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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Flipping properties and huge cardinals *

by

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Abstract. We develop flipping properties related to hugeness. This involves the consideration of notions strictly between n-hugeness and $n+1$-hugeness, and generalizations of closed and unbounded to this context. We also consider natural ideals associated with these properties, and relate this to generalized notions of ineffability.

0. Introduction. Flipping properties were first defined and studied by Abramson, Harrington, Kleinberg, and Zwicker in [1]. They showed how strongly inaccessible, weakly compact, ineffable, and measurable cardinals may be characterized by flipping properties. Henle and Kleinberg [8] established a flipping characterization of Ramsey cardinals. Di Prisco and Zwicker [7] showed that supercompact and strongly compact cardinals can also be characterized by flipping properties. In this paper, we present flipping properties related to huge cardinals.

In section one, we show how the standard method of using ultrafilters to characterize elementary embeddings witnessing n-hugeness (for some positive integer n) or supercompactness, can be generalized so that we can use ultrafilters to characterize elementary embeddings witnessing something strictly between n-hugeness and $n+1$-hugeness. In section two, we review and generalize results of Di Prisco and Marek [5] regarding a notion of closed and unbounded related to hugeness. In section three, we use the ideas of the first two sections to define and study flipping properties related to hugeness. In section four, we study flipping properties related to almost hugeness and almost supercompactness. In section five, we show that there are natural ideals associated with our flipping properties, and conclude by showing that our flipping properties, and their associated ideals, are related to generalized notions of ineffability.

Our set-theoretic notation is quite standard. V denotes the universe of all sets. Greek letters α, β and δ refer to ordinals, while γ, η, $\varkappa$, λ and σ are reserved for infinite cardinals. ORD denotes the class of all ordinals, and CARD denotes the class of all cardinals. For any set X, $|X|$ denotes the cardinality of X, and, if X is a set of ordinals, $\overline{X}$ denotes the order type of X. By the term "inner model", we shall always

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mean a transitive class which satisfies ZFC. If M is an inner model and λ is an infinite cardinal, we say that M is *closed under λ-sequences* if and only if for any $X \subseteq M$, if $|X| \leqslant \lambda$, then $X \in M$. $\lambda^{\underline{\gamma}}$ denotes $\sup(\lambda^{\gamma})$. By the term "inaccessible", we shall always mean "strongly inaccessible".

1. Between n-huge and $n+1$-huge. For $\varkappa < \lambda$, we define $P_{=\varkappa}(\lambda) = \{X \subseteq \lambda : |X| = \varkappa\}$, and for $\varkappa \leqslant \lambda$, we define $P_\varkappa(\lambda) = \{X \subseteq \lambda : |X| < \varkappa\}$. Then, $\varkappa$ is huge with target λ if and only if there exists a normal, fine, $\varkappa$-complete ultrafilter on $P_{=\varkappa}(\lambda)$, and $\varkappa$ is λ-supercompact if and only if there exists a normal, fine $\varkappa$-complete ultrafilter on $P_\varkappa(\lambda)$.

Hugeness and supercompactness can also be characterized by embedding properties. $\varkappa$ is huge with target λ if and only if there exists an elementary embedding $i: V \to M$, where M is an inner model closed under λ-sequences, $\varkappa$ is the first cardinal moved by i, and $i(\varkappa) = \lambda$. $\varkappa$ is λ-supercompact if and only if there exists an elementary embedding $i: V \to M$ as above, except that $i(\varkappa) > \lambda$.

The notion of hugeness can be generalized. Let n be a positive integer. We say that $\varkappa_0$ is *n-huge with targets $\varkappa_1, \varkappa_2, \ldots, \varkappa_n$* if and only if there exists an elementary embedding $i: V \to M$ where M is an inner model closed under $\varkappa_n$-sequences, $\varkappa_0$ is the first cardinal moved by i, and $i(\varkappa_0) = \varkappa_1$, $i(\varkappa_1) = \varkappa_2, \ldots, i(\varkappa_{n-1}) = \varkappa_n$. This property is equivalent to the existence of a normal, fine, $\varkappa_0$-complete ultrafilter U on $P_{=\varkappa_{n-1}}(\varkappa_n)$ satisfying that

$$\{X \in P_{=\varkappa_{n-1}}(\varkappa_n) : \text{for each } r = 1, 2, \ldots, n, \ |X \cap \varkappa_r| = \varkappa_{r-1}\} \in U.$$

A study of n-hugeness and supercompactness can be found in [12]. We assume familiarity with the standard techniques involving these notions.

It will be convenient to consider a cardinal to be 0-huge if and only if it is measurable. For our present purposes, we need to study notions strictly between n-hugeness and $n+1$-hugeness, for $n < \omega$. Consider an elementary embedding $i: V \to M$, where M is an inner model closed under λ-sequences for some cardinal $\lambda > \varkappa_0$, where $\varkappa_0$ is the first cardinal moved by i, and $i(\varkappa_0) = \varkappa_1$, $i(\varkappa_1) = \varkappa_2, \ldots$

By a result of Kunen [10] we cannot have $\varkappa_n < \lambda$ for each $n < \omega$. Fix n such that $\varkappa_n \leqslant \lambda < \varkappa_{n+1}$. We show that this situation can be characterized by the existence of an ultrafilter. We consider two cases:

Case 1. $\lambda = i(\varkappa)$ for some cardinal $\varkappa$. Then, for fixed $n \geqslant 1$ and cardinals $\varkappa_0, \varkappa_1, \ldots, \varkappa_n, \varkappa$, and λ, the existence of such an embedding is equivalent to the existence of a normal, fine $\varkappa_0$-complete ultrafilter U on $P_{=\varkappa}(\lambda)$ satisfying that $\{X \in P_{=\varkappa}(\lambda) : \text{for each } r = 1, 2, \ldots, n, \ |X \cap \varkappa_r| = \varkappa_{r-1}\} \in U$. We write

$$H(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$$

to indicate that such an ultrafilter and elementary embedding exist.

Case 2. λ is not in the range of i. Let $\varkappa$ be the least cardinal such that $i(\varkappa) > \lambda$. Then, for fixed n and cardinals $\varkappa_0, \varkappa_1, \ldots, \varkappa_n, \varkappa$, and λ, the existence of such an embedding is equivalent to the existence of a normal, fine $\varkappa_0$-complete ultrafilter U on $P_\varkappa(\lambda)$ which, for $n \geqslant 1$, satisfies the following two conditions:

a. $\{X \in P_\varkappa(\lambda) : \text{for each } r = 1, 2, \ldots, n, \ |X \cap \varkappa_r| = \varkappa_{r-1}\} \in U$ and
b. For any $\gamma < \varkappa$, $\{X \in P_\varkappa(\lambda) : |X| > \gamma\} \in U$.

For $n = 0$, we must have $\varkappa = \varkappa_0$ and hence our embedding simply witnesses that $\varkappa$ is λ-supercompact. In either event, we write $S(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ to indicate that such an ultrafilter and elementary embedding exist. We note that condition b above guarantees that U does not concentrate on $P_\gamma(\lambda)$ for any $\gamma < \varkappa$. In terms of the associated embedding, this condition guarantees that the "leastness" assumption on $\varkappa$ holds.

The "H" and "S" in our notation are meant to indicate that the strength of the embedding above n-hugeness is, in the first case, "huge-like", and in the second case, "supercompact-like". Although we shall not consider it here, it is also possible to study strengthenings above n-hugeness which are "strongly compact-like".

We note that $\varkappa_0$ is n-huge with targets $\varkappa_1, \varkappa_2, \ldots, \varkappa_n$ if and only if $H(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa_{n-1}, \varkappa_n)$, $\varkappa_0$ is λ-supercompact if and only if $S(\varkappa_0; \varkappa_0, \lambda)$, and $\varkappa_0$ is measurable if and only if $S(\varkappa_0; \varkappa_0, \varkappa_0)$.

The proofs of the equivalence between the ultrafilter and elementary embedding characterizations involve standard techniques. Given an ultrafilter U of the appropriate type, let M be the transitive collapse of the ultrapower $\Pi V/U$, and let i be the composition of the canonical embedding from V into $\Pi V/U$ by constant functions, and the collapsing isomorphism from $\Pi V/U$ into M. For the reverse direction, given an inner model M and an elementary embedding $i: V \to M$ of the type described, define U as follows:

For $A \subseteq P_{=\varkappa}(\lambda)$ (for case 1) or $A \subseteq P_\varkappa(\lambda)$ (for case 2), $A \in U$ if and only if $i[\lambda] \in i(A)$.

We note that $i[\lambda] \in M$ by closure considerations. The details are similar to those found in [12].

We wish to thank J. Henle for helpful discussions regarding the contents of this section.

