Simplexwise linear and piecewise linear near self-homeomorphisms of surfaces

by

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Abstract. Let $K \subset \boldsymbol{R}^2$ be a triangulated 2-disk; a map $f\colon K \to \boldsymbol{R}^2$ is called *simplexwise linear* (SL) if $f|\sigma$ is an affine linear map for every (closed) simplex σ in K. Let $L(K) = \{$SL homeomorphisms $K \to K$ fixing ∂K pointwise$\}$, and let $\overline{L(K)}$ denote its closure in the space of all SL maps $K \to \boldsymbol{R}^2$. Some criteria are given for determining when an SL map $K \to K$ fixing ∂K pointwise is in $\overline{L(K)}$, strengthening previous results. Similar criteria are given for determining when a PL map of a compact, oriented surface to itself is a near-homeomorphism.

1. Introduction. Let $K \subset \boldsymbol{R}^2$ be a triangulated 2-disk. We will use K to denote both the simplicial complex and its underlying topological space. A map $f\colon K \to \boldsymbol{R}^2$ is called *simplexwise linear* (abbreviated SL) if $f|\sigma$ is an affine linear map for every (closed) simplex σ in K. Note that an SL map is determined by what it does to the vertices. Let $L(K) = \{$SL homeomorphisms $K \to K$ fixing ∂K pointwise$\}$. If K has k interior vertices, then $L(K)$ can be identified with an open subset of $\boldsymbol{R}^{2k}$; if K is convex, it is proved in [BCH] that $L(K)$ is in fact homeomorphic to an open ball. In the proof of this fact, as well as in subsequent work (see [B1], [B2], [B3]), it became necessary to use the closure $\overline{L(K)}$ of $L(K)$, and to characterize those SL maps $K \to K$ fixing ∂K which are in $L(K)$ and $\overline{L(K)}$. Our main result is Theorem 1.2, stated below, which gives some characterizations of elements in $\overline{L(K)}$ (strengthening Theorem 1.2 of [B1]). Because Theorem 1.2 has an analog for PL maps of compact, oriented surfaces, we first state and sketch the proof of this analog, both as motivation, and because part of its proof is used in the proof of Theorem 1.2; also, it does not appear to be in print, in spite of its straightforward proof.

First some conventions and a definition. Throughout this paper all manifolds will be PL, compact and oriented, all subsets of manifolds will be PL, and all maps between manifolds will be PL and orientation preserving.

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* Partially supported by NSF Contract DMS-8503388.

1980 Mathematics Subject Classification. Primary 57N12; Secondary 57Q35, 57N35. Key words and phrases. Simplexwise linear, near-homeomorphism, surface, piecewise linear.

DEFINITION. Let $f\colon M^n \to N^n$ be a map of n-manifolds, let $D \subset M^n$ be an n-ball, and let $x \in \operatorname{int} D$ be any point. We define $s(f, x, D)$ to be either an integer or ∞ as follows. If $f(x) \in f(\partial D)$, then let $s(f, x, D) = \infty$. If $f(x) \notin f(\partial D)$, but $f(x) \in \partial N$, then let $s(f, x, D) = 0$. If $f(x) \notin f(\partial D) \cup \partial N$, choose an n-ball $Q \subset N^n$ such that $f(x) \in \operatorname{int} Q$ and $Q \cap f(\partial D) = \emptyset$. Let $\Sigma_1 = D/\partial D$ and $\Sigma_2 = Q/\partial Q$ be the n-spheres obtained by collapsing the indicated boundaries to points; clearly f induces a continuous map $f^*\colon \Sigma_1 \to \Sigma_2$. Finally, let $s(f, x, D) = \deg f^*$. It can be seen from standard results about the degree of a map that the definition of $s(f, x, D)$ is independent of the choice of n-ball Q. We say that the map f has the *simple surrounding property* (abbreviated SSP) if for all n-balls $D \subset M^n$, and all $x \in \operatorname{Int} D$, $s(f, x, D) = 1$ or ∞; if $f(x) \in \partial N$, it is further required that $s(f, x, D) = \infty$. Similarly, f has $-$SSP if for all n-balls $D \subset M^n$, and all $x \in \operatorname{Int} D$, $s(f, x, D) = -1$ or ∞, and the added requirement whenever $f(x) \in \partial N$.

It is well known that various conditions on a map $f\colon M^n \to N^n$ of manifolds will imply that f is a near-homeomorphism (i.e. it is the the limit of homeomorphisms); the usual condition is that the sets $f^{-1}(x)$ are well behaved. (See [C], [S], [A] or [Q]). In particular, for PL maps of surfaces the condition is that the sets $f^{-1}(x)$ are 1-connected. However, if one starts with a self-map of a manifold, then it is conceivable that weaker conditions on the map suffice to imply that the map is a near-homeomorphism. The following theorem indicates that this is indeed the case for PL self-maps of compact, orientable surfaces.

THEOREM 1.1. *Let M be a compact, orientable surface, and let $f\colon M \to M$ be a* PL *map, with $f(\partial M) \subset \partial M$. The following are equivalent*:

(1) *f is a near-homeomorphism*;
(2) *f is surjective and has* SSP *or* -SSP;
(3) *f is surjective and $f^{-1}f(x)$ is connected for all $x \in M$.*

Remark. The example in § 2 of this paper shows that the surjectivity condition in (2) of the theorem is necessary.

Our main result is the following theorem, an analog of Theorem 1.1 for SL maps of 2-disks. The problem, as always when dealing with SL maps, is going from being approximated by topological or PL homeomorphisms, to being approximated by SL homeomorphisms. The main tool for doing this is Theorem 1.2 of [B1].

