots, \varphi_n\}\) (those certainly generate the rth sequence for \(\varphi\) by the lemma) or the rth sequence for \(\varphi\) is generated somewhere earlier in the procedure.

The number \(L\) is the biggest in the following situation: for every \(i = 1, \ldots, n\) the 0th sequence for \(\varphi_i\) does not appear in any K-closure of \([t, -1, 0)\) until all the sequences for \(\varphi_1, \ldots, \varphi_{i-1}\) occurring in the K-closure of \([t, -1, 0)\) w.r.t. \(\{\varphi_1, \ldots, \varphi_{i-1}\}\) are generated. Neither does the rth sequence for \(\varphi\) appear until a K-closure of \([t, -1, 0)\) w.r.t. \(\{\varphi_1, \ldots, \varphi_n\}\) is generated.

Assume that we are dealing with the above (worst) situation. Then we define \(L_{ij} \in \mathbb{N}\) to be the first number such that in the \(L_{ij}\)th K-closure of \([t, -1, 0)\) w.r.t. \(\emptyset\) there appears the jth sequence for \(\varphi_i\) (\(i = 1, \ldots, n\)). Then \(L_{ij} = 1,\ldots, j+1\).

Define the functions \(f(t)\) bounding the elements of the \(L_{ij}\)th K-closure of \([t, -1, 0)\) w.r.t. \(\emptyset\):

1. Assume \(i = 1\). We have to consider the supersets of the polynomials whose numbers are less than \(K\) iterated \(K\) times. If \(u > 1\) then there is an \(m \in \mathbb{N}\) such that whenever \(x < u\) and \(q\) is a polynomial whose number is less than \(K\) then \(q(x) < u^m\). Hence, since \(L_{1,0} = 1\), we can define \(f_1(t)^{L_{1,0}} = (\ldots (t^n)^{L_{1,0}})^{L_{1,0}} = t^n\). Then \(f_1^{L_{1,0}}(t) < F_K^i(t)\) for almost all \(t \in X\).

2. Assume \(i = 2\). Note that the function \(f_2^{L_{2,0}}(t)\) bounds the elements of the \(L_{2,0}\)th K-closure of \([t, -1, 0)\) w.r.t. \(\emptyset\) (since \(L_{2,0} = j+1\)). Let \(H\) denote the K-closure of \([t, -1, 0)\) w.r.t. \(\varphi_2\). To find the number \(L_{2,0}\) it is enough to bound the elements of \(H\). By the definition of a K-closure, \(H = H_0 \cup \ldots \cup H_K\). By the definition of a K-closure it follows that the greatest element of \(H\) is the rth sequence for \(\varphi_2\) (we can assume that it is greater than the values of the polynomials having numbers less than \(K\) at \(t\)). Hence if \(a \in H\), then \(a < f_2^{L_{2,0}}(t)\). Similarly, if \(a \in H_2\), then \(a < f_2^{L_{2,0}+1}(t)\) (the \(t^{\text{th}}\) sequence for \(\varphi_2\)). And so on. If \(a \in H^{L_{2,0}}\) then \(a < f_2^{L_{2,0}+1}(t)\).

Hence we can define \(f_2(t)\) to be equal to the above expression. Applying the same reasoning to the set \([t, -1, 0)\) where \(j\) is the 0th sequence for \(\varphi_2\) we infer that the function \(f_2(t)\) bounds the elements of the \(H_2\)th K-closure of \([t, -1, 0)\) w.r.t. \(\emptyset\). Repeating the same argument \(j\) times, we infer that the function \(f_2^{L_{2,0}+1}(t)\) bounds the elements of the \(L_{2,0}\)th K-closure of \([t, -1, 0)\) w.r.t. \(\emptyset\).

(3) The general case. Assume that we have defined the function \(f(t)\). Similarly as before, we note that the function \(f_2^{L_{2,0}+1}(t)\) bounds the elements of the \(L_{2,0}\)th K-closure of \([t, -1, 0)\) w.r.t. \(\emptyset\). Let \(H\) denote the K-closure of \([t, -1, 0)\) w.r.t. \(\{\varphi_1, \ldots, \varphi_n\}\). By the definition of a K-closure. To define the function \(f_2^{L_{2,0}+1}(t)\) it suffices to bound the elements of \(H\). By the definition of a K-closure we obtain successively: if \(a \in H\), then \(a < f_2^{L_{2,0}+1}(t)\) and \(a < f_2^{L_{2,0}+1}(t)\).

We can define \(f_2(t)\) to be equal to the above expression.