2. Generalizing closed and unbounded. The notion of a closed and unbounded subset of an uncountable cardinal has been important to set theorists for many years. In [9], Jech showed that this notion may be generalized to the context of $P_\varkappa(\lambda)$ in a natural way, so long as $\varkappa$ is regular. If $C \subseteq P_\varkappa(\lambda)$, we say that C is *closed* if and only if for any directed $A \subseteq C$, if $\bigcup A \in P_\varkappa(\lambda)$, then $\bigcup A \in C$. Equivalently (see, for example, [11]) C is closed if and only if for any $\eta < \varkappa$ and any increasing η-sequence $\langle X_\alpha : \alpha < \eta \rangle$ of elements of C, $\bigcup_{\alpha<\eta} X_\alpha \in C$. C is unbounded if and only if given any $X \in P_\varkappa(\lambda)$, there exists $Y_x \in C$ such that $X \subseteq Y_x$. Jech showed that every normal, fine, $\varkappa$-complete ultrafilter on $P_\varkappa(\lambda)$ contains every closed and unbounded subset of $P_\varkappa(\lambda)$. We define $D \subseteq P_\varkappa(\lambda)$ to be *stationary* if and only if $D \cap C \neq \varnothing$ for every closed and unbounded $C \subseteq P_\kappa(\lambda)$. Clearly, every element of a normal, fine κ-complete ultrafilter on $P_\kappa(\lambda)$ is stationary.

In [5], DiPrisco and Marek generalized these notions to the $P_{=\kappa}(\lambda)$ context. For each $A \subseteq P_\kappa(\lambda)$, let $A^* = \{UB\colon B \subseteq A,\ B \text{ is directed, and } UB \in P_{=\kappa}(\lambda)\}$. Define $F_{\kappa\lambda} = \{E \in P_{=\kappa}(\lambda)\colon A^* \subseteq E \text{ for some closed and unbounded } A \subseteq P_\kappa(\lambda)\}$. Then $F_{\kappa\lambda}$ is a normal, fine, κ-complete filter on $P_{=\kappa}(\lambda)$, and every normal, fine, κ-complete ultrafilter on $P_{=\kappa}(\lambda)$ contains every element of $F_{\kappa\lambda}$. Di Prisco and Marek also show that $F_{\kappa\lambda}$ remains unchanged if we change the definition of A^* to

$$A^* = \{\bigcup_{\alpha<\kappa} X_\alpha\colon \langle X_\alpha\colon \alpha<\kappa\rangle \text{ is an increasing } \kappa\text{-sequence of elements of } A\}.$$

It is appropriate, at this point, to comment on our choice of the definition of $P_{=\kappa}(\lambda)$. If we wish only to look at the ultrafilters and embeddings associated with hugeness, we could equally well have defined $P_{=\kappa}(\lambda)$ to be the collection of all subsets of λ having order type κ, instead of cardinality κ. For our present purposes, the difference is crucial. As is shown in [5], if we had used the "order type" definition, the construction of $F_{\kappa\lambda}$ would not work. In particular, $\{X \in P_{=\kappa}(\lambda)\colon \overline{X} = \kappa\} \notin F_{\kappa\lambda}$.

We wish to generalize this notion of DiPrisco and Marek to the context of $H(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ and $S(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$. We consider the following index sets, where $\kappa_0 < \kappa_1 < \ldots < \kappa_{n-1} < \kappa_n$:

First, suppose $\kappa_{n-1} \leqslant \kappa < \kappa_n \leqslant \lambda$, where $\kappa_{n-1} = \kappa$ if and only if $\kappa_n = \lambda$. Then, we define $I_h(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda) = \{X \in P_{=\kappa}(\lambda)\colon \text{for each } r = 1, 2, \ldots, n,\ |X \cap \kappa_r| = \kappa_{r-1}\}$.

Next, suppose $\kappa_{n-1} < \kappa \leqslant \kappa_n < \lambda$. Then, we define

$$I_s(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda) = \{X \in P_\kappa(\lambda)\colon \text{for each } r = 1, 2, \ldots, n,\ |X \cap \kappa_r| = \kappa_{r-1}\}.$$

In the definition of $I_h(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$, we insist that $n > 0$. In the definition of $I_s(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ we allow the possibility that $n = 0$. In this case, we insist that $\kappa = \kappa_0$, and we ignore both the "$|X \cap \kappa_r| = \kappa_{r-1}$" condition and the "$\kappa_{n-1}$" in the preceeding inequality. Hence, we have that $I_h(\kappa, \lambda; \kappa, \lambda) = P_{=\kappa}(\lambda)$ and $I_s(\kappa; \kappa, \lambda) = P_\kappa(\lambda)$.

By definition, if $H(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ or $S(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$, then every ultrafilter witnessing this concentrates on $I_h(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ or $I_s(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ respectively. Hence, if we wish to define such ultrafilters, we may restrict our attention to subsets of these index sets.

For each $A \subseteq P_{\kappa_0}(\lambda)$ we define $A_h^*(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ and $A_s^*(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ as follows:

$$A_h^*(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda) = \{UB\colon B \subseteq A,\ B \neq \varnothing,\ B \text{ is directed, and } UB \in I_h(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)\} \text{ and},$$

$$A_s^*(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda) = \{UB\colon B \subseteq A,\ B \neq \varnothing,\ B \text{ is directed, and } UB \in I_s(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)\}.$$

We shall write A_h^* or A_s^* for short, when the index set is clear by context, and shall call these sets the *sets generated by A*.

We note that, in contrast with the $P_{=\kappa}(\lambda)$ and $P_\kappa(\lambda)$ situation, we apparently do not have the option of considering unions of increasing chains from A, rather than unions of directed subsets.

We shall abuse terminology slightly, and say $C \subseteq I_h(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ or $C \subseteq I_s(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ is *closed* and *unbounded* if and only if $C = A_h^*$ or $C = A_s$ respectively, for some closed and unbounded $A \subseteq P_{\kappa_0}(\lambda)$. If $D \subseteq I_h(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ or $D \subseteq I_s(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$, we say that D is *stationary* if and only if $D \cap C \neq \varnothing$ for every closed and unbounded $C \subseteq I_h(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ or $C \subseteq I_s(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$, respectively. It is straigthforward to show that every stationary subset D of $I_h(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ or $I_s(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ is unbounded in the sense that for any $X \in P_{\kappa_0}(\lambda)$, there exists $Y_x \in D$ such that $X \subseteq Y_x$.

Suppose that $\langle A_\alpha\colon \alpha < \lambda\rangle$ is a sequence of subsets of either $I_h(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ or $I_s(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$. The *diagonal intersection* $\Delta_{\alpha<\lambda} A_\alpha$, of this sequence, is defined as follows:

$$X \in \Delta_{\alpha<\lambda} A_\alpha \text{ if and only if } X \neq \varnothing \text{ and, for every } \alpha \in X,\ X \in A_\alpha.$$

Standard techniques show that any ultrafilter witnessing

$$H(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda) \quad \text{or} \quad S(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$$

is closed under diagonal intersections.

The collection of all closed and unbounded subsets of $I_h(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ or $I_s(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ generates a κ_0-complete, normal filter. κ_0-completeness follows easily from the fact that the filter generated by the collection of closed and unbounded subsets of $P_{\kappa_0}(\lambda)$ is κ_0-complete. The proof that these filters are normal is a direct generalization of the proof of Theorem 1c of [5].

Unfortunately, we do not know whether, in general, given a closed and unbounded $A \subseteq P_{\kappa_0}(\lambda)$, it is necessarily the case that $A_h^* \neq 0$ or $A_s^* \neq 0$. Hence, we do not know whether the filters under discussion are necessarily nontrivial for $n \geqslant 1$. However, the methods of Di Prisco and Marek [5] generalize to show that any ultrafilter witnessing $H(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ or $S(\kappa_0, \kappa_1, \ldots, \kappa_n; \kappa, \lambda)$ extends the corresponding filter. Hence, in these cases, these filters must be non-trivial.

3. Flipping properties. If λ is a cardinal, I is a set, and $t\colon \lambda \to P(I)$, we define t' to be a flip of t, written $t' \sim t$ if and only if for each $\alpha < \lambda$, either $t'(\alpha) = t(\alpha)$ or $t'(\alpha) = I \setminus t(\alpha)$. In this section we show how the techniques of Di Prisco and Zwicker ([7]) can be generalized to establish connections between flipping properties and the properties considered in sections 1 and 2.

Let $\varkappa_0$ be a regular cardinal. We write $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ to indicate that for any $t\colon \lambda \to \boldsymbol{P}(I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda))$ there exists $t' \sim t$ such that $\underset{\alpha<\lambda}{\Delta} t'(\alpha)$ is a stationary subset of $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa; \varkappa, \lambda)$. $\mathrm{Flip}_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ indicates that for any $t\colon \lambda \to \boldsymbol{P}(I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda))$, there exists $t' \sim t$ such that for every

$$\gamma < \varkappa, \quad \underset{\alpha<\lambda}{\Delta} t'(\alpha) \cap \{X \in I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)\colon |X| > \gamma\}$$

is a stationary subset of $I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$.