THEOREM 1.2. *Let $K \subset \mathbf{R}^2$ be a strictly convex triangulated 2-disk, and let $f\colon K \to K$ be an* SL *map fixing ∂K pointwise. The following are equivalent.*

(1) $f \in \overline{L(K)}$;
(2) *f has* SSP;
(3) *$s(f, v, D) = 1$ or ∞ for all disks $D \subset K$ such that $\partial D \cap K^0 = \emptyset$, and all vertices $v \in \operatorname{int} D$*;
(4) *$f^{-1}f(v)$ is connected for all interior vertices v of K.*

We then obtain

COROLLARY 1.3. *Let $K \subset \mathbf{R}^2$ be a strictly convex triangulated 2-disk, and let $f\colon K \to K$ be an* SL *map fixing ∂K pointwise. Then $f \in L(K)$ iff $f^{-1}f(v) = \{v\}$ for all interior vertices v of K.*

Remarks. (1) Both Theorem 1.2 and Corollary 1.3 are false if one considers SL maps $f\colon K \to \mathbf{R}^2$ fixing ∂K pointwise. Consider for example K having one interior vertex v, and $f(v)$ is outside of K.

(2) It is not clear whether Condition (3) of Theorem 1.2 can be replaced by the condition: $s(f, v, L) = 1$ or ∞ for all disks $L \subset K$ such that L is a subcomplex of K, and all vertices $v \in \operatorname{Int} L$.

(3) Both Theorem 1.2 and Corollary 1.3 are false in all dimensions higher than 2. In dimension 3 a counterexample is constructed as follows. Let $\Delta \subset \mathbf{R}^3$ be a 3-simplex with vertices $\{a, b, p, q\}$ such that the line segments ab and pq are perpendicular, both parallel to the yz coordinate plain, and with a, b having negative x-coordinates, and p, q having positive x-coordinates. Let $K \subset \mathbf{R}^3$ be a strictly convex triangulated 3-ball containing Δ in its interior, and no other interior vertices. Let $f\colon K \to K$ be the SL map fixing ∂K pointwise, which projects Δ orthogonally into the yz plane. Then $f^{-1}f(v) = \{v\}$ for all interior vertices of K^3, and yet f is not a homeomorphism; if K is chosen carefully, f will be a near homeomorphism, with only Δ collapsed. Clearly there are similar examples in all dimensions greater than 3.

We do not give any information here about the interest in SL maps, and known results, since such information can be found in [BCH], [CHHS], [BS], [B1], [B2], [B3]. This paper is organized as follows: § 2 has preliminaries about SSP, § 3 has the proof of Theorem 1.1, § 4 has some lemmas about SL maps, and § 5 has the proofs of Theorem 1.2 and Corollary 1.3.

2. Simple surrounding property. This section dicusses some basic properties of SSP. Throughout this section, assume that M and N are compact, orientable PL n-manifolds, and all maps $M \to N$ are PL, with $f(\partial M) \subset \partial N$.

LEMMA 2.1. *Let M, N and maps $f, g\colon M \to N$ be as above. Then*

(i) *let $D \subset M$ be an n-ball, and $x \in \operatorname{Int} D$ be any point. If $f(x) \notin f(\partial D) \cup \partial N$ then $s(f, x, D) = \deg_{f(x)}[f/U]$, for any sufficiently small neighborhood U of $f^{-1}f(x) \cap \operatorname{int} D$, (where the local degree of $f|U$ over $f(x)$ is as defined in* [D] VIII § 4).

(ii) *Suppose $x \in M$ is such that $f(x) \notin \partial N$, and $f^{-1}f(x)$ has two distinct components K_1 and K_2. Suppose further that there exist disjoint n-balls D_1 and D_2 containing K_1 and K_2 in their respective interiors, and an n-ball D containing $D_1 \cup D_2$, and with $D \cap f^{-1}f(x) = K_1 \cup K_2$. If $x_i \in K_i$ for $i = 1, 2$, then $s(f, x_1, D) = s(f, x_2, D) = s(f, x_1, D_1) + s(f, x_2, D_2)$. (Note that none of these numbers is ∞.)*

Proof. (i) $f(x) \notin f(\partial D)$ implies that $f^{-1}f(x) \cap \operatorname{int} D$ is compact. $f(x) \notin \partial N$ implies that for any sufficiently small neighborhood U of $f^{-1}f(x) \cap \operatorname{int} D$,

$f(U) \cap \partial N = \emptyset$; therefore $f|U$ is a map into $\operatorname{Int} N$, and so local degree (as in [D]) is well defined. The proof is now straightforward.

(ii) This follows from (i), and Theorem VIII 4.7 of [D]. ■

LEMMA 2.2. *Let $f\colon M \to N$ have* SSP *or* -SSP. *Then* $\deg f = \pm 1$ *iff f is surjective.*

Proof. This is straightforward, using the fact that f is PL, and the relation between local and global degrees (as discussed in [D, VIII § 4]). ■

LEMMA 2.3. *Let $f\colon M \to M$ be as above, and suppose $\partial M = \emptyset$. If f is a near-homeomorphism, then f has* SSP *or* -SSP.

