References

[A] S. Armentrout, *Cellular decompositions of* 3-*manifolds that yield* 3-*manifolds*, Memoir 107, Amer. Math. Soc. (1971).

[BS] R. H. Bing and M. Starbird, *Linear isotopies in* E^2, Trans. Amer. Math. Soc. 237 (1978), 205–222.

[B1] E. D. Bloch, *Simplexwise linear near-embeddings of a* 2-*disk into* R^2, Trans. Amer. Math. Soc. 288 (1985), 701–722.

[B2] — *Strictly convex simplexwise linear near-embeddings of a* 2-*disk*, Trans. Amer. Math. Soc. 288 (1985), 723–737.

[B3] — *Simplexwise linear near-embeddings of a* 2-*disk into* R^2, II, Topology Appl. 25 (1987), 93–101.

[BCH] E. D. Bloch, R. Connelly and D. W. Henderson, *The space of simplexwise linear homeomorphisms of a convex* 2-*disk*, Topology 23 (1984), 161–175.

[C] M. Cohen, *Homeomorphisms between homotopy manifolds and their resolutions*, Invent. Math. 10 (1970), 239–250.

[CHHS] R. Connelly, D. W. Henderson, C.-W. Ho and M. Starbird, *On the problems related to linear homeomorphisms, embeddings, and isotopies*, in *Continua, Decompositions, Manifolds*, Univ. of Texas Press. Austin 1983.

[D] A. Dold, *Lectures on Algebraic Topology*, Spring-Verlag, Berlin 1972.

[G] B. Grünbaum, *Convex Polytopes*, Wiley, New York 1967.

[L] H. Levine, *Homotopic curves on surfaces*, Proc. Amer. Math. Soc. 14 (1963), 986–990.

[MKS] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, 2nd ed., Dover, New York 1976.

[Q] F. Quinn, *Ends of maps III: dimensions 4 and 5*, J. Differ. Geom. 17 (1982), 503–521.

[S] L. Siebenmann, *Approximating cellular maps by homeomorphisms*, Topology 11 (1972), 271–294.

BARD
A COLLEGE OF THE LIBERAL ARTS AND SCIENCES
Annandale-on-Hudson
New York 12504

Received 29 June 1987

Subparacompactness in locally nice spaces

by

Nobuyuki Kemoto (Okinawa)

Dedicated to Kiyoshi-Iséki on the 70-th birthday

Abstract. It is known from P. Daniels [Da] that normal locally compact zero-dimensional metacompact spaces are subparacompact. It is also known an example of locally compact metacompact space which is not subparacompact, see [Bu, 4.2]. In this paper, we shall give some characterizations of subparacompactness in locally Lindelöf (or locally ω_1-spread) spaces. As a corollary, it will be shown that locally ω_1-spread (or normal locally Lindelöf) submetacompact spaces are subparacompact.

1. Introduction. In this paper, all spaces are assumed to be regular T_1. P. Daniels proved that normal locally compact zero-dimensional metacompact spaces are subparacompact, see [Da]. It is also known an example of locally compact metacompact space which is not subparacompact, see [Bu, 4.2]. In this paper, we shall characterize subparacompactness of locally Lindelöf (or locally ω_1-spread) spaces. As a corollary, it will be shown that locally ω_1-spread (or normal locally Lindelöf) submetacompact spaces are subparacompact.

In the rest of this section, we remind some basic definitions and introduce some notations.

For a regular uncountable cardinal $\varkappa$, a subset of $\varkappa$ is said to be closed unbounded (abbreviated as *cub*) if it is closed and unbounded in its order topology, and a subset of $\varkappa$ is said to be *stationary* if it intersects with every cub set of $\varkappa$. For a set X and cardinal $\varkappa$, we shall use the next notations, $[X]^{\varkappa} = \{Y \subset X: |Y| = \varkappa\}$, $[X]^{\leqslant\varkappa} = \{Y \subset X: |Y| \leqslant \varkappa\}$, and similarly $[X]^{<\varkappa}$. For a collection $\mathscr{C}$ of subsets of a set X and $x \in X$, $(\mathscr{C})_x$ denotes the collection $\{C \in \mathscr{C}: x \in C\}$.