THEOREM 1. a. *If* $H(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$, *then* $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$.
b. *If* $S(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$, *then* $\mathrm{Flip}_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$.

Proof. The proof is straightforward, and is similar to the technique of [7]. In each case, we simply flip according to the associated ultrafilter. For part b, we note that by the definition of $S(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$, for every

$$\gamma < \varkappa, \quad \underset{\alpha<\lambda}{\Delta} t'(\alpha) \cap \{X \in I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)\colon |X| > \gamma\}$$

is in the ultrafilter, and hence is stationary. ∎

COROLLARY. a. *If $\varkappa$ is huge with target λ, then, for every $t\colon \lambda \to \boldsymbol{P}(P_{=\varkappa}(\lambda))$, there exists $t' \sim t$ such that $\underset{\alpha<\lambda}{\Delta} t'(\alpha)$ is stationary.*

b. (Di Prisco, Zwicker [7]). *If $\varkappa$ is λ-supercompact, then, for every $t\colon \lambda \to \boldsymbol{P}(P_\varkappa(\lambda))$, there exists $t' \sim t$ such that $\underset{\alpha<\lambda}{\Delta} t'(\alpha)$ is stationary.*

Proof. This is precisely Theorem 1 with, for part a, $n = 1$, $\varkappa_0 = \varkappa$, and $\varkappa_1 = \lambda$ and, for part b, $n = 0$ and $\varkappa_0 = \varkappa$. ∎

The converse to Theorem 1 is not true. To see this for part a, suppose, for example, that $\varkappa_0$ is the least cardinal such that, for some $\varkappa_1, \varkappa_2, \ldots, \varkappa_n, \varkappa$, and λ, $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ holds. In addition suppose, by way of contradiction, that $H(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ and $i\colon V \to M$ witnesses this. By closure considerations, it follows that $M \models \mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$. But this contradicts the fact that $i(\varkappa_0) > \varkappa_0$ and, by elementarity, $M \models i(\varkappa_0)$ is the least cardinal such that for some cardinals $\varkappa_1, \varkappa_2, \ldots, \varkappa_n, \varkappa$, and λ, $\mathrm{Flip}_h(i(\varkappa_0), \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda))$ holds.

Similarly, it can be shown that the converse to part b of Theorem 1 does not hold.

We are able to obtain a type of converse to Theorem 1, by considering a stronger flipping property. To do this, we must extend our notation. Assume $\varkappa_0 < \varkappa_1 < \ldots < \varkappa_n$.

First, suppose $\varkappa_{n-1} \leqslant \varkappa \leqslant \varkappa' \leqslant \varkappa_n \leqslant \lambda \leqslant \lambda'$ where $\varkappa_{n-1} = \varkappa$ if and only if $\varkappa_n = \lambda$ and also where $\varkappa = \varkappa'$ if and only if $\lambda = \lambda'$. We define $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda') = \{X \in P_{=\varkappa'}(\lambda')\colon |X \cap \lambda| = \varkappa$ and, for each $r = 1, 2, \ldots, n$, $|X \cap \varkappa_r| = \varkappa_{r-1}\}$.

Next, suppose $\varkappa_{n-1} < \varkappa \leqslant \varkappa' \leqslant \varkappa_n < \lambda \leqslant \lambda'$ where, if $\lambda = \lambda'$ then $\varkappa = \varkappa'$. We define $I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda') = \{X \in P_{\varkappa'}(\lambda')\colon |X \cap \lambda| < \varkappa$ and for each $r = 1, 2, \ldots, n$, $|X \cap \varkappa_r| = \varkappa_{r-1}\}$.

Again, in the definition of $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$, we insist that $n > 0$. In the definition of $I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$ we allow the possibility that $n = 0$. In this case, we insist that $\varkappa = \varkappa' = \varkappa_0$, and we ignore both the "$|X \cap \varkappa_r| = \varkappa_{r-1}$" condition and the "$\varkappa_{n-1}$" in the preceeding inequality.

Closed and unbounded subsets of these index sets, stationary subsets of these index sets, $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$, and $\mathrm{Flip}_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$ are defined in a manner entirely analogous to our previous definitions. In particular, for $\mathrm{Flip}_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$, we insist that for every $\gamma < \varkappa$ and $\gamma' < \varkappa'$, $\underset{\alpha<\lambda}{\Delta} t'(\alpha) \cap \{X \in I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')\colon |X \cap \lambda| > \gamma$ and $|X| > \gamma'\}$ is a stationary subset of $I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$.

It is straightforward to show, using the technique of Theorem 1, that $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$ follows from the existence of a normal, fine, $\varkappa_0$-complete ultrafilter U on $P_{=\varkappa'}(\lambda')$ which concentrates on $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$, or, equivalently, an elementary embedding $i\colon V \to M$ where M is closed under λ'-sequences, $i(\varkappa_0) = \varkappa_1$, $i(\varkappa_1) = \varkappa_2, \ldots, i(\varkappa_{n-1}) = \varkappa_n$, $i(\varkappa) = \lambda$, and $i(\varkappa') = \lambda'$. $\mathrm{Flip}_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$ follows from analogous properties (which will involve certain minimality assumptions on $\varkappa$ and $\varkappa'$).

THEOREM 2. a. *If* $\lambda' \geqslant 2^{\lambda^\varkappa}$ *and* $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$ *then*

$$H(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda).$$

b. *If* $\lambda' \geqslant 2^{\lambda^{\underline{\varkappa}}}$ *and* $\mathrm{Flip}_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$ *then*

$$S(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda).$$

Proof. We begin by noting that the assumptions of the theorem imply that for any closed and unbounded $A \subseteq P_{\varkappa_0}(\lambda)$, $A_h^* \neq 0$ and $A_s^* \neq 0$. This is so since if either of these sets were empty, the empty set would be considered a closed and unbounded subset of $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$ or $I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$ respectively. Hence, there would exist no stationary subsets of these sets. But, by definition, the assumed flipping properties imply the existence of stationary sets.

The proof is similar to the technique of [7]. For part a, we first note that $|I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)| \leqslant |P_{=\varkappa}(\lambda)| = \lambda^\varkappa$. Let $\langle A_\alpha\colon \alpha < \lambda' \rangle$ be an enumeration of all subsets of $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$, with repetitions, if necessary.

Define $t\colon \lambda' \to \boldsymbol{P}(I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda'))$ by, for each $\alpha < \lambda'$, $t(\alpha) = \{X \in I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')\colon X \cap \lambda \in A_\alpha\}$. Let $t' \sim t$ be such that $\underset{\alpha<\lambda'}{\Delta} t'(\alpha)$ is a stationary subset of $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$.

Define U by, for each $\alpha < \lambda'$, $A_\alpha \in U$ if and only if $t'(\alpha) = t(\alpha)$. The proof that U is a normal, fine, $\varkappa_0$-complete ultrafilter on $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ is similar to that in [7], and we omit it.

The definition of the ultrafilter for part b is similar to that for part a. In this case, our t' is such that for every $\gamma < \varkappa$ and $\gamma' < \varkappa'$,

$$\mathop{\Delta}_{\alpha<\lambda'} t'(\alpha) \cap \{X \in I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda') \colon |X \cap \lambda| > \gamma \text{ and } |X| > \gamma'\}$$

is a stationary subset of $I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda')$. The proof that the ultrafilter we obtain is normal, fine, and $\varkappa_0$-complete is also similar to that in [7]. Finally, suppose, by way of contradiction that for some $\gamma < \varkappa$, $\{X \in I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda) \colon |X| \leqslant \gamma\} \in U$. This set is A_β for some $\beta < \lambda'$. Since $A_\beta \in U$, $t'(\beta) = t(\beta)$. The fact that

$$\mathop{\Delta}_{\alpha<\lambda'} t'(\alpha) \cap \{X \in I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda') \colon |X \cap \lambda| > \gamma\}$$

is stationary (we do not care about γ' here), implies that

$$t'(\beta) \cap \{X \in I_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda; \varkappa', \lambda') \colon |X \cap \lambda| > \gamma\}$$

is stationary. But this is a contradiction, since this set is clearly empty. ■

COROLLARY. a. *Suppose, for some cardinal* $\varkappa'$, *that for every* $t\colon 2^{\lambda^\varkappa} \to \boldsymbol{P}(I_h(\varkappa, \lambda; \varkappa', 2^{\lambda^\varkappa}))$, *there exists* $t' \sim t$ *such that* $\mathop{\Delta}_{\alpha<2^{\lambda^\varkappa}} t'(\alpha)$ *is a stationary subset of* $I_h(\varkappa, \lambda; \varkappa', 2^{\lambda^\varkappa})$. *Then* $\varkappa$ *is huge with target* λ.

b (Di Prisco and Zwicker [7]). *Assume* $\varkappa$ *is a limit cardinal, and suppose that for every* $t\colon 2^{\lambda^{\underset{\sim}{\varkappa}}} \to \boldsymbol{P}(P_\varkappa(2^{\lambda^{\underset{\sim}{\varkappa}}}))$, *there exists* $t' \sim t$ *such that* $\mathop{\Delta}_{\alpha<2^{\lambda^{\underset{\sim}{\varkappa}}}} t'(\alpha)$ *is a stationary subset of* $P_\varkappa(2^{\lambda^{\underset{\sim}{\varkappa}}})$. *Then,* $\varkappa$ *is* λ-*supercompact*.