Proof. Suppose $\{f_m\}$ is a sequence of homeomorphisms of M converging to f. Let x and D be such that $s(f, x, D) \neq \infty$. By compactness, there is an n-ball $Q \subset M$ such that $f(x) \in \operatorname{int} Q$, $Q \cap f(\partial D) = \emptyset$, and $Q \cap f_m(\partial D) = \emptyset$ for all sufficiently large m. Let Σ_1 and Σ_2 be as in the definition of $s(f, x, D)$. It is easy to see that the induced maps $\{f_m^*\colon \Sigma_1 \to \Sigma_2\}$ converge to $f^*\colon \Sigma_1 \to \Sigma_2$. Since $\deg f_m^* = 1$ for all m, or $\deg f_m^* = -1$ for all m, it follows that $\deg f^* = 1$ or -1. The result now follows easily. ■

Remark. The hypothesis $\partial M = \emptyset$ in the above is not necessary, but is used to simplify the proof.

An example which shows the necessity of assuming surjectivity in parts (2) of Theorem 1.1 is as follows. For any $n \geq 2$, let $M = S^{n-1} \times S^1$, and let $f\colon S^{n-1} \times S^1 \to S^{n-1} \times S^1$ be given by $f(S^{n-1} \times \{y\}) = (*, y)$, for some point $* \in S^{n-1}$. Suppose $D \subset S^{n-1} \times S^1$ is an n-ball, and $(x, y) \in \operatorname{Int} D$. D cannot contain all of $S^{n-1} \times \{y\}$, since $S^{n-1} \times \{y\}$ cannot be contracted to a point in $S^{n-1} \times S^1$. Therefore $S^{n-1} \times \{y\} \cap \partial D \neq \emptyset$, and so $f(x, y) \in f(\partial D)$. It follows that $s(f, (x, y), D) = \infty$ for all (x, y) and D, so that f has SSP and -SSP as desired. On the other hand, f is not a near-homeomorphism, having degree 0.

QUESTION. Could condition (2) of Theorem 1.1 be replaced by the condition: f has SSP or -SSP but not both (i.e. $s(f, xD) = \pm 1$ for some x and D)?

3. Proof of Theorem 1.1

LEMMA 3.1. *Let M be a closed, orientable surface, and let $f\colon M \to M$ be a* PL *map, with* $\deg f = \pm 1$.

(i) *If $X \in M$ is such that $f^{-1}f(x)$ has a non-simply connected component K, then there is a disk $D \subset M$ such that $\partial D \subset K$, but ∂D does not bound a disk in K.*

(ii) *If $x \in M$ is such that $f^{-1}f(x)$ has distinct components $K_1, \ldots, K_n$, then there are closed disks $D_1, \ldots, D_n$ in M such that $K_i \subset \operatorname{int} D_i$ and $\partial D_i \cap \partial D_j = \emptyset$ for all i, j.*

Proof. (i) f is PL, and hence K is a subcomplex of some triangulation of M. It is therefore seen that K contains a simple closed curve C which is not null-homotopic in K. By a covering space argument, $\deg f = \pm 1$ implies that $f_*\colon \pi_1(M) \to \pi_1(M)$ is surjective; the Hopfian property of surface groups then implies that f_* is injective (see [MKS] § 6.5). Since $f(K)$ is a point, it follows that the map $\pi_1(K) \to \pi_1(M)$ induced by inclusion must be trivial. Thus C is null-homotopic in M. By a theorem of [L], C bounds a disk D in M.

(ii) This is straightforward, using regular neighborhoods of the K_i, and the theorem of [L] referred to above. ■

Proof of Theorem 1.1. There are two cases.

Case 1. $\partial M = \emptyset$.

(1)⇒(2). Surjectivity is clear, and SSP or -SSP follows from Lemma 2.3.

(2)⇒(3). Assume f has SSP; the other case is similar. By Lemma 2.2, $\deg f = \pm 1$. We need to show that $f^{-1}f(x)$ is connected for all x; suppose otherwise for some point $y \in M$. Let K_1 and K_2 be two distinct components of $f^{-1}f(y)$. By Lemma 3.1 (ii) there exist disks $D_1, D_2 \subset M$ such that $K_i \subset \operatorname{int} D_i$ for $i = 1, 2$, and $\partial D_1 \cap \partial D_2 = \emptyset$. First, suppose D_1 and D_2 can be chosen to be disjoint. Then by choosing innermost components of $f^{-1}f(y)$, we may assume that $D_i \cap f^{-1}f(y) = K_i$ for $i = 1, 2$. By connecting D_1 and D_2 with an appropriately chosen thin strip, one obtains a disk D satisfying the hypothesis of Lemma 2.1 (ii). Let $y_1 \in K_i$ be any point for $i = 1, 2$, and so Lemma 2.1 (ii) implies that $s(f, y_1, D_1) = s(f, y_1, D_1) + s(f, y_2, D_2)$. It follows that not all three of $s(f, y_1, D)$, $s(f, y_1, D_1)$, $s(f, y_2, D_2)$ are equal to 1. None of these numbers is ∞, however, and a contradiction to SSP is obtained.

Next, assume that D_1 and D_2 can not be chosen to be disjoint. Two things follow: (1) M is not the 2-sphere S^2, and (2) one of the K_i is not simply connected; (the first fact holds because the complement of a PL disk in S^2 is a PL disk, and the second fact holds by considering regular neighborhoods of the K_i, which would be disks if the K_i were simply connected). Assume K_1 is not simply connected. Let D be a disk as given by Lemma 3.1 (i). Note that $f|D$ factors through maps $D \to D/\partial D \approx S^2 \to M$, since $f(\partial D)$ is a point. Since M is not S^2, $\pi_2(M) = 1$, and so $f|D$ is null homotopic rel ∂D. Choose any point $z \in D \setminus f^{-1}f(y)$; then $\deg_{f(z)}[f|\operatorname{int} D] = 0$. Since $f(z) \notin f(\partial D)$, it follows from the remark after the definition of SSP that $s(f, x, D) = 0$. This contradicts the fact that f has SSP.