For a pairwise disjoint family $\mathscr{F} = \{F_\alpha: \alpha \in A\}$ of subsets of a space, an *expansion* $\mathscr{U} = \{U_\alpha: \alpha \in A\}$ of $\mathscr{F}$ is a family of subsets such that $F_\alpha \subset U_\alpha$ for every $\alpha \in A$ and $U_\alpha \cap F_\beta = 0$ for every $\alpha, \beta \in A$ with $\alpha \neq \beta$. An *open expansion* is an expansion whose elements are open. A (*open*) *separation* is a pairwise disjoint (open) expansion. A subspace Y of a space is *discrete* if there is an open expansion of $\{\{y\}: y \in Y\}$. A disjoint family $\mathscr{F}$ of a space is said to be *separated* if it has an open separation. A subspace Y of a space is said to be separated if $\{\{y\}: y \in Y\}$ has an open separation.

Let $\varkappa$ be a cardinal. A space X is *(strongly) $\varkappa$-collectionwise Hausdorff* (abbreviated as (strongly) $\varkappa$-CWH) if every closed discrete subspace of cardinality $\varkappa$ has an (discrete, respectively) open separation. When $\varkappa$ is a regular uncountable, a space X is said to be *(strongly) stationary $\varkappa$-collectionwise Hausdorff* (abbreviated as (strongly) $\varkappa$-SCWH) if for every stationary set S of $\varkappa$ and closed discrete subspace $\{x_\alpha: \alpha \in S\}$ of distinct points indexed by S, there is a stationary subset S' of $\varkappa$ such that $S' \subset S$ and $\{x_\alpha: \alpha \in S'\}$ has an (discrete, respectively) open separation. A space is *(strongly)* CWH if it is (strongly) $\varkappa$-CWH for every cardinal $\varkappa$. A space is *(strongly)* SCWH if it is (strongly) $\varkappa$-SCWH for every *regular uncountable* cardinal $\varkappa$.

A space is *countable chain condition* (abbreviated as ccc) if there is not pairwise disjoint family of uncountably many non-empty open sets. A space is *ω_1-compact* (*ω_1-spread*) if there is not a closed discrete (discrete) subspace of size ω_1. Then the implications "Lindelöf $\to$ ω_1-compact $\leftarrow$ ω_1-spread $\to$ ccc" hold.

Let P be a topological property. A space is said to be *locally P* if every point has an open neighborhood whose closure has the property P. Note that if a space is locally ω_1-spread, then so is every subspace.

A family $\mathscr{V}$ of subsets of X is called a *weak refinement* of a cover $\mathscr{U}$ of X provided for each V in $\mathscr{V}$, there is an U in $\mathscr{U}$ with $V \subset U$. If in addition $\mathscr{V}$ is also a cover, $\mathscr{V}$ is called a *refinement* of $\mathscr{U}$. A space X is said to be submetaLindelöf (submetacompact) if for every open cover $\mathscr{U}$ of X there is a sequence $\{\mathscr{U}_n: n \in \omega\}$ of open refinements of $\mathscr{U}$ such that for every point x in X, there is an $n \in \omega$ such that $|(\mathscr{U}_n)_x| \leqslant \omega$ ($<\omega$, respectively). Clearly submetacompact spaces are submetaLindelöf.

2. Results. Z. Balogh showed that locally Lindelöf (locally ccc), strongly CWH (CWH respectively), submetaLindelöf spaces are paracompact, see [Ba]. By a similar argument, we can prove the next result.

THEOREM 1. *Locally Lindelöf (locally ccc), strongly* SCWH (SCWH, *respectively*), *submetaLindelöf spaces are strongly paracompact.*

DEFINITION. Let m be a natural number. A family $\mathscr{U}$ of subsets of a space X is said to be *point-m* if $|(\mathscr{U})_x| \leqslant m$ for every x in X.