Proof. For part a, we note that $I_h(\varkappa, \lambda; \varkappa', 2^{\lambda^\varkappa}) = I_h(\varkappa, \lambda; \varkappa, \lambda; \varkappa', 2^{\lambda^\varkappa})$, and hence the result follows from part a of the theorem with $n = 1$, $\varkappa_0 = \varkappa$, $\varkappa_1 = \lambda$, and $\lambda' = 2^{\lambda^\varkappa}$.

For part b, we note that since $\varkappa$ is a limit cardinal, it follows that for any $\gamma < \varkappa$, $\{X \in P_\varkappa(2^{\lambda^{\underset{\sim}{\varkappa}}}) \colon |X \cap \lambda| > \gamma\}$ is a closed and unbounded subset of $P_\varkappa(2^{\lambda^{\underset{\sim}{\varkappa}}})$. Hence, our assumption implies $\mathrm{Flip}_s(\varkappa; \varkappa, \lambda; \varkappa, 2^{\lambda^{\underset{\sim}{\varkappa}}})$. The result then follows from part b of the theorem with $n = 0$, $\varkappa_0 = \varkappa = \varkappa'$, and $\lambda' = 2^{\lambda^{\underset{\sim}{\varkappa}}}$. ■

In part b, our assumption that $\varkappa$ is a limit cardinal is actually not necessary, since it is implied by the given flipping property. We choose not to show this because the given statement follows so easily from the theorem.

There are many possible "if and only if" statements that follow from Theorem 1 and 2. Here are two examples:

COROLLARY. a. (DiPrisco and Zwicker [7]): *$\varkappa$ is supercompact if and only if $\varkappa$ is a limit cardinal and for every $\lambda \geqslant \varkappa$, and every $t\colon \lambda \to \boldsymbol{P}(P_\varkappa(\lambda))$, there exists $t' \sim t$ such that $\mathop{\Delta}_{\alpha<\lambda} t'(\alpha)$ is stationary.*

b. *Fix inaccessible cardinals $\varkappa_0 < \varkappa_1 < \varkappa_2 < \ldots$ For each $n < \omega$, $\varkappa_0$ is n-huge with targets $\varkappa_1, \varkappa_2, \ldots, \varkappa_n$ if and only if for each $n < \omega$ and every $t\colon \varkappa_n \to \boldsymbol{P}(I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa_{n-1}, \varkappa_n))$, there exists $t' \sim t$ such that $\mathop{\Delta}_{\alpha<\varkappa_n} t'(\alpha)$ is a stationary subset of $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa_{n-1}, \varkappa_n)$.*

Proof. Part a follows immediately from part b of each of the corollaries to Theorems 1 and 2.

For the forward direction of part b, fix $n < \omega$. We must establish the given flipping property for this n. This is immediate from Theorem 1 since, by assumption, $H(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa_{n-1}, \varkappa_n)$.

For the reverse direction of part b, fix $n < \omega$. We must establish that $\varkappa_0$ is n-huge with targets $\varkappa_1, \varkappa_2, \ldots, \varkappa_n$. By assumption, $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_{n+1}; \varkappa_n, \varkappa_{n+1})$. Since $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_{n+1}; \varkappa_n, \varkappa_{n+1}) = I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa_{n-1}, \varkappa_n; \varkappa_n, \varkappa_{n+1})$, we have

$$\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa_{n-1}, \varkappa_n; \varkappa_n, \varkappa_{n+1}).$$

Then, since by the inaccessibility of $\varkappa_{n+1}$, $2^{\varkappa_n^{\varkappa_{n-1}}} < \varkappa_{n+1}$, our conclusion follows from Theorem 2. ■

We note that, by elementarity and closure considerations, the targets of n-huge cardinals are inaccessible. Hence, for the forward direction of part c, our assumption of inaccessibility is redundant. For the reverse direction, we assume inaccessibility for simplicity. A weaker assumption would suffice.

4. Almost hugeness and almost supercompactness. Suppose M is an inner model, and $i\colon V \to M$ is an elementary embedding such that $\varkappa_0$ is the first cardinal moved by i, $i(\varkappa_0) = \varkappa_1$, $i(\varkappa_1) = \varkappa_2, \ldots$ In addition, suppose that $\lambda > \varkappa_0$ is a limit cardinal such that for each $\gamma < \lambda$, M is closed under γ-sequences. Then, for some $n < \omega$, $\varkappa_n < \lambda \leqslant \varkappa_{n+1}$.

Let $\varkappa$ be the least cardinal such that $i(\varkappa) \geqslant \lambda$. If M is closed under λ-sequences, then we have a situation exactly as in Section 1, where case 1 applies if $i(\varkappa) = \lambda$ and case 2 applies if $i(\varkappa) > \lambda$.

Now, we do not assume that M is closed under λ-sequences. For simplicity, we restrict our attention to the case of $n = 0$ and $\varkappa = \varkappa_0$. Also, instead of considering the distinct cases of $i(\varkappa) = \lambda$ and $i(\varkappa) > \lambda$, we consider the general case of $i(\varkappa) \geqslant \lambda$ and the special case of $i(\varkappa) = \lambda$.

We shall say that $\varkappa$ is *almost λ-supercompact* if and only if there exists an elementary embedding $i\colon V \to M$ where M is an inner model closed under γ-sequences for each $\gamma < \lambda$, $\varkappa$ is the first cardinal moved by i, and $i(\varkappa) \geqslant \lambda$. $\varkappa$ is almost huge with target λ if and only if there exists $i\colon V \to M$ as above, except that $i(\varkappa) = \lambda$. It should be noted that the assertion that $\varkappa$ is almost λ-supercompact is strictly stronger than the assertion that $\varkappa$ is γ-supercompact for each γ with $\varkappa \leqslant \gamma < \lambda$ (see [2], p. 106).

In this section, we first show that almost supercompactness and almost hugeness are both implied by a certain flipping property. Then, we consider possible converses.

THEOREM 3. *Let $\lambda > \varkappa$ be an inaccessible cardinal.*

a. *If, for every $t\colon \lambda \to P(P_\varkappa(\lambda))$, there exists $t' \sim t$ such that $\triangle_{\alpha<\lambda} t'(\alpha)$ is a stationary subset of $P_\varkappa(\lambda)$, then $\varkappa$ is almost λ-supercompact.*

b. *If, for every $t\colon \lambda \to P(P_{=\varkappa}(\lambda))$, there exists $t' \sim t$ such that $\triangle_{\alpha<\lambda} t'(\alpha)$ is a stationary subset of $P_{=\varkappa}(\lambda)$, then $\varkappa$ is almost huge with target λ.*

Before beginning the proof of Theorem 3, we must consider ultrafilter characterizations of almost supercompactness and almost hugeness, and to do this we must first consider some additional notation and terminology.

Suppose $\varkappa \leqslant \gamma_1 < \gamma_2 < \lambda$ and U_{γ_1} and U_{γ_2} are normal, fine, $\varkappa$-complete ultrafilters on $P_\varkappa(\gamma_1)$ and $P_\varkappa(\gamma_2)$ respectively. We say that U_{γ_1} is a *restriction of* U_{γ_2}, and write $U_{\gamma_1} = U_{\gamma_2} \upharpoonright \gamma_1$, if, for every $A \subseteq P_\varkappa(\gamma_1)$, $A \in U_{\gamma_1}$ if and only if $\{X \in P_\varkappa(\gamma_2);\ X \cap \gamma_1 \in A\} \in U_{\gamma_2}$. For λ a limit cardinal we say that $\langle U_\gamma\colon \varkappa \leqslant \gamma < \lambda\rangle$ is a *coherent sequence* if, for each γ with $\varkappa \leqslant \gamma < \lambda$, U_γ is a normal, fine, $\varkappa$-complete ultrafilter on $P_\varkappa(\gamma)$ and, whenever $\varkappa \leqslant \gamma_1 < \gamma_2 < \lambda$, $U_{\gamma_1} = U_{\gamma_2} \upharpoonright \gamma_1$. We will also use the fact that in this situation, there is a canonical elementary embedding $k_{\gamma_1\gamma_2}\colon M_{\gamma_1} \to M_{\gamma_2}$, where M_{γ_1} and M_{γ_2} are the inner models corresponding to U_{γ_1} and U_{γ_2} respectively, such that for each $\alpha \leqslant \gamma_1$, $k_{\gamma_1\gamma_2}(\alpha) = \alpha$. For details see [12].