(3)⇒(1). It is easy to check that $\deg f = \pm 1$. It is known that f is a PL near-homeomorphism iff $f^{-1}f(x)$ is contractible for all $x \in M$ (see [S]; in the PL case cellular implies contractible). Hence, it suffices to prove that $f^{-1}f(x)$ is simply connected for all $x \in M$. Suppose otherwise, so that there is some $z \in M$ such that $f^{-1}f(z)$ is not simply connected. Some component K of $f^{-1}f(z)$ must be non-simply connected. Let D be as in Lemma 3.1 (i). Suppose M is not S^2. Then as in the proof of (2)⇒(3), $f|D\colon D \to M$ is null homotopic rel ∂D. Therefore f is homotopic to the map $g\colon M \to M$ given by

$$g(x) = \begin{cases} f(x) & \text{if } x \in M \setminus \operatorname{int} D, \\ f(z) & \text{if } x \in D. \end{cases}$$

It follows that deg $g = \deg f = \pm 1$; in particular g is surjective. Consequently, $M \setminus \{f(z)\} \subset g(M \setminus D) = f(M \setminus D)$. It now follows that ∂D separates $f^{-1}f(x)$ for some $x \in \operatorname{int} D$, a contradiction.

Now suppose $M \approx S^2$. If $f|D: D \to M$ is null homotopic rel ∂D, the proof is just like the previous case, so assume otherwise. Consider the disk $B = M \setminus \operatorname{int} D$, (so $\partial B = \partial D$). If $B \subset f^{-1}f(z)$, then by connectivity $B \subset K$, contradicting the hypothesis on D. Hence $B \not\subset f^{-1}f(z)$. If $f|B: B \to M$ is null homotopic rel ∂B, we are once again just as in the previous case, so assume otherwise. It follows that both of $f|D$ and $f|B$ are surjective. Since $\partial D = \partial B$ separates M, this contradicts the assumption that $f^{-1}f(x)$ is connected for all $x \in M$.

Case 2. $\partial M \neq \emptyset$. Let K and L be triangulations of M such that $f: K \to L$ is simplicial. Suppose $\partial K = \bigcup_{i=1,p} C_i$ and $\partial L = \bigcup_{j=1,r} D_j$ where the C_i and D_j are triangulated circles. Let K' and L' be the abstract triangulations $K' = K \cup \bigcup_{i=1,p} (a_i * C_i)$ and $L' = L \cup \bigcup_{j=1,r} (b_j * D_j)$, where $*$ denotes join, and the a_i and b_j are distinct points. Then K' and L' are both triangulations of the surface M' obtained by coning on each of the components of ∂M. Moreover, the extension $f': K' \to L'$ of $f: K \to L$, obtained by defining $f(a_i) = b_j$ iff $f(C_i) \subset D_j$, is a simplicial map representing a PL map $f': M' \to M'$. To reduce this case to the previous one, it remains to note that straightforward arguments show that each of conditions (1), (2) and (3) holds for f iff it holds for f'. ■

4. Simplexwise linear maps. In this section we prove some lemmas concerning SL maps. The analysis is very similar to Sections 2 and 3 of [B1], although the present situation is much simpler; we will use definitions and results from these sections of [B1] without restating them. Throughout the rest of the paper, we assume that $K \subset \mathbf{R}^2$ is a triangulated 2-disk, and all maps are SL. Let K^i denote the set of all (closed) i-simplices of K, and let $\operatorname{int} K^0$ and ∂K^0 denote the interior and boundary vertices of K respectively.

LEMMA 4.1. *Let K be strictly convex, and let $f: K \to K$ be an* SL *map fixing ∂K pointwise. Suppose $f^{-1}f(v)$ is connected for every vertex $v \in K$. Then*

(i) *$f^{-1}f(x)$ is simply connected for all $x \in K$.*

(ii) *Let $D \subset K$ be a disk, and suppose $f(\partial D)$ is a point or a line segment; then $f(D) = f(\partial D)$.*

Proof. (i) Suppose $C \subset K$ is a simple closed curve (not necessarily a subcomplex), such that $f(C)$ is a point. Then any 1-simplex which intersects C in at least two points, or any 2-simplex which intersects C in at least three non-colinear points is also mapped to a point. If the disk bounded by C contains no vertices in its interior, then it follows that the whole disk must be mapped to a point. The result now follows easily.

(ii) $f(\partial D)$ is convex; hence, if all vertices in $\operatorname{int} D$ are mapped into $f(\partial D)$, then $f(D) = f(\partial D)$. Therefore, if we suppose $f(D) \not\subset f(\partial D)$, then $f(v) \notin f(\partial D)$ for some vertex $v \in \operatorname{int} D$. It is easy to see that this implies that $f^{-1}f(v)$ has points both inside and outside of D, a contradiction to the connectivity of $f^{-1}f(v)$. ■

DEFINITION. *Let* $\mathrm{SC}(K) = \{f: K \to K | f$ is SL, fixes ∂K pointwise, and every 2-simplex $\delta \in K$ is either not collapsed or of type SC$\}$. See [B1] p. 704 for definitions.

Throughout the rest of this section, assume $f \in \mathrm{SC}(K)$ is given. Let $\delta \in K^2$ be of type SC. Define $\hat{\Lambda}(\delta)$ as on p. 706 of [B1]. We then obtain the following analog of Lemma 3.1 of [B1].