LEMMA 2. *Let D be a closed discrete subspace of a space X and n be a natural number. If D has a point-n open expansion $\mathscr{U} = \{U_x: x \in D\}$ such that each member of $\mathscr{U}$ is ccc, then D is separated.*

Proof. Assume that D has such an expansion $\mathscr{U}$. We shall show that $\mathscr{U}$ is star-countable (i.e. for every U of $\mathscr{U}$, $\{U' \in \mathscr{U}: U \cap U' \neq 0\}$ is countable). Then by the usual chaining argument, D is separated. Fix a point x in D. For every natural number i with $1 \leqslant i \leqslant n$, put $\mathscr{A}_i = \{\mathscr{V} \in [\mathscr{U}]^i: U_x \in \mathscr{V}, \bigcap \mathscr{V} \neq 0\}$. By downward induction, we shall prove that $\mathscr{A}_2$ is countable, thus $\mathscr{U}$ is star-countable. First, to show that $\mathscr{A}_n$ is countable, take $\mathscr{V}, \mathscr{V}' \in \mathscr{A}_n$ with $\mathscr{V} \neq \mathscr{V}'$. Since $\mathscr{U}$ is point-n, $\bigcap \mathscr{V}$ and $\bigcap \mathscr{V}'$ are disjoint non-empty open subsets of U_x. Thus $\mathscr{A}_n$ is countable, since U_x is ccc. Now assume that $\mathscr{A}_{i+1}$ is countable, where $2 < i+1 \leqslant n$. $\mathscr{A}_i$ can be decomposed into two parts $\mathscr{B}_i$ and $\mathscr{C}_i$:

$$\mathscr{B}_i = \bigcup \{\{\mathscr{V}' \in [\mathscr{V}]^i: U_x \in \mathscr{V}'\}: \mathscr{V} \in \mathscr{A}_{i+1}\},$$

$$\mathscr{C}_i = \{\mathscr{V} \in [\mathscr{U}]^i: U_x \in \mathscr{V}, \bigcap \mathscr{V} \neq 0, \text{ and } \forall V \in \mathscr{U} - \mathscr{V}((\bigcap \mathscr{V}) \cap V = 0)\}.$$

By the inductive assumption, $\mathscr{B}_i$ is countable. As in the first step, $\mathscr{C}_i$ is also countable, since U_x is ccc. Thus $\mathscr{A}_i = \mathscr{B}_i \cup \mathscr{C}_i$ is countable. Therefore $\mathscr{A}_2$ is countable.

The above lemma implies that every closed discrete subspace having a point-n open expansion for some natural number n of locally ccc spaces is separated. Next we shall prove a similar result for normal locally Lindelöf spaces. But to do it, we need more delicate arguments.

LEMMA 3. *Let X be a normal space and $Y = \{x_\alpha: \alpha \in S\}$ be a closed discrete subspace of distinct points, where S is stationary in a regular uncountable cardinal $\varkappa$. Assume Y has an open expansion $\mathscr{U} = \{U_\alpha: \alpha \in S\}$ such that $\mathscr{U}$ covers X, each $\mathrm{cl}\, U_\alpha$ is Lindelöf, and for every $\alpha, \beta \in S$ with $\alpha \neq \beta$, $\{\gamma \in S: U_\alpha \cap U_\beta \cap U_\gamma \neq 0\}$ is countable. Then there is a stationary set $S' \subset S$ in $\varkappa$ such that $\{x_\alpha: \alpha \in S'\}$ is separated.*

Proof. Identify Y with S. For each $\alpha \in S$, define $H_\alpha = X - \bigcup \{U_\beta: \beta \in S, \beta \neq \alpha\}$. Since $\mathscr{U}$ is a cover of X, $\{H_\alpha: \alpha \in S\}$ is a discrete family of Lindelöf closed sets.

CLAIM. *$S' = \{\alpha \in S: \alpha \in \mathrm{cl}(\bigcup \{U_\beta: \beta \in S, \beta < \alpha\})\}$ is not stationary.*

Proof. Assume on the contrary that S' is stationary. First we shall define $g: S' \to \{0, 1\}$ as follows. Let $g(\min S') = 0$, and assume that for all $\beta < \alpha$ with $\beta \in S'$, $g(\beta)$ are already defined, where $\alpha \in S'$. Here $\min S'$ denotes the minimum of S' under the usual order on $\varkappa$. Define $g(\alpha) = 1$ if $\alpha \in \mathrm{cl}(\bigcup \{U_\beta: g(\beta) = 0, \beta < \alpha, \beta \in S'\}$, and $g(\alpha) = 0$ otherwise. Set $S_i = \{\beta \in S': g(\beta) = i\}$ and