Let us show by induction w.r.t. \(t\) that, for every \(i = 1, \ldots, n\), \(f_i^{L_{1,0}}(t) < F_K^{i+1}(t)\) for almost all \(t\).

If \(i = 1\) then \(f_1^{L_{1,0}}(t) < F_K^{i+1}(t)\) as we have already noticed. Assume that \(F_K^{i+1}(t) < F_K^{i+1}(t)\) for almost all \(t\) and we shall show that \(f_i^{L_{1,0}}(t) < F_K^{i+1}(t)\) for almost all \(t\).

We make the following estimation for \(t > Y\):

\[
K\times f_i^{L_{1,0}+1}(t) < F_K^{i+1}(t) < F_K^{i+1}(t) < F_K^{i+1}(t)
\]

In this way we obtain \(f_i(t) < F_K^{i+1}(t)\) for almost all \(t\). Taking a K-closure of \([t)\) w.r.t. \(\{\varphi_1, \ldots, \varphi_n\}\) and reasoning as before we conclude that \(f_i(t) < F_K^{i+1}(t)\) for almost all \(t\).

**Proof.** Suppose that it is finitely axiomatizable and let \(T\) be a finite set of sentences axiomatizing it. Then there is a finite set of \(\Sigma_i\) formulas \(\{\varphi_1, \ldots, \varphi_n\}\) such

\[
f_i(t) < F_K^{i+1}(t)\]
that Ind$\phi_1 \& \ldots \& Ind\phi_n \vdash T$. Since all the functions $F_i(t)$ are provably total in $T + \Sigma_1$, $T$ proves that $F_{n+2}$ is total. By our theorem, $F_{n+2} < F_{n+1}$ almost everywhere, contradiction.

References

[A] Z. Adamowicz, Algebraic approach to $\exists_2$ induction, Seminarbericht Nr. 70, Proceedings of the Third Easter Conference on Model Theory, Girol Kôris, April 8–13, 1985, pp. 5-15


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An isomorphism theorem of Hurewicz type in the proper homotopy category

by

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Abstract. Numerous mathematicians have proved theorems of Hurewicz type in different contexts shape theory, pro-categories, coherent categories. In this paper we obtain a Hurewicz Theorem in the proper homotopy category. In particular, we prove:

Theorem. Let $(X, A)$ be a proper pair such that $\pi_0(X, A), \pi_1(X, A), \pi_2(A)$ are trivial. Suppose that for $n > 2(X, A)$ and $(X, A)$ 3-connected and $(X, A)$ 4-connected. Then for each proper ray $a$ in $A$, $\varphi_3(\pi,X,A)\cong G(\pi,X,A)$ is an isomorphism. In the case where $(X, A)$ is 4-connected, for example if $\pi(A, a) = 0$, then $\varphi_3(\pi,X,A)\cong J\pi(X,A)$ is an isomorphism.

1. Introduction. A natural relationship between singular homology groups and Hurewicz homotopy groups is displayed by Hurewicz's Theorem. This theorem was established in terms of simplicial homology and absolute homotopy groups by Hurewicz [11] in 1935 for simply connected polyhedra. In 1944, Eilenberg proved that the fundamental group modulo the commutator subgroup is the first singular homology group. Blakers [2], the proposer of the concept of relative homotopy groups, proved in 1948 the Hurewicz Theorem in the relative case given the kernel of a homomorphism.

There are more Hurewicz type theorems in other homotopy theories. For example, in 1969 Arlin and Mazur [1] proved a Hurewicz Theorem in the category pro-$\mathcal{A}_0$, where $\mathcal{A}_0$ is the pointed homotopy category of connected pointed CW-complexes, and pro-$\mathcal{A}_0$ is the category of inverse systems of objects of $\mathcal{A}_0$. Relative Hurewicz type theorems for pro-$\mathcal{A}_0$ and $\mathcal{S}^1$ were proved by Mardski and Ungar [15] and independently by Morita [16]. Raussen [17] proved a Hurewicz type theorem in pro-$\mathcal{H}(\mathcal{D}_0)$, where $\mathcal{H}(\mathcal{D}_0)$ is the homotopy category of pointed topological spaces. In 1972, Kuperberg [13] proved another Hurewicz type Theorem between the homotopy groups defined by Borsuk and the Vietoris–Čech homology groups. In 1979, Kodama and Koyama [12] proved a Hurewicz type theorem between the Quigley approaching groups and the Steenrod homology groups. In a recent paper, Koyama proved a Hurewicz Theorem in the coherent homotopy category of inverse systems of spaces CPHTop.