LEMMA 4.2. *Suppose $f \in \mathrm{SC}(K)$, and $f^{-1}f(v)$ is connected for every vertex $v \in K$. Then*

(i) *Let $x \in K$; if $f^{-1}f(x) \cap K^0 = \emptyset$, then $f^{-1}f(x)$ is the union of arcs and points, with each point in the boundary of two 2-simplices of K which are not collapsed by f, and each endpoint of every arc is in the boundary of one such 2-simplex; if $f^{-1}f(x) \cap K^0 \neq \emptyset$, then $f^{-1}f(x)$ is either a point or a compact, connected simplicial tree (not a subcomplex of K), with each endpoint of the tree in the boundary of a 2-simplex of K which is not collapsed by f (unless the endpoint is in ∂K).*

(ii) *$\tilde{\Lambda}(\delta)$ is a 1-connected subcomplex of K.*

Proof. (i) This is straightforward, using the facts that f is affine linear on each simplex of K, and no 1-simplices are collapsed (since $f \in \mathrm{SC}(K)$).

(ii) First, let η and ν be 2-simplices in $\hat{\Lambda}(\delta)$ such that η rel ν (in the notation of p. 706 of [B1]), so that $f(\eta) \cap f(\nu)$ is a line segment; we claim that there is a finite sequence of two simplices $\eta = \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r = \nu$ in $\hat{\Lambda}(\delta)$ such that $\varepsilon_i \cap \varepsilon_{i+1}$ is a 1-simplex. It is easy to see that there are at least two vertices $v, w \in \eta \cup \nu$ such that $f(v)$, $f(w) \in f(\eta) \cap f(\nu)$, and $f(v) \neq f(w)$. By (i), $f^{-1}f(v)$ and $f^{-1}f(w)$ are disjoint connected trees which intersect both η and ν. Parts of these trees, together with appropriate subsets of $\partial\eta$ and $\partial\nu$, bound a disk D in K. See Figure 1. Clearly $f(\partial D)$ is a line segment contained in $f(\hat{\Lambda}(\delta))$. Lemma 4.1(ii) implies that $f(D) \subset f(\hat{\Lambda}(\delta))$; hence every 2-simplex of K that intersects $\operatorname{int} D$ is in $\hat{\Lambda}(\delta)$. The claim now follows.

To prove $\hat{\Lambda}(\delta)$ is connected, note that by definition, if ϱ, τ are 2-simplices in $\hat{\Lambda}(\delta)$, then there is a finite sequence of two simplices $\varrho = \beta_1, \beta_2, \ldots, \beta_r = \tau$ in $\hat{\Lambda}(\delta)$ such that β_i rel β_{i+1}. It follows from the above claim that there is a path in $\hat{\Lambda}(\delta)$ from any point in β_i to any point in β_{i+1}; thus there is a path in $\hat{\Lambda}(\delta)$ from any point in ϱ to any point in τ.

The simple connectivity of $(\Lambda\delta)$ follows easily from Lemma 4.1(i). ■

It follows from the above lemma that $\hat{\Lambda}(\delta) = \Lambda(\delta)$, in the notation of [B1] § 3. The analog in our situation of Lemma 3.3 of [B1] is

LEMMA 4.3. *Suppose $f \in \mathrm{SC}(K)$, and $f^{-1}f(v)$ is connected for every vertex $v \in K$. Then*

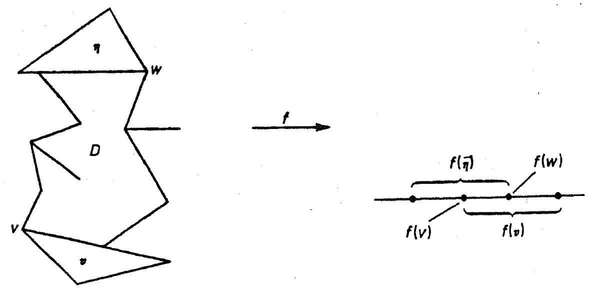

Fig. 1

(i) *All f-segment complexes are simple (in the terminology of* [B1], § 3), *with each E_i a single vertex, and $f|S_i$ injective, for $i = 1, 2$. In particular, f-segment complexes are 2-disks.*

(ii) *If $\delta \in K^2$ is of type* SC, *and $x \in \Lambda(\delta)$, then $f^{-1}f(x) \cap \Lambda(\delta)$ is an arc with one endpoint in each of S_1, S_2, (unless $x = E_1$ or E_2, in which case $f^{-1}f(x) \cap \cap \Lambda(\delta) = \{x\}$).*