$$T_i = \bigcup \{H_\alpha: \alpha \in S_i\}$$

for $i = 0, 1$. Then T_0 and T_1 are disjoint closed sets in the normal space X. Thus we can take disjoint open sets W_0 and W_1 such that $T_i \subset W_i$ for $i = 0, 1$. Assume that S_0 is stationary (the rest case is similar). Let α be in S_0, then

$$\alpha \notin \mathrm{cl}(\bigcup \{U_\beta: g(\beta) = 0, \beta < \alpha, \beta \in S'\}) = \mathrm{cl}(\bigcup \{U_\beta: \beta \in S_0 \cap \alpha\})$$

holds. Since α is in S', we have $\alpha \in \mathrm{cl}(\bigcup \{U_\beta: \beta \in S_1 \cap \alpha\})$. For $\alpha \in S_0$, define $V_\alpha = U_\alpha \cap W_0$. Since each V_α is an open neighborhood of α, there is a $\beta(\alpha)$ in $S_1 \cap \alpha$ such that $V_\alpha \cap U_{\beta(\alpha)} \neq 0$. Noting $\beta(\alpha) < \alpha$ for each $\alpha \in S_0$, and using the pressing down lemma, we can take a stationary set $S_0' \subset S_0$ in $\varkappa$ and a β is S_1 such that $U_\beta \cap V_\alpha \neq 0$ for each α in S_0'. Since $\mathrm{cl}\, U_\beta - W_1$ is Lindelöf and $\mathscr{U}$ is an open cover of X, there is an $F \in [S - \{\beta\}]^{\leqslant\omega}$ such that $\{U_\gamma: \gamma \in F\}$ covers $\mathrm{cl}\, U_\beta - W_1$. For each γ in F, define $F_\gamma = \{\delta \in S \ \ U_\beta \cap U_\gamma \cap U_\delta \neq 0\}$. By the assumption, each F_γ is countable. Next take α in $S_0' - (\{\beta\} \cup (\bigcup_{\gamma \in F} F_\gamma))$, and take x in $V_\alpha \cap U_\beta$. Since $V_\alpha \subset W_0$, $x \in \mathrm{cl}\, U_\beta - W_1$. Hence there is a $\gamma \in F$ such that $x \in U_\gamma$. Thus $x \in U_\beta \cap U_\gamma \cap U_\alpha$.

Since $\alpha \notin F_\gamma$, $U_\beta \cap U_\gamma \cap U_\alpha = 0$ holds. This is a contradiction. The proof of the claim is complete.

Next using the claim, take a cub set C in $\varkappa$ such that $S' \cap C = 0$. Then since for every α in $S \cap C$, which is stationary, α is not in $\mathrm{cl}(\bigcup \{U_\beta \colon \beta \in S, \beta < \alpha\})$, we can take an open set V_α such that $\alpha \in V_\alpha \subset U_\alpha$ and V_α misses U_β's $(\beta \in S, \beta < \alpha)$. Then $\{V_\alpha \colon \alpha \in S \cap C\}$ is an open separation of $S \cap C$. Thus the proof of the lemma is complete.

Lemma 4. *Let n be a natural number, X be a normal space and $T = \{x_\alpha \colon \alpha \in S\}$ be a closed discrete subspace of X of distinct points, where S is stationary in a regular uncountable cardinal $\varkappa$. Assume T has a point-n open expansion $\mathscr{U} = \{U_\alpha \colon \alpha \in S\}$ such that each $\mathrm{cl}\, U_\alpha$ is Lindelöf. Then there is a stationary set $S' \subset S$ in $\varkappa$ such that $\{x_\alpha \colon \alpha \in S'\}$ is separated.*