LEMMA. *Let λ be an innaccessible cardinal.*

a. *If there exists a coherent sequence $\langle U_\gamma\colon \varkappa \leqslant \gamma < \lambda\rangle$, then $\varkappa$ is almost λ-supercompact.*

b. *Suppose there exists a coherent sequence $\langle U_\gamma\colon \varkappa \leqslant \gamma < \lambda\rangle$ satisfying the following condition:*

If, for some γ with $\varkappa \leqslant \gamma < \lambda$, $f\colon P_\varkappa(\gamma) \to \mathrm{ORD}$ is such that $\{X \in P_\varkappa(\gamma)\colon f(X) < \varkappa\} \in U_\gamma$, then, for some η with $\gamma < \eta < \lambda$, $\{X \in P_\varkappa(\eta)\colon f(X \cap \gamma) \leqslant |X|\} \in U_\eta$.

Then, $\varkappa$ is almost huge with target λ.

Proof. For part a, we are given the coherent sequence $\langle U_\gamma\colon \varkappa \leqslant \gamma < \lambda\rangle$. Then, $\langle M_{\gamma_1}, k_{\gamma_1\gamma_2}\colon \varkappa \leqslant \gamma_1 < \gamma_2 < \lambda\rangle$ is a directed system. Since λ is regular, the direct limit of this system is well-founded. Let M be its transitive collapse. Then, using the regularity of λ again, it follows that M is closed under γ-sequences for each $\gamma < \lambda$. There is a canonical elementary embedding $i\colon V \to M$. It is clear that $i(\varkappa) \geqslant \lambda$, and hence, this embedding witnesses that $\varkappa$ is almost λ-supercompact. For details, see [2] or [12].

For part b, we consider the directed system of inner models and elementary embeddings, and obtain $i\colon V \to M$ as above. We must show that $i(\varkappa) = \lambda$.

For each γ with $\varkappa \leqslant \gamma < \lambda$, let $\pi_\gamma\colon \Pi V/U_\gamma \to M_\gamma$ be the collapsing isomorphism, and let $i_\gamma\colon V \to M_\gamma$ and $h_\gamma\colon M_\gamma \to M$ be the canonical elementary embeddings. We note that for $\alpha \leqslant \gamma$, $h_\gamma(\alpha) = \alpha$.

Suppose, by way of contradiction, that $i(\varkappa) > \lambda$. Then, for some γ with $\varkappa \leqslant \gamma < \lambda$, there exists $\alpha < i_\gamma(\varkappa)$ such that $h_\gamma(\alpha) \geqslant \lambda$. Let $f\colon P_\varkappa(\gamma) \to \mathrm{ORD}$ be such that $\pi_\gamma(f) = \alpha$. Then, since $\alpha < i_\gamma(\varkappa)$, $\{X \in P_\varkappa(\gamma)\colon f(X) < \varkappa\} \in U_\gamma$. By assumption, this implies that for some η with $\gamma < \eta < \lambda$, $\{X \in P_\varkappa(\eta)\colon f(X \cap \gamma) \leqslant |X| \in U_\eta$.

Define $g\colon P_\varkappa(\eta) \to \mathrm{ORD}$ by $g(X) = f(X \cap \gamma)$. Then $h_\eta(\pi_\eta(g)) = h_\gamma(\pi_\gamma(f))$. Also, the fact that $\{X \in P_\varkappa(\eta)\colon f(X \cap \gamma) \leqslant |X|\} \in U_\eta$ tells us that $\pi_\eta(g) \leqslant \eta$. Putting this all together, we have $\lambda \leqslant h_\gamma(\alpha) = h_\gamma(\pi_\gamma(f)) = h_\eta(\pi_\eta(g)) \leqslant h_\eta(\eta) = \eta < \lambda$. This is a contradiction. ∎

Solovay, Reinhardt, and Kanamori [12] have a different technical condition associated with almost hugeness. It is not hard to show that their condition is equivalent to ours. We presented our version because it will make the proof of Theorem 3 easier.

We note that the converses to parts a and b of the lemma are both true, but are not needed here, and we omit the proofs.

Proof of Theorem 3. For part a, we first note that since λ is inaccessible, there are λ many total subsets of the $P_\varkappa(\gamma)$ for $\varkappa \leqslant \gamma < \lambda$. Let $\langle (A_\alpha, \gamma_\alpha)\colon \alpha < \lambda\rangle$ be an enumeration of all pairs such that $A_\alpha \subseteq P_\varkappa(\gamma_\alpha)$ and $\varkappa \leqslant \gamma_\alpha < \lambda$. Define $t\colon \lambda \to P(P_\varkappa(\lambda))$ by, for each $\alpha < \lambda$, $t(\alpha) = \{X \in P_\varkappa(\lambda)\colon X \cap \gamma_\alpha \in A_\alpha\}$. Let $t' \sim t$ be such that $\triangle_{\alpha<\lambda} t'(\alpha)$ is a stationary subset of $P_\varkappa(\lambda)$. Then, for each γ with $\varkappa \leqslant \gamma < \lambda$, we define $U_\gamma = \{A \subseteq P_\varkappa(\gamma)\colon$ there exists $\alpha < \lambda$ such that $(A, \gamma) = (A_\alpha, \gamma_\alpha)$ and $t'(\alpha) = t(\alpha)\}$.

It is straightforward to show that each such U_γ is a normal, fine, $\varkappa$-complete ultrafilter on $P_\varkappa(\gamma)$. By the lemma, it suffices to show that $\langle U_\gamma\colon \varkappa \leqslant \gamma < \lambda\rangle$ is a coherent sequence.

Suppose $\varkappa \leqslant \eta_1 < \eta_2 < \lambda$ We show that $U_{\eta_1} = U_{\eta_2} \upharpoonright \eta_1$. Pick $A \subseteq P_\varkappa(\eta_1)$. Let $\alpha_1 < \lambda$ be such that $A_{\alpha_1} = A$ and $\gamma_{\alpha_1} = \eta_1$. Let $\alpha_2 < \lambda$ be such that $A_{\alpha_2} = \{X \in P_\varkappa(\eta_2)\colon X \cap \eta_1 \in A_{\alpha_1}\}$ and $\gamma_{\alpha_2} = \eta_2$. Then,

$$\begin{aligned} t(\alpha_2) &= \{X \in P_\varkappa(\lambda)\colon X \cap \gamma_{\alpha_2} \in A_{\alpha_2}\} = \{X \in P_\varkappa(\lambda)\colon X \cap \eta_2 \in A_{\alpha_2}\} \\ &= \{X \in P_\varkappa(\lambda)\colon X \cap \eta_1 \in A_{\alpha_1}\} = \{X \in P_\varkappa(\lambda)\colon X \cap \gamma_{\alpha_1} \in A_{\alpha_1}\} = t(\alpha_1). \end{aligned}$$

Next, we note that $A_{\alpha_1} \in U_{\eta_2} \upharpoonright \eta_1$ if and only if $\{X \in P_\varkappa(\eta_2)\colon X \cap \eta_1 \in A_{\alpha_1}\} \in U_{\eta_2}$ if and only if $A_{\alpha_2} \in U_{\eta_2}$ if and only if $t'(\alpha_2) = t(\alpha_2)$. Also $A_{\alpha_1} \in U_{\eta_1}$ if and only if $t'(\alpha_1) = t(\alpha_1)$.

Thus, in order to show that $A \in U_{\eta_2} \upharpoonright \eta_1$ if and only if $A \in U_{\eta_1}$, we must show that $t'(\alpha_2) = t(\alpha_2)$ if and only if $t'(\alpha_1) = t(\alpha_1)$. But we have already shown that $t(\alpha_2) = t(\alpha_1)$. Hence, we must show that $t'(\alpha_2) = t'(\alpha_1)$.

Since $\triangle_{\alpha<\lambda} t'(\alpha)$ is stationary, it is unbounded. Pick $X \in \triangle_{\alpha<\lambda} t'(\alpha)$ such that $\{\alpha_1, \alpha_2\} \subseteq X$. It follows that $X \in t'(\alpha_1) \cap t'(\alpha_2)$. But clearly, since $t(\alpha_1) = t(\alpha_2)$, we know that $t'(\alpha_1)$ and $t'(\alpha_2)$ are either disjoint or identical. We conclude that $t'(\alpha_1) = t'(\alpha_2)$.