Proof. (i) Let $\Lambda(\delta)$ be an f-segment complex. Lemma 4.2(ii) implies that $\Lambda(\delta)$ is a 2-disk. Let e_1, e_2 be the endpoints of $f(\Lambda(\delta))$ (which is a line segment). We can write $\partial\Lambda(\delta)$ as $\partial\Lambda(\delta) = E_1 \cup S_1 \cup \ldots \cup E_n \cup S_n$ for some $n \geqslant 2$, as in the proof of Lemma 3.3 of [B1]. Since $f \in \mathrm{SC}(K)$, each E_i is a single vertex. First, we show $f|S_i$ is injective for each i; suppose otherwise for some S_i. Suppose WLOG that $f(S_i)$ is in the x-axis, with $e_1 < e_2$. Let v be the first vertex of S_i, as S_i is traversed from E_i to E_{i+1}, such that $f|\mathrm{star}(v, S_i)$ is not injective. Let w be the first vertex of S_i after v (still going toward E_{i+1}), such that $f|\mathrm{star}(w, S_i)$ is not injective. Then there exist points $x, y \in S_i$ such that x is between E_i and v, y is between w and E_{i+1}, and $f(y) = f(v)$, $f(x) = f(w)$. See Figure 2. By hypothesis $f^{-1}(v)$ and $f^{-1}f(w)$ are connected; hence, there exists paths A and C in $f^{-1}f(v)$ and $f^{-1}f(w)$, respectively, connecting y to v and x to w, respectively. Because $f \in \mathrm{SC}(K)$, these two paths intersect $\partial\Lambda(\delta)$ in finitely many points each (intersecting each 1-simplex at most once), and they do not intersect S_i between v and w. Also, $A \cap C = \emptyset$. Because of the order in which the points x, v, w, y lie in S_i, it is seen that at least one of A or C is not entirely contained in $\Lambda(\delta)$; assume that C is not entirely contained in $\Lambda(\delta)$. C has a subarc C' such that $C \cap \Lambda(\delta) = \{r, t\}$, for some points $r, s \in \partial\Lambda(\delta)$. C', together with one of the arcs in $\partial\Lambda(\delta)$ from r to t, bounds a disk Q such that $\mathrm{int}\, Q \cap \Lambda(\delta) = \emptyset$; (if $r = t$, consider $\{r\}$ to be a degenerate arc). $f(\partial Q) \subset f(\Lambda(\delta))$ by definition of C. It follows from Lemma 4.1(ii) that $f(Q) \subset f(\Lambda(\delta))$. This implies that any 2-simplices of K intersecting $\mathrm{int}\, Q$ must be in $\Lambda(\delta)$, by the definition of $\partial\Lambda(\delta)$, contradicting the fact that $\mathrm{int}\, Q \cap \Lambda(\delta) = \emptyset$. Hence $f|S_i$ is injective.

It follows from the above that the E_i are mapped alternatively to e_1 and e_2. Since each $f^{-1}f(E_i)$ is connected, $n \leqslant 2$, and the proof is complete.

(ii) This follows easily from Lemma 4.2(i). ■

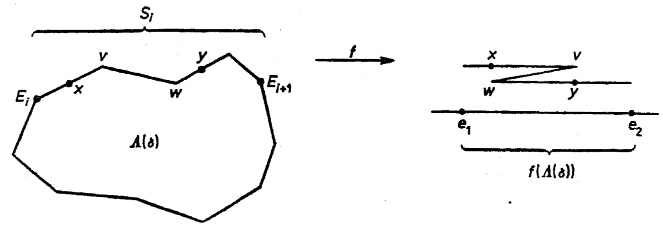

Fig. 2

The next Lemma is proved similarly to Lemma 4.3.

LEMMA 4.4. *Let $D \subset \mathbf{R}^2$ be a triangulated 2-disk such that ∂D has three vertices $\{a, b, c\}$. Let h: $D \to \mathbf{R}$ be an* SL *map such that*

(1) $h(a) = h(b) \neq h(c)$;
(2) $h(D) \subset h(\partial D)$;
(3) *$h^{-1}h(x)$ is simply connected for all $x \in D$;*
(4) *h is injective on every 1-simplex of D other than $\langle a, b\rangle$.*

Then for all $x \in D$, $h^{-1}h(x)$ is an arc with on end in each of $\langle a, c\rangle$ and $\langle b, c\rangle$ (unless $x = c$, in which case $h^{-1}h(c) = \{c\}$).

5. Proof of Theorem 1.2. We start by setting up some notation. As before, assume $K \subset \mathbf{R}^2$ is a tringulated 2-disk.

DEFINITION. Let $R(K)$ be the space of maps as on p. 702 of [B1], with the added requirement that all maps are $K \to K$ fixing ∂K pointwise.

Remark. Theorem 1.2 of [B1] holds with $L(K)$, $\overline{L(K)}$ replacing $E(K)$, $\overline{E(K)}$ respectivelly, and with the present $R(K)$ replacing the $R(K)$ in [B1].

DEFINITION. Let V, V_I, V_B, E, E_I, E_B, F denote the number of vertices, interior vertices, boundary vertices, edges, interior edges, boundary edges, and faces of K. If α denotes an angle of some 2-simplex η of K, and $f|\eta$ is injective, let $f(\alpha)$ denote the signed radian measure of the corresponding angle of $f(\eta)$ (the sign depends on whether $f|\eta$ is orientation preserving or reversing).

Proof of Theorem 1.2. (1)⇒(2). This follows from Lemma 2.3, the coning argument in the last case in the proof of Theorem 1.1, and the fact that $\deg f = 1$.

(2)⇒(3). This is trivial.

(3)$\Rightarrow$(4). By using the coning argument in the last case in the proof of Theorem 1.1, this is proved just like the implication (2)$\Rightarrow$(3) in the proof of Theorem 1.1, noting that the disks D_1, D_2, D can be budged so that their boundaries do not intersect K^0.

(4)$\Rightarrow$(1). Let $c(f) = \# \{\delta \in K^2 | \det(f|\delta) = 0\}$ and

$$r(f) = \# \{\delta \in K^2 | \det(f|\delta) < 0\},$$

where $\det(f|\delta)$ is defined in [B1] p. 702; note that $f \in R(K)$ iff $r(f) = 0$ by the $L(K)$ analog of Lemma 1.1(ii) of [B1]. Using Lemma 4.1(i), it follows from the $L(K)$ analog of Theorem 1.2 (5) of [B1] that to prove our theorem it suffices to prove that $f \in R(K)$ i.e. $r(f) = 0$. This is proved by induction on $c(f)$.