Proof. We shall use induction on n. If $n = 1$, point-n means "disjoint". Thus the lemma for $n = 1$ is valid. Assume the lemma for $i = 1, \ldots, n$. We shall prove for $n+1$. We identify $\{x_\alpha \colon \alpha \in S\}$ with S: Let $\mathscr{U} = \{U_\alpha \colon \alpha \in S\}$ be a point-$n+1$ open expansion of S such that each $\mathrm{cl}\, U_\alpha$ is Lindelöf. Define $Y = \{x \in X \colon |(\mathscr{U})_x| \leqslant n\}$. Since Y is closed in X, Y is normal. Then $\{Y \cap U_\alpha \colon \alpha \in S\}$ is a point-n open (in Y) expansion of S such that each $\mathrm{cl}_Y(Y \cap U_\alpha)$ is Lindelöf. By the inductive assumption, there is a stationary subset $S_0 \subset S$ in $\varkappa$ such that S_0 is separated in Y. Using the normality of X, for each α in S_0, we can take an open set W_α in X such that $\alpha \in W_\alpha \subset \mathrm{cl}_X W_\alpha \subset U_\alpha$ and $\{\mathrm{cl}_X W_\alpha \cap Y \colon \alpha \in S_0\}$ is pairwise disjoint. Take an open set Z' in X such that $S_0 \subset Z' \subset \mathrm{cl}_X Z' \subset \bigcup \{W_\alpha \colon \alpha \in S_0\}$. Then in $Z = \mathrm{cl}_X Z'$, $\{Z \cap W_\alpha \colon \alpha \in S_0\}$ is an open expansion of S_0 and covers Z, and $\mathrm{cl}_Z(Z \cap W_\alpha)$ is Lindelöf. We shall show that $\{Z \cap W_\alpha \colon \alpha \in S_0\}$ satisfies the properties of the hypothesis of Lemma 3.

Claim. *For each $\alpha, \beta \in S_0$ with $\alpha \neq \beta$, $\{\gamma \in S_0 \colon Z \cap W_\alpha \cap W_\beta \cap W_\gamma \neq 0\}$ is countable.*

Proof. Fix $\alpha, \beta \in S_0$ with $\alpha \neq \beta$. Let $A = \mathrm{cl}_Z(Z \cap W_\alpha) \cap \mathrm{cl}_Z(Z \cap W_\beta)$. Since $\mathrm{cl}_X W_\alpha \cap \mathrm{cl}_X W_\beta \cap Y = 0$, $A \cap Y = 0$ holds. Put

$$F = \{S' \in [S - \{\alpha, \beta\}]^{n-1} \colon A \cap \bigcap \{U_\gamma \colon \gamma \in S'\} \neq 0\}.$$

And for S' in F, define $V_{S'} = \bigcap \{U_\gamma \colon \gamma \in S'\}$. Then it is not hard to show that $\{V_{S'} \colon S' \in F\}$ is a disjoint open cover of A. Since A is Lindelöf, F is countable. Then for every γ in $S_0 - \bigcup F$, $Z \cap W_\alpha \cap W_\beta \cap W_\gamma = 0$. Because, if there is a point x in $Z \cap W_\alpha \cap W_\beta \cap W_\gamma$, then x is not in Y. Thus there is an S' in $[S - \{\alpha, \beta\}]^{n-1}$ with $\gamma \in S'$ such that $x \in A \cap V_{S'}$. Therefore $S' \in F$ and thus $\gamma \in \bigcup F$. This is a contradiction. Thus the proof of the claim is complete.

Now we can apply Lemma 3 to take a stationary subset $S_1 \subset S_0$ in $\varkappa$ such that S_1 is separated in $Z = \mathrm{cl} Z'$. Since $S_1 \subset Z'$ and Z' is open in X, actually S_1 is separated in X. Thus the proof is complete.

Now we are prepared to give characterizations of subparacompactness in locally nice spaces.

Theorem 5. *Let X be a locally Lindelöf space. Then the following assertions are equivalent.*

(1) *X is the countable closed sum of (strongly) paracompact subspaces (i.e. $X = \bigcup_{n \in \omega} X_n$, where each X_n is closed in X and (strongly) paracompact).*