For part b, we begin as in part a by letting $\langle (A_\alpha, \gamma_\alpha): \alpha < \lambda \rangle$ be an enumeration of all pairs such that $A_\alpha \subseteq P_\varkappa(\gamma_\alpha)$ and $\varkappa \leqslant \gamma_\alpha < \lambda$. Define $t: \lambda \to P(P_{=\varkappa}(\lambda))$ by, for each $\alpha < \lambda$, $t(\alpha) = \{X \in P_{=\varkappa}(\lambda): X \cap \gamma_\alpha \in A_\alpha\}$. Let $t' \sim t$ be such that $\triangle_{\alpha<\lambda} t'(\alpha)$ is a stationary subset of $P_{=\varkappa}(\lambda)$. For each γ with $\varkappa \leqslant \gamma < \lambda$, U_γ is defined exactly as in part a. Then, $\langle U_\gamma: \varkappa \leqslant \gamma < \lambda \rangle$ is again a coherent sequence. We must show that the additional condition of part b of the lemma is satisfied.

Suppose $\varkappa \leqslant \gamma < \lambda$ and $f: P_\varkappa(\gamma) \to \mathrm{ORD}$ is such that $B_\gamma = \{X \in P_\varkappa(\gamma): f(X) < \varkappa\} \in U_\gamma$. Assume by way of contradiction that, for each η with $\gamma < \eta < \gamma$, $B_\eta = \{X \in P_\varkappa(\eta): f(X \cap \gamma) \geqslant |X|\} \in U_\eta$.

Define $g: \{\eta: \gamma \leqslant \eta < \gamma\} \to \lambda$ such that for each η with $\gamma \leqslant \eta < \lambda$, $B_\eta = A_{g(\eta)}$ and $\gamma_{g(\eta)} = \eta$. Let $C = \{X \in P_\varkappa(\lambda)$: for each $\eta \in X \setminus \gamma$, $g(\eta) \in X\}$. It is straightforward to show that C is closed and unbounded. Let C^* be the closed and unbounded subset of $P_{=\varkappa}(\lambda)$ generated by C. Then also, if $X \in C^*$ and $\eta \in X \setminus \gamma$, $g(\eta) \in X$.

Let $D = \{X \in P_\varkappa(\lambda): \gamma \in X$ and $|(X \setminus \gamma) \cap \mathrm{CARD}| = |X|\}$. It is straightforward to verify that D is closed and unbounded and that if D^* is the closed and unbounded subset of $P_{=\varkappa}(\lambda)$ generated by D, and $X \in D^*$, then $\gamma \in X$ and $|(X \setminus \gamma) \cap \mathrm{CARD}| = \varkappa$.

Fix $X \in \triangle_{\alpha<\lambda} t'(\alpha) \cap C^* \cap D^*$. Since $\gamma \in X$ and $X \in C^*$, $g(\gamma) \in X$ and hence, since $X \in \triangle_{\alpha<\lambda} t'(\alpha)$, $X \in t'(g(\gamma))$. But $A_{g(\gamma)} = B_\gamma \in U_\gamma$ and consequently $t(g(\gamma)) = t'(g(\gamma))$. It follows that $X \in t(g(\gamma))$. This implies that $X \cap \gamma_{g(\gamma)} \in A_{g(\gamma)} = B_\gamma$. But $\gamma_{g(\gamma)} = \gamma$. Hence $X \cap \gamma \in B_\gamma$, and thus $f(X \cap \gamma) < \varkappa$.

For any η with $\gamma < \eta < \lambda$, $A_{g(\eta)} = B_\eta \in U_\eta$, and hence $t'(g(\eta)) = t(g(\eta))$. If $\eta \in X$, $g(\eta) \in X$ and so $X \in t'(g(\eta))$. But then $X \in t(g(\eta))$. This implies that $X \cap \gamma_{g(\eta)} \in A_{g(\eta)} = B_\eta$. But $\gamma_{g(\eta)} = \eta$. Hence, $X \cap \eta \in B_\eta$, and thus $f(X \cap \gamma) > |X \cap \eta|$.

Since $X \in D^*$, $|(X \setminus \gamma) \cap \mathrm{CARD}| = \varkappa$. Then, since $f(X \cap \gamma) < \varkappa$, there exists $\eta \in X \setminus \gamma$ such that $|X \cap \eta| > f(X \cap \gamma)$. This is a contradiction. Hence, by the lemma, we conclude that $\varkappa$ is almost huge with target λ. ∎

We do not know whether the exact converses to Theorem 3 hold. We can establish a converse to part a if we make a somewhat stronger assumption on λ. After establishing this result, we discuss the relationship of this result to the converse of part b.

THEOREM 4. *Let λ be an ineffable cardinal. If $\varkappa$ is almost λ-supercompact, then, for every $t: \lambda \to P(P_\varkappa(\lambda))$, there exists $t' \sim t$ such that $\triangle_{\alpha<\lambda} t'(\alpha)$ is a stationary subset of $P_\varkappa(\lambda)$.*

For the definition and standard facts on ineffable cardinals, see Baumgartner [3]. We shall use the flipping characterization of ineffability given in [1]:

λ is ineffable if and only if given any $s: \lambda \to P(\lambda)$, there exists $s' \sim s$ such that $\triangle_{\alpha<\lambda} s'(\alpha)$ is a stationary subset of λ.

Before proving Theorem 4, we establish a lemma, which may be of interest in its own right.

LEMMA. *Suppose $C \subseteq P_\varkappa(\lambda)$ is closed and unbounded, λ is a regular cardinal, and for every $\delta < \lambda$, $|\delta|^{\underset{\smile}{\varkappa}} < \lambda$. Then $\{\delta < \lambda: C \cap P_\varkappa(\delta)$ is a closed and unbounded subset of $P_\varkappa(\delta)\}$ is a closed and unbounded subset of λ.*

Proof. Assume $C \subseteq P_\varkappa(\lambda)$ and λ are as in the statement of the lemma. Let $D = \{\delta < \lambda: C \cap P_\varkappa(\delta)$ is a closed and unbounded subset of $P_\varkappa(\delta)\}$.

First we show that D is closed. Suppose $\langle \delta_\alpha: \alpha < \eta \rangle$ is an increasing sequence of elements of D, where $\eta < \lambda$. Let $\delta = \sup_{\alpha<\eta} \delta_\alpha$. We must show that $C \cap P_\varkappa(\delta)$ is a closed and unbounded subset of $P_\varkappa(\delta)$.

It is straightforward to show that $C \cap P_\varkappa(\delta)$ is closed. To see that $C \cap P_\varkappa(\delta)$ is unbounded, pick $X \in P_\varkappa(\delta)$. If $X \in P_\varkappa(\delta_\alpha)$ for some $\alpha < \eta$, then there exists $Y_x \in C \cap P_\varkappa(\delta_\alpha)$ such that $X \subseteq Y_x$. But then $Y_x \in C \cap P_\varkappa(\delta)$. Suppose then that X is unbounded below δ. Let $\langle \beta_\alpha: \alpha < \overline{X} \rangle$ be a nondecreasing subsequence of $\langle \delta_\alpha: \alpha < \eta \rangle$ such that for each $\alpha < \overline{X}$, β_α is greater than the αth element of X. Then, since for each $\alpha < \overline{X}$, $C \cap P_\varkappa(\beta_\alpha)$ is an unbounded subset of $P_\varkappa(\beta_\alpha)$, we can inductively define an increasing sequence $\langle Y_\alpha: \alpha < \overline{X} \rangle$ of elements of $C \cap P_\varkappa(\delta)$ such that for each $\alpha < \overline{X}$, $X \cap \beta_\alpha \subseteq Y_\alpha \in C \cap P_\varkappa(\beta_\alpha)$. If we let $Y_x = \bigcup_{\alpha<\overline{X}} Y_\alpha$, we have $Y_x \in C \cap P_\varkappa(\delta)$. Hence, $X = \bigcup_{\alpha<\overline{X}} (X \cap \beta_\alpha) \subseteq Y_x$ and we have shown that $C \cap P_\varkappa(\delta)$ is unbounded. This establishes that D is closed.

To show that D is unbounded, pick $\delta_0 < \lambda$. We must find $\delta \in D$ with $\delta \geqslant \delta_0$. For each $X \in P_\varkappa(\lambda)$, let $Y_x \in C$ be such that $X \subseteq Y_x$. Using our assumptions on λ, we may inductively define $\delta_{i+1} < \lambda$, for each $i < \omega$, as follows: δ_{i+1} is the least ordinal such that for each $X \in P_\varkappa(\delta_i)$, $Y_x \in P_\varkappa(\delta_{i+1})$. Let $\delta = \sup_{i<\omega} \delta_i$. Then $\delta_0 \leqslant \delta < \lambda$, and $C \cap P_\varkappa(\delta)$ is a closed and unbounded subset of $P_\varkappa(\delta)$. The proof is similar to that contained in the previous paragraph, and we omit it. This establishes that D is unbounded. ∎

We wish to thank W. Zwicker for contributing to the proof of this lemma. In addition, Zwicker pointed out that our assumptions on λ can be weakened somewhat. In our definition of the δ_i's, the regularity of λ is certainly necessary, but something weaker than requiring that $|\delta|^{\underset{\smile}{\varkappa}} < \lambda$ for each $\delta < \lambda$ will suffice to complete the argument. The point is that in defining each δ_{i+1}, we need not consider every element of each $P_\varkappa(\delta_i)$, but need only consider every element of some unbounded subset of $P_\varkappa(\delta_i)$, and such a subset can have strictly smaller cardinality than does $P_\varkappa(\delta_i)$. Hence, we need only require that for each $\delta < \lambda$, λ is greater than the cardinality of some unbounded subset of $P_\varkappa(\delta)$. For more on the distinction between the size of $P_\varkappa(\delta)$ and the size of unbounded subsets of $P_\varkappa(\delta)$, see [13].