$c(f) = 0$. This part of the proof involves a polyhedral Gauss–Bonnet type argument. If $v \in \operatorname{int} K^0$, then the *curvature* di_v of $f(K)$ at v is $di_v = 2\pi - \Sigma_{v \in \alpha} |f(\alpha)|$; if $v \in \partial K^0$, then the curvature db_v of $f(K)$ at v is $db_v = \pi - \Sigma_{v \in \alpha} |f(\alpha)|$. The *signed curvatures* are $si_v = 2\pi - \Sigma_{v \in \alpha} f(\alpha)$ and $sb_v = \pi - \Sigma_{v \in \alpha} f(\alpha)$ respectively. The *total curvature* of $f(K)$ is $d = \Sigma_{v \in \operatorname{int} K} di_v + \Sigma_{v \in \partial K} db_v$, and the *total signed curvature* is $s = \Sigma_{v \in \operatorname{int} K} si_v + \Sigma_{v \in \partial K} sb_v$. Note that $E_B = V_B$, and $3F = 2E_I + E_B$. The Gauss–Bonnet Theorem is

$$\begin{aligned} d &= \Sigma_{v \in \operatorname{int} K} di_v + \Sigma_{v \in \partial K} db_v \\ &= \Sigma_{v \in \operatorname{int} K}(2\pi - \Sigma_{v \in \alpha} |f(\alpha)|) + \Sigma_{v \in \partial K}(\pi - \Sigma_{v \in \alpha} |f(\alpha)|) \\ &= 2\pi V_I + \pi V_B - \Sigma_{v \in \alpha} |f(\alpha)| \\ &= 2\pi V_I + \pi V_B - \pi F = \pi(2V_I + 2V_B - V_B - 3F + 2F) \\ &= \pi(2V - V_B - 2E_I - E_B + 2F) = \pi(2V - 2E + 2F) = 2\pi. \end{aligned}$$

For any $v \in \operatorname{int} K^0$, let $\deg_v f$ denote the local degree of f at v. Since $c(f) = 0$, it is seen that $\deg_v f = (1/2\pi) \Sigma_{v \in \alpha} f(\alpha)$, and so $si_v = 2\pi - \Sigma_{v \in \alpha} f(\alpha) = 2\pi(1 - \deg_v f)$. By considering exterior angles, it is seen that $\Sigma_{v \in \partial K} sb_v = 2\pi$. Consider the quantity $s - d$. Clearly the only non-zero contributions to $s - d$ come from 2-simplices δ such that $f|\delta$ is orientation reversing; consequently, it is easy to see that $s - d = 2\pi r(f)$. On the other hand, the Gauss–Bonnet Theorem just proved, and the observations just made about degree, imply that

$$\begin{aligned} 2\pi r(f) &= s - d = \Sigma_{v \in \operatorname{int} K} si_v + \Sigma_{v \in \partial K} sb_v - 2\pi \\ &= \Sigma_{v \in \operatorname{int} K} 2\pi(1 - \deg_v f) + 2\pi - 2\pi = 2\pi \Sigma_{v \in \operatorname{int} K}(1 - \deg_v f). \end{aligned}$$

Hence $r(f) = \Sigma_{v \in \operatorname{int} K}(1 - \deg_v f)$. Now, the hypothesis that $c(f) = 0$, together with the fact that $f^{-1}f(v)$ is connected for every $v \in \operatorname{int} K^0$, implies that in fact $f^{-1}f(v) = \{v\}$. Hence, since $\deg f = 1$, it follows that $\deg_v f = 1$ for all $v \in \operatorname{int} K^0$. The equation for $r(f)$ then shows that $r(f) = 0$, as desired.

Inductive step. Suppose that $c(f) > 0$, and that the result holds for all triangulated 2-disks K' and all $g\colon K' \to K'$ with $c(g) < c(f)$. There are now two cases.

Case 1. *There is a* 1-*simplex* $\langle a, b\rangle$ *in* K *such that* $f(\langle a, b\rangle)$ *is a point.* By hypothesis on f, $\langle a, b\rangle$ is not in ∂K. Let c and d be the two vertices of K such that $\langle a, c\rangle$, $\langle b, c\rangle$ and $\langle a, d\rangle$, $\langle b, d\rangle$ are 1-simplices in K (although the triangles with vertices $\{a, b, c\}$ and $\{a, b, d\}$ might not be 2-simplices in K), and every other such vertices are contained in the interiors of the triangles $\{a, b, c\}$ and $\{a, b, d\}$. Moreover, by choosing an innermost $\langle a, b\rangle$, we may assume that f is injective on all 1-simplices inside $\{a, b, c\}$ and $\{a, b, d\}$. We now construct an abstract triangulation K' of a 2-disk by deleting all the vertices in the interiors of the triangles $\{a, b, c\}$ and $\{a, b, d\}$, and then forming the quotient space obtained by identifying vertices a and b, and extending this identification linearly over the triangles $\{a, b, c\}$ and $\{a, b, d\}$. Note that $\partial K'$ is simplicially isomorphic to ∂K in the obvious way. K' can thus be embedded in $\boldsymbol{R}^2$ with $\partial K'$ identified with ∂K, using Theorem 2.2 of [BS]. f induces an SL map $f'\colon K' \to K'$ fixing $\partial K'$.

