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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Inaccessibility, essential maps, and shape theory

by

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Abstract. The accessibility of a point p of a compactum $X\subset E^n$ from a complementary domain $U\subset E^n-X$ can be characterized in terms of the homotopy classes of certain maps. Let π_p be the projection from p of $E^n-\{p\}$ radially onto an S^{n-1} with p as center. Let $q\in U$. Then p is accessible from U iff $\pi_p|X-\{p\}\simeq\pi_q|X-\{p\}$. In particular, if U is the unbounded complementary domain of X, then p is accessible from U iff $\pi_p|X-\{p\}$ is inessential. As an application, suppose X is a cellular plane continuum with an inaccessible point p (for example the pseudo-arc). Then X has trivial shape, but $X-\{p\}$ admits an essential map to S^1.

Nevertheless, $X-\{p\}$ is shape incomparable to S^1 in the weak and strong shape theories of Borsuk and the shape theory of Fox. It also follows that in the strong shape theory of Borsuk and in the shape theory of Fox, $X-\{p\}$ does not have trivial shape.

1. Introduction. We obtain a characterization, in terms of the homotopy classes of certain maps, of the accessibility of a point p of a compactum X in E^n from a complementary domain $U\subset E^n-X$ $(n\geqslant 2)$. Theorem 3.1 essentially says the following: Let U be a complementary domain of X in E^n and let q be a point in U. Let π_p denote the projection from p of $E^n-\{p\}$ radially onto an S^{n-1} with p as center. Then p is accessible from U iff $\pi_p|X-\{p\}\simeq\pi_q|X-\{p\}$. As a special case, Theorem 3.2 says that if U is the unbounded complementary domain of X in E^n, then p is accessible from U iff $\pi_p|X-\{p\}$ is inessential.

David Bellamy has asked us in conversation, the following two questions: (1) Does the pseudo-arc minus a point admit an essential map to S^1? (2) Is there a space with trivial shape which admits an essential map to S^1? An affirmative answer to (1) follows from the above results on accessibility.

It seemed likely that the pseudo-arc minus an (end) point had trivial shape, since it is the intersection of a tower of open 2-balls. However, in Section 5, where we consider several different extensions of shape theory to noncompact metric spaces, we show that the pseudo-arc minus a point does not have the shape of a point in any version we consider.

The answer to the second question may depend upon the version of shape theory one considers; in the version of shape theory due to Fox, that X has trivial shape is equivalent to there being no essential map from X to any ANR.

In Section 4, we determine sufficient conditions for a continuum X minus a point

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p to admit an essential map to S^1, regardless of the embeddability of X in E^2, or the accessibility of point p.

In Section 5, we obtain for each integer $n \geqslant 1$, an example of a cellular continuum X_n in E^{n+1} such that for a certain point $p_n \in X_n$, $X_n - \{p_n\}$ admits an essential map to S^n, has nontrivial shape, and, for $n = 1$, is shape incomparable to any compact, noncontractible ANR, e.g. S^1. We also review the definitions and elementary properties of several versions of shape theory, including Borsuk's strong and weak shape theories and Fox's shape theory, all of which agree on *compact* metric spaces. Of course, $X_n - \{p_n\}$ is *not* compact, which leads to some counter-intuitive results. We confine our attention to metric spaces throughout.

Understanding the construction of the pseudo-arc is *not* required for this paper, however, the interested reader is refered to [B] or [Mo].

We wish to thank David Bellamy, David Wilson, and Juan Toledo for helpful conversations. We also wish to thank the referee for extensive remarks which led to improvements in the paper, in particular, shortening the proof of Theorem 3.1, simplifying the arguments in Sections 5.4–5.4.4, and clarifying the proof of Lemma 5.4.6.

2. Preliminary definitions and theorems. A *compactum* is a compact (subset of a) metric space, and a *continuum* is a connected compactum. A *domain* is a connected open set in E^n (S^n). The double arrow in $f\colon X \twoheadrightarrow Y$ means that f is an onto map. All maps are continuous. By $f \simeq q$ we mean that the map f is homotopic to the map g. By $f \simeq 0$ we mean that the map f is homotopic to a constant map. A map $f\colon X \to S^n$, $n \geqslant 1$, is called *inessential* iff $f \simeq 0$; otherwise f is called *essential.* Unless stated otherwise, assume $n \geqslant 2$.

Let $Y \subset E^n$ (S^n). By $\mathrm{Cl}(Y)$, $\mathrm{Bd}(Y)$, and $\mathrm{Int}(Y)$ we mean the closure, boundary, and interior of Y, respectively, as a subset of E^n (S^n). If $\mathrm{Cl}(Y) \subset E^n$ is compact, we mean by $\mathrm{Ext}(Y)$ the unbounded complementary domain of $\mathrm{Cl}(Y)$ in E^n.