Proof of Theorem 4. We assume that $\varkappa$ is almost λ-supercompact and λ is ineffable, and we consider $t\colon \lambda \to \boldsymbol{P}(P_\varkappa(\lambda))$. For each γ with $\varkappa \leqslant \gamma < \lambda$, define $t_\gamma(\alpha) = t(\alpha) \cap P_\varkappa(\gamma)$ for each $\alpha < \gamma$. Since $\varkappa$ is almost λ-supercompact, $\varkappa$ is γ-supercompact for each such γ. Then, by Corollary b of Theorem 1, we may let $t'_\gamma \sim t_\gamma$ be such that $\triangle_{\alpha<\gamma} t'_\gamma(\alpha)$ is a stationary subset of $P_\varkappa(\gamma)$. Define $s\colon \lambda \to \boldsymbol{P}(\lambda)$ by $s(\alpha) = \{\gamma\colon \varkappa \leqslant \gamma < \lambda$ and $t'_\gamma(\alpha) = t_\gamma(\alpha)\}$. By the ineffability of λ, suppose $s' \sim s$ is such that $\triangle_{\alpha<\lambda} s'(\alpha)$ is a stationary subset of λ. Now, define $t' \sim t$ by, for each $\alpha < \lambda$, $t'(\alpha) = t(\alpha)$ if and only if $s'(\alpha) = s(\alpha)$. We must show that $\triangle_{\alpha<\lambda} t'(\alpha)$ is a stationary subset of $P_\varkappa(\lambda)$.

Suppose by way of contradiction that $C \subseteq P_\varkappa(\lambda)$ is closed and unbounded and $C \cap \triangle_{\alpha<\lambda} t'(\alpha) = \emptyset$. Let $D = \{\gamma\colon \varkappa \leqslant \gamma < \lambda$ and $C \cap P_\varkappa(\gamma)$ is a closed and unbounded subset of $P_\varkappa(\gamma)\}$. Since λ is ineffable, it is inaccessible, and thus it certainly satisfies the assumptions of the lemma. Hence, D is a closed and unbounded subset of λ.

Pick $\gamma \in D \cap \triangle_{\alpha<\lambda} s'(\alpha)$. Then, $C \cap P_\varkappa(\gamma)$ is a closed and unbounded subset of $P_\varkappa(\gamma)$. Since $\triangle_{\alpha<\gamma} t'_\gamma(\alpha)$ is a stationary subset of $P_\varkappa(\gamma)$, we may pick $X \in C \cap P_\varkappa(\gamma) \cap \triangle_{\alpha<\gamma} t'_\gamma(\alpha)$. We claim that $X \in \triangle_{\alpha<\lambda} t'(\alpha)$. Pick $\delta \in X$. We must show that $X \in t'(\delta)$.

We note that since $\delta \in X$, $\delta < \gamma$ and hence, since $\gamma \in \triangle_{\alpha<\lambda} s'(\alpha)$, $\gamma \in s'(\delta)$. Also, since $\delta \in X$ and $X \in \triangle_{\alpha<\gamma} t'_\gamma(\alpha)$, $X \in t'_\gamma(\delta)$.

Suppose first that $s'(\delta) = s(\delta)$. It follows that $\gamma \in s(\delta)$ and this implies that $t'_\gamma(\delta) = t_\gamma(\delta)$. Then, we have $X \in t_\gamma(\delta)$. Hence $X \in t(\delta)$. But, since $s'(\delta) = s(\delta)$, we know that $t'(\delta) = t(\delta)$, and consequently, $X \in t'(\delta)$.

Suppose then that $s'(\delta) \neq s(\delta)$. Then $\gamma \notin s(\delta)$, and hence $t'_\gamma(\delta) \neq t_\gamma(\delta)$. This implies that $X \notin t_\gamma(\delta)$, which tells us that $X \notin t(\delta)$. Then, since $s'(\delta) \neq s(\delta)$, we have $t'(\delta) \neq t(\delta)$, and we conclude that $X \in t'(\delta)$.

We have shown that $X \in C \cap \triangle_{\alpha<\lambda} t'(\alpha)$. This contradicts our assumption that $C \cap \triangle_{\alpha<\lambda} t'(\alpha) = \emptyset$. ■

We note that an indirect proof of Theorem 4 does already exist in the literature. DiPrisco and Zwicker ([6]) showed that the assumptions of our Theorem 4 imply that a certain partition property holds for $P_\varkappa(\lambda)$. Magidor ([11]) showed that this partition property implies a certain ineffability property for $P_\varkappa(\lambda)$, and DiPrisco and Zwicker ([7]) showed that this ineffability property implies the flipping property given in the conclusion of Theorem 4.

In trying to apply the methods of Theorem 4 to the almost huge case, we find that a converse to part b of Theorem 3 is not what we obtain. What we do obtain is:

THEOREM 5. *Suppose λ is an ineffable cardinal, I is the ineffable ideal on λ, and $\{\gamma\colon \varkappa < \gamma < \lambda$ and $\varkappa$ is huge with target $\gamma\} \in I^+$. Then, for every $t\colon \lambda \to \boldsymbol{P}(P_{=\varkappa}(\lambda))$, there exists $t' \sim t$ such that $\triangle_{\alpha<\lambda} t'(\alpha)$ is a stationary subset of $P_{=\varkappa}(\lambda)$.*

For the definition and basic facts on the ineffable ideal, see [3]. The proof uses the same method as the proof of Theorem 4, and we omit it.

5. On ideals and ineffability properties. In this section we consider natural ideals associated with the flipping properties discussed in this paper, and show how these ideals connect with certain ineffability properties. For simplicity, in this section, we only study ideals associated with $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$. The analogous study for $\mathrm{Flip}_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ would be similar, but slightly more involved because of the slightly harder definition of this flipping property.

Suppose $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$. We wish to define an ideal $J_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ on $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$. For simplicity, when the context is clear, we write J in place of $J_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$. As usual, J^+ denotes the collection of sets of positive measure, and J^* denotes the collection of sets of measure one. We note that we certainly have the option of considering such a J to be an ideal on all of $P_{=\varkappa}(\lambda)$ by declaring that $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda) \in J^*$. Our perspective of viewing J as an ideal on $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ will make the proof of Theorem 6 slightly easier.

We define J as follows: For $A \subseteq I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$, we set $A \in J^+$ if and only if for every $t\colon \lambda \to \boldsymbol{P}(A)$, there exists $t' \sim t$ (where flips are taken with respect to A) such that $\triangle_{\alpha<\lambda} t'(\alpha)$ is a stationary subset of $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$.

THEOREM 6. *If $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$, then $J_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ is a $\varkappa_0$-complete, normal ideal on $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$.*

Proof. We assume $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ and we show that $J = J(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ is a $\varkappa_0$-complete, normal ideal.

The verification that J is an ideal is straightforward.

To show that J is $\varkappa_0$-complete, let us suppose that $\langle A_\beta\colon \beta < \eta\rangle$ is a sequence of disjoint elements of J, where $\eta < \varkappa_0$. Let $A = \bigcup_{\beta<\eta} A_\beta$. We must show that $A \in J$.

For each $\beta < \eta$, let $t_\beta\colon \lambda \to \boldsymbol{P}(A_\beta)$ be such that for no $t'_\beta \sim t_\beta$ (where flips are taken with respect to A_β) do we have that $\triangle_{\alpha<\lambda} t'_\beta(\alpha)$ is stationary.

For each $X \in A$, let $\beta_X < \eta$ be such that $X \in A_{\beta_X}$. (Note that the choice of β_X is unique.) Define $t\colon \lambda \to \boldsymbol{P}(A)$ by, for each $\alpha < \lambda$ and $X \in A$, $X \in t(\alpha)$ if and only if $X \in t_{\beta_X}(\alpha)$. It suffices for us to show that for no $t' \sim t$ (where flips are taken with respect to A) do we have that $\triangle_{\alpha<\lambda} t'(\alpha)$ is stationary.

Suppose, by way of contradiction, that $t' \sim t$ and $\triangle_{\alpha<\lambda} t'(\alpha)$ is stationary. Then, since $\langle \triangle_{\alpha<\lambda} t'(\alpha) \cap A_\beta\colon \beta < \eta\rangle$ is a partition of $\triangle_{\alpha<\lambda} t'(\alpha)$ into less than $\varkappa_0$ many disjoint subsets, we know that for some $\beta_0 < \eta$, $\triangle_{\alpha<\lambda} t'(\alpha) \cap A_{\beta_0}$ is stationary.