(2) *X is the countable closed sum of normal metacompact subspaces.*

(3) *X is the countable closed sum of normal submetacompact subspaces.*

(4) *X is subparacompact.*

Proof. (1)→(2) and (2)→(3) are evident. To prove (3)→(4), let X be the union of X_n's $(n \in \omega)$, where each X_n is normal submetacompact and closed in X. Since local Lindelöfness is closed hereditary, each X_n is normal submetacompact and locally Lindelöf. We shall show that each X_n is the countable closed sum of paracompact subspaces, since the countable closed sum of subparacompact subspaces is subparacompact. From now on, we proceed in X_n. By submetacompactness and local Lindelöfness, there exists a sequence $\{\mathscr{U}_m \colon m \in \omega\}$ of open covers of X_n such that each member of $\mathscr{U} = \bigcup_{m \in \omega} \mathscr{U}_m$ has the Lindelöf closure in X_n, and such that for each x in X_n, there is an $m(x)$ in ω with $|(\mathscr{U}_{m(x)})_x| < \omega$. For every m, k in ω, define $X_{nmk} = \{x \in X_n \colon |(\mathscr{U}_m)_x| \leqslant k\}$, which is closed in X_n (hence in X). Thus each X_{nmk} is normal, submetacompact and locally Lindelöf. Thus if we can show that each X_{nmk} is SCWH, then we know that each X_{nmk} is strongly paracompact by Theorem 1. Assume that $\{x_\alpha \colon \alpha \in S\}$ is a closed discrete subspace of X_{nmk} of distinct points, where S is a stationary subset in a regular uncountable cardinal $\varkappa$. Identify $\{x_\alpha \colon \alpha \in S\}$ with S. Since each member U of $\mathscr{U}_m$ has the Lindelöf closure, $S \cap U$ is countable. Furthermore since $S \subset X_{nmk}$, for every α in S, $|(\mathscr{U}_m)_\alpha| \leqslant k$. Define an equivalence relation $\simeq$ on S by $\alpha \simeq \alpha'$ iff there is finite elements $U_0, \ldots, U_j$ of $\mathscr{U}_m$ such that $\alpha \in U_0$, $\alpha' \in U_j$, and $U_i \cap U_{i+1} \cap S \neq 0$. Let $S/\simeq$ be the quotient of S by $\simeq$. It is easy to show that the cardinality of each equivalence class is at most countable. Thus $S_0 = \{\min E \colon E \in S/\simeq\}$ is stationary in $\varkappa$ and $S_0 \subset S$. For each $\alpha \in S_0$, choose a $V_\alpha \in (\mathscr{U}_m)_\alpha$. Then $\{V_\alpha \cap X_{nmk} \colon \alpha \in S_0\}$ is a point-k open expansion of S_0 in X_{nmk}, and each $V_\alpha \cap X_{nmk}$ has the Lindelöf closure in X_{nmk}. Thus by Lemma 4, there is a stationary set $S_1 \subset S_0$ in $\varkappa$ such that S_1 is separated in X_{nmk}. Hence X_{nmk} is SCWH. This completes the proof (3)→(4).

To prove (4)→(1), assume that X is subparacompact and locally Lindelöf. By subparacompactness and local Lindelöfness, take a σ-discrete cover $\bigcup \{\mathscr{F}_n \colon n \in \omega\}$ by Lindelöf closed sets (i.e. each $\mathscr{F}_n$ is a discrete family of Lindelöf closed sets). Since each $X_n = \bigcup \mathscr{F}_n$ is the free union of Lindelöf closed sets, each X_n is strongly paracompact. Furthermore each X_n is closed, since $\mathscr{F}_n$ is discrete. Thus X is the countable closed sum of (strongly) paracompact subspaces.

Remark. Since ω_1-compact submetaLindelöf spaces are Lindelöf ([Au]), the above equivalences hold for locally ω_1-compact spaces.

Noting that the countable closed sum of submetacompact spaces are submetacompact, we can prove the next result by a similar argument.

THEOREM 6. *Let X be a locally ω_1-spread space. Then the following assertions are equivalent.*

(1) *X is the countable closed sum of (strongly) paracompact subspaces.*
(2) *X is the countable closed sum of metacompact subspaces.*
(3) *X is submetacompact.*
(4) *X is subparacompact.*

The equivalence (3) $\leftrightarrow$ (4) of Theorem 3 or 4 implies the following corollary.

COROLLARY 7. *Locally ω_1-spread (or normal, locally Lindelöf) submetacompact spaces are subparacompact.*

Remark. The Example 4.2 of [Bu] is locally compact 2-boundedly metacompact (for definition, see below), but neither subparacompact nor locally ω_1-spread. The example (ii) of 4.9 of [Bu] is normal metacompact but not subparacompact, hence not locally Lindelöf.

In the rest of this paper, we shall look at paracompactness of locally nice spaces. It is known that normal, locally compact, boundedly metacompact (or normal, locally Lindelöf, screenable) spaces are paracompact, see [Da] ([Ba], respectively).