We want to check that f' satisfies the inductive hypothesis. Clearly $c(f') < c(f)$, so it only needs to be seen that $f^{-1}f(v)$ is connected for all interior vertices v of K'. This is evident, however, since the transition from K to K' at changes the sets $f^{-1}f(v)$ by at most collapsing some arcs or disks to points, (using Lemma 4.4 with $f(c) \neq f(a) = f(b)$ or $f(d) \neq f(a) = f(b)$; that this lemma applies follows from Lemma 4.1). Such collapsing can not change the connectivity of the $f^{-1}f(v)$. By induction $r(f') = 0$. However, Lemma 4.1(ii) implies that $r(f) = r(f')$, and this case is complete.

Case 2. *There is no* 1-*simplex* $\langle a, b\rangle$ *in* K *such that* $f(\langle a, b\rangle)$ *is a point.* In other words, $f \in SC(K)$. Since $c(f) > 0$, there is some $\delta \in K^2$ of type SC. By Lemma 4.3(i), $\Lambda(\delta)$ is a 2-disk. Construct an abstract triangulation K' of a 2-disk by collapsing $\Lambda(\delta)$ to an arc, which is done by identifying each set $f^{-1}f(x) \cap \Lambda(\delta)$ to a point, $x \in \Lambda(\delta)$. By Lemma 4.3(ii) this procedure simply collapses arcs to points. Triangulate K' by giving it all the vertices of $K - \operatorname{int}\Lambda(\delta)$, and adding any 1-simplices as necessary. As in the previous case, K' can be embedded in $\boldsymbol{R}^2$ with $\partial K'$ identified with ∂K, and f induces an SL map $f'\colon K' \to K'$ fixing $\partial K'$ (noting that $f(\Lambda(\delta))$ is a line segment). The rest of the proof for this case is just like the previous one. ■

Proof of Corollary 1.3. (i) If f is a homeomorphism, then trivially $f^{-1}f(v) = \{v\}$. Now assume f is not a homeomorphism; either $f \in \overline{L(K)}$ or it is not. In the latter case, it follows from Theorem 1.2 that there is a interior vertex $w \in K^0$ such that $f^{-1}f(w)$ is not connected; in particular, $f^{-1}f(w) \neq \{w\}$. Now suppose that $f \in \overline{L(K)}$. Then $f \in R(K)$, since $\overline{L(K)} \subset R(K)$ (seen similarly to the remark following Lemma 1.1 of [B1]). Thus $\det(f|\sigma) \geqslant 0$ for all $\sigma \in K^2$ (using the notation of p. 702 in [B1]). Since f is not a homeomorphism, it follows from the $L(K)$ analog of Lemma 1.1(i) of [B1] that $\det(f|\delta) = 0$ for some $\delta \in K^2$. By noting the three generic ways in which f can map δ (in Section 2 of [B1]), it is seen that for at least one of the vertices w of δ, $f^{-1}f(w) \neq \{w\}$. Since K is strictly convex, one such vertex in δ must be in the interior of K. ■

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Received 29 June 1987

Subparacompactness in locally nice spaces

by

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Dedicated to Kiyoshi-Iséki on the 70-th birthday

Abstract. It is known from P. Daniels [Da] that normal locally compact zero-dimensional metacompact spaces are subparacompact. It is also known an example of locally compact metacompact space which is not subparacompact, see [Bu, 4.2]. In this paper, we shall give some characterizations of subparacompactness in locally Lindelöf (or locally ω_1-spread) spaces. As a corollary, it will be shown that locally ω_1-spread (or normal locally Lindelöf) submetacompact spaces are subparacompact.

1. Introduction. In this paper, all spaces are assumed to be regular T_1. P. Daniels proved that normal locally compact zero-dimensional metacompact spaces are subparacompact, see [Da]. It is also known an example of locally compact metacompact space which is not subparacompact, see [Bu, 4.2]. In this paper, we shall characterize subparacompactness of locally Lindelöf (or locally ω_1-spread) spaces. As a corollary, it will be shown that locally ω_1-spread (or normal locally Lindelöf) submetacompact spaces are subparacompact.

In the rest of this section, we remind some basic definitions and introduce some notations.

For a regular uncountable cardinal $\varkappa$, a subset of $\varkappa$ is said to be closed unbounded (abbreviated as *cub*) if it is closed and unbounded in its order topology, and a subset of $\varkappa$ is said to be *stationary* if it intersects with every cub set of $\varkappa$. For a set X and cardinal $\varkappa$, we shall use the next notations, $[X]^{\varkappa} = \{Y \subset X: |Y| = \varkappa\}$, $[X]^{\leqslant\varkappa} = \{Y \subset X: |Y| \leqslant \varkappa\}$, and similarly $[X]^{<\varkappa}$. For a collection $\mathscr{C}$ of subsets of a set X and $x \in X$, $(\mathscr{C})_x$ denotes the collection $\{C \in \mathscr{C}: x \in C\}$.

For a pairwise disjoint family $\mathscr{F} = \{F_\alpha: \alpha \in A\}$ of subsets of a space, an *expansion* $\mathscr{U} = \{U_\alpha: \alpha \in A\}$ of $\mathscr{F}$ is a family of subsets such that $F_\alpha \subset U_\alpha$ for every $\alpha \in A$ and $U_\alpha \cap F_\beta = 0$ for every $\alpha, \beta \in A$ with $\alpha \neq \beta$. An *open expansion* is an expansion whose elements are open. A (*open*) *separation* is a pairwise disjoint (open) expansion. A subspace Y of a space is *discrete* if there is an open expansion of $\{\{y\}: y \in Y\}$. A disjoint family $\mathscr{F}$ of a space is said to be *separated* if it has an open separation. A subspace Y of a space is said to be separated if $\{\{y\}: y \in Y\}$ has an open separation.