Define $t'_{\beta_0} \sim t_{\beta_0}$ by, for each $\alpha < \lambda$, $t'_{\beta_0}(\alpha) = t_{\beta_0}(\alpha)$ if and only if $t'(\alpha) = t(\alpha)$. By assumption, $\underset{\alpha<\lambda}{\triangle} t'_{\beta_0}(\alpha)$ is non-stationary. But this is a contradiction, since clearly $\underset{\alpha<\lambda}{\triangle} t'(\alpha) \cap A_{\beta_0} = \underset{\alpha<\lambda}{\triangle} t'_{\beta_0}(\alpha)$.

To show that J is normal, we establish that J is closed under diagonal unions. Let $\langle A_\beta \colon \beta < \lambda \rangle$ be a sequence of elements of J. The diagonal union of this sequence, written $\underset{\beta<\lambda}{\nabla} A_\beta$, is defined as follows:

For $X \in P_{=\varkappa}(\lambda)$, $X \in \underset{\beta<\lambda}{\nabla} A_\beta$ if and only if for some $\beta \in X$, $X \in A_\beta$.

Let $A = \underset{\beta<\lambda}{\nabla} A_\beta$. We must show $A \in J$. By assumption, for each $\beta < \lambda$, there exists $t_\beta \colon \lambda \to \boldsymbol{P}(A_\beta)$ such that for no $t'_\beta \sim t_\beta$ (where flips are taken with respect to A_β) do we have that $\underset{\alpha<\lambda}{\triangle} t'_\beta(\alpha)$ is stationary.

For each $X \in A$, let $\beta_x \in X$ be minimal such that $X \in A_{\beta_x}$. Define $t \colon \lambda \to \boldsymbol{P}(A)$ by, for each $\alpha < \lambda$ and $X \in A$, $X \in t(\alpha)$ if and only if $X \in t_{\beta_x}(\alpha)$. It suffices for us to show that for no $t' \sim t$ (where flips are taken with respect to A) do we have that $\underset{\alpha<\lambda}{\triangle} t'(\alpha)$ is stationary.

Suppose, by way of contradiction, that $t' \sim t$ and $\underset{\alpha<\lambda}{\triangle} t'(\alpha)$ is stationary. For each $\beta < \lambda$, define $t'_\beta \sim t_\beta$ by, for each $\alpha < \lambda$ $t'_\beta(\alpha) = t_\beta(\alpha)$ if and only if $t'(\alpha) = t(\alpha)$. By assumption, for each $\beta < \lambda$, $\underset{\alpha<\lambda}{\triangle} t'_\beta(\alpha)$ is non-stationary. For each such β, let C_β be a closed and unbounded subset of $P_{=\varkappa}(\lambda)$ such that $\underset{\alpha<\lambda}{\triangle} t'_\beta(\alpha) \cap C_\beta = \emptyset$. Let $C = \underset{\beta<\lambda}{\triangle} C_\beta$. Then C is closed and unbounded. Also, if $X \in C$, then, for each $\beta \in X$, there exists $\alpha \in X$ such that $X \notin t'_\beta(\alpha)$.

Since $\underset{\alpha<\lambda}{\triangle} t'(\alpha)$ is stationary, we may pick $X \in \underset{\alpha<\lambda}{\triangle} t'(\alpha) \cap C$. Since $X \in C$ and $\beta_x \in X$, it follows from the above that there exists $\alpha_x \in X$ such that $X \notin t'_{\beta_x}(\alpha_x)$.

Since $X \in \underset{\alpha<\lambda}{\triangle} t'(\alpha)$ and $\alpha_x \in X$, we have $X \in t'(\alpha_x)$. By definition, $X \in t(\alpha_x)$ if and only if $X \in t_{\beta_x}(\alpha_x)$. Also, by definition, $t'_{\beta_x}(\alpha_x) = t_{\beta_x}(\alpha_x)$ if and only if $t'(\alpha_x) = t(\alpha_x)$. Hence, since $X \in t'(\alpha_x)$, we conclude that $X \in t'_{\beta_x}(\alpha_x)$. This is a contradiction. ■

Next, we consider certain ineffability properties that are related to the flipping properties we have studied. These properties generalize notions that have previously been studied for $P_\varkappa(\lambda)$.

We write $\mathrm{Inef}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ to denote the following property:

For every function f with domain $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$, satisfying that $f(X) \subseteq X$ for each $X \in I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$, there exists $T \subseteq \lambda$ such that

$$\{X \in I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda) \colon f(X) = X \cap T\}$$

is a stationary subset of $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$.

By direct generalizations of techniques of DiPrisco and Zwicker ([7]), one can show that $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ if and only if $\mathrm{Inef}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$.

There is a natural way to associate ideals with these ineffability properties. These ideals turn out to be the same as those associated with out flipping properties. In particular, if $\mathrm{Inef}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$, we define an ideal J on $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ as follows:

For $A \subseteq I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$, we set $A \in J^+$ if and only if, for every function f with domain A, satisfying that $f(X) \subseteq X$ for each $X \in A$, there exists $T \subseteq \lambda$ such that $\{X \in A \colon f(X) = X \cap T\}$ is a stationary subset of $I_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$.

By following through the proof that $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ if and only if $\mathrm{Inef}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$, it is straightforward to verify that the J we have just defined is the same as the ideal $J_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ associated with $\mathrm{Flip}_h(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$.

As we mentioned at the beginning of this section, all of the definitions and results of the section carry over to the $\mathrm{Flip}_s(\varkappa_0, \varkappa_1, \ldots, \varkappa_n; \varkappa, \lambda)$ case. In the special case that $n = 0$ and $\varkappa = \varkappa_0$, this ideal is the ineffable ideal on $P_\varkappa(\lambda)$. This ideal has been studied by Carr in [4].

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Functions provably total in $I^-\Sigma_1$

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Abstract. We estimate the rapidity of the growth of recursive functions which are provably total in a finite fragment of Σ_1 parameter-free induction subject to the size of the fragment.

The aim of this paper is to bound the rapidity of the growth of recursive (Σ_1 definable) functions which are provably total in $I^-\Sigma_1$ (induction for parameter free Σ_1 formulas). We show that if in the proof of the totality of a recursive function f from N to N Σ_1 induction is applied n times then the function can be bounded by the $n+1$'s function in the Wainer hierarchy (see [W]).

The result is proved by means of a proof-theoretic analysis of proofs of sentences of the form $(\forall t)_{\geqslant 0}\varphi(t)$ in $I^-\Sigma_1$ (an analogous analysis for $\exists_1$ formulas and $I^-\exists_1$ can be found in [A]). We consider here Σ_1 formulas φ without parameters.

Here PA^- denotes the theory of discretely ordered rings. If φ is a formula then $\mathrm{Ind}\,\varphi$ denotes the following sentence:

$$\mathrm{PA}^- \,\&\, [\varphi(0) \,\&\, \big((\forall t)_{\geqslant 0}\varphi(t) \Rightarrow \varphi(t+1)\big) \Rightarrow (\forall t)_{\geqslant 0}\varphi(t)]\,.$$

To simplify the notation we will assume that for every formula of the form $\varphi(y, \vec{x})$ the sentence $\varphi(-1, 0, \ldots, 0)$ is true. Formally, this can be assumed since we can replace φ be the formula φ^* defined as $\big(y \geqslant 0 \,\&\, \varphi(y, x)\big) \vee y < 0$. Then φ is equivalent to φ^* for all non-negative y's we are interested in. Without causing confusion we identify φ and φ^*.

DEFINITION 1. Let the formulas $\varphi_1, \ldots, \varphi_n$ be of the form

$$\varphi_i(t) = (\exists \vec{s})\,\varphi_i'(t, \vec{s}) \quad \text{where } \varphi_i' \in \Delta_0,\ i = 1, \ldots, n.$$

We assume that the quantifiers in the formulas φ_i' are bounded by the free variable or by one of the variables of $\vec{s}$. Let $M \models \mathrm{PA}^-$, $v_1, \ldots, v_m \in M$. Assume that we have a fixed enumeration of polynomials. Let $K \in N$. A set $H \subseteq M$ is called a *K-closure of* $\{v_1, \ldots, v_m\}$ *with respect to* $\{\varphi_1, \ldots, \varphi_n\}$ if there exist sets $H_0, \ldots, H_K$ such that

1. $H = H_0 \cup H_1 \cup \ldots \cup H_K$ and $\{v_1, \ldots, v_m\} = H_0$.
2. If $x \in H_j$ for a certain $j < K$ then for every $i \in \{1, \ldots, n\}$ there is an $\vec{y} \in H_{j+1}$ such that $M \models \varphi_i'(x, \vec{y})$.