DEFINITION. Let m be a natural number.

(1) A space is *m-boundedly metacompact* if every open cover has a point-m open refinement.

(2) A space is *boundedly metacompact* if every open cover has a point-m open refinement for some m in ω.

(3) A space is *σ-boundedly metacompact* if for every open cover $\mathscr{U}$, there are a sequence $\{\mathscr{U}_n\colon n\in\omega\}$ of weak open refinements of $\mathscr{U}$ and a sequence $\{m(n)\colon n\in\omega\}$ of natural numbers such that each $\mathscr{U}_n$ is point-$m(n)$ and $\bigcup\{\mathscr{U}_n\colon n\in\omega\}$ covers X.

Note that bounded metacompactness (or screenability) implies σ-bounded metacompactness and also that σ-bounded metacompactness implies (sub)metaLindelöfness.

THEOREM 8. *Locally ccc, σ-boundedly metacompact spaces are* SCWH (*thus strongly paracompact by Theorem* 1).

Proof. Let X be a locally ccc, σ-boundedly metacompact space and $\{x_\alpha\colon \alpha\in S\}$ be a closed discrete subspace of distinct points indexed by S, where S is stationary in some regular uncountable cardinal $\varkappa$. Identify $\{x_\alpha\colon \alpha\in S\}$ with S. Let $\mathscr{U}=\{U_\alpha\colon \alpha\in S\}$ be a ccc open expansion (i.e. each U_α is ccc) of S. Since X is σ-boundedly metacompact, take $\{\mathscr{U}_n\colon n\in\omega\}$ of weak open refinements of $\mathscr{U}\cup\{X-S\}$ and $\{m(n)\colon n\in\omega\}$ of natural numbers such that each $\mathscr{U}_n$ is point-$m(n)$ and $\bigcup\{\mathscr{U}_n\colon n\in\omega\}$ covers X. Put $S_n = S\cap(\bigcup\mathscr{U}_n)$ for each n in ω. Then there is an n in ω such that S_n is stationary. For each α in S_n, take V_α in $\mathscr{U}_n$ such that $\alpha\in V_\alpha$. Then $\mathscr{V}_n=\{V_\alpha\colon \alpha\in S_n\}$ is a point-$m(n)$ open expansion of S_n and each V_α is ccc. Thus S_n is separated by Lemma 2.

In a similar way, we can prove:

THEOREM 9. *Normal, locally Lindelöf, σ-boundedly metacompact spaces are* SCWH (*thus strongly paracompact*).

Remark. Using Theorem 8, 9 and the Dowker Theorem (in the sense of [En, 7.2.3]), we can prove that a locally ccc (or normal, locally Lindelöf) space X is paracompact and $\dim X\leqslant n-1$ if and only if X is n-boundedly metacompact.

Remark. S. Watson proved that it is consistent that there is a locally compact perfectly normal metaLindelöf space which is not paracompact, see [Wa]. Thus we can not replace σ-bounded metacompactness by metaLindelöfness in the above theorems. Here note that perfectly normal locally compact spaces are locally ccc.

References

[Au] C. E. Aull, *A generalization of a theorem of Aquaro*, Bull. Austral. Math. Soc. 9 (1973) 105–108.

[Ba] Z. Balogh, *Paracompactness in locally Lindelöf spaces*, Canadian J. Math. 38 (3) (1986) 719–727.

[Bu] D. K. Burke, *Covering properties*, *Handbook of Set Theoretic Topology*, North-Holland, Amsterdam (1984) 347–422.

[Da] P. Daniels, *Normal locally compact boundedly metacompact spaces are paracompact*, Canadian J. Math. 35 (5) (1983) 807–823.

[En] R. Engelking, *General topology*, Polish Scientific Publishers, Warszawa 1977.

[Wa] S. Watson, *Locally compact normal metaLindelöf spaces may not be paracompact: an application of uniformization and Suslin lines*, Proc. Amer. Math. Soc. 98 (4) (1986) 676–680.

DEPARTMENT OF MATHEMATICS
COLLEGE OF EDUCATION
RYUKYU UNIVERSITY
Nishihara-Cho, Okinawa
Japan

Received 27 July 1987;
in revised form 16 November 1987