Let X be a compactum in E^n (S^n), and let p be a point in X. We say p is *accessible* iff there is an arc $A \subset E^n$ such that $A \cap X = \{p\}$. Otherwise p is called *inaccessible.* If X separates E^n (S^n), we say that p is *accessible from* a complementary domain U of X in E^n (S^n) provided that $A \subset U \cup \{p\}$. If X separates E^n into exactly two complementary domains, we denote the unbounded domain by $\mathrm{Ext}(X)$ and the bounded domain by $\mathrm{Int}(X)$. Though we use "Ext" and "Int" in two distinct ways, context will make clear which is intended.

DEFINITION OF THE MAP π_p (see p. 97 of [H–W]). Let S^{n-1} be the $(n-1)$-sphere in E^n of radius 1 centered at the origin 0. For each point $p \in E^n$, we define a map $\pi_p\colon E^n - \{p\} \twoheadrightarrow S^{n-1}$ as follows: for each point $x \in E^n - \{p\}$, $\pi_p(x)$ is the projection of the point $\bar{x} - \bar{p}$, in vector terminology, radially from 0 onto S^{n-1}; that is:

$$\pi_p(x) = \frac{\bar{x} - \bar{p}}{\|\bar{x} - \bar{p}\|}.$$

2.1. THEOREM (Theorem VI. 10 in [H–W]). *If X is a compactum in E^n, then points p and q in $E^n - X$ are separated by X iff $\pi_p|X \not\simeq \pi_q|X$.*

2.2. COROLLARY. *If X is a compactum in E^n, and point p lies in the unbounded complementary domain of X in E^n, then $\pi_p|X \simeq 0$.*

Proof. Let B be a closed polyhedral ball in E^n containing X in $\mathrm{Int}(B)$ and let $q \in E^n - B$. Clearly, $\pi_q|B \simeq 0$. Since $X \subset B$, $\pi_q|X \simeq 0$. Since both p and q lie in the same complementary domain of X, $\pi_p|X \simeq 0$. ■

2.3. THEOREM (Theorem IV. 5.1 of [W]). *If X is a compactum in E^n, U is a complementary domain of X, and there is a nondegenerate closed, connected subset $L \subset \mathrm{CL}(U)$ meeting X only in p, then p is accessible from U.*

2.4. LEMMA. *Let X be a compactum and U and V domains such that $X \subset U \subset V \subset E^n$. Suppose $q_0 \in V - \mathrm{Cl}(U)$, $p \in U - X$, and $q_1 \in U - X$, with an arc $A = [q_0, q_1] \subset V - X$. Then X separates q_0 from p in V iff X separates q_1 from p in U.*

2.5. THEOREM. *Let X be a compactum in E^n, U a complementary domain of X, and $\{p\}$ a component of X. Suppose that no closed subset of X missing p separates p from a point of U. Then p is accessible from U.*

Proof. Let $q_0 \in U$. We show that there is an arc A from q_0 to p so that $A - \{p\} \subset U$. Since p is a component of X, there is a separation $X = (X \cap U_1) \cup (X \cap V_1)$ of X by open sets U_1 and V_1 in E^n whose boundaries miss X, with $p \in U_1$ and $\mathrm{diam}(U_1) < \frac{1}{2}$. Since E^n is locally connected, the components of U_1 are open. Hence we may suppose that U_1 is connected, and also that $q_0 \notin U_1$. Let $X_1 = U_1 \cap X$ and $Y_1 = V_1 \cap X$. Note that U_1 is a domain and $\mathrm{diam}(U_1) < \frac{1}{2}$.

Since Y_1 is a closed subset of X, Y_1 does not separate q_0 from p in E^n. Let A_0' be an arc from q_0 to p in $E^n - Y_1$. Give A_0' its natural order with initial point q_0. Observe that A_0' meets $\mathrm{Bd}(U_1)$. Let q_1' be the first point of A_0' in $\mathrm{Bd}(U_1)$. For sufficiently small ε, there is an ε-ball W about q_1' so that $W \cap X = \emptyset$. Let q_1 be a point of $W \cap U_1$ and let A_0'' be an arc from q_1' to q_1 in W. Let A_0 be an arc in $A_0' \cup A_0''$ from q_0 to q_1. Then $A_0 \cap X = \emptyset$.

Now X_1 is a compactum in U_1, $q_1 \in U_1 - X_1$, $\{p\}$ is a component of X_1, so by Lemma 2.4, no closed subset of X_1 separates p from q_1 in U_1. Thus we can apply the above argument with X_1 in place of X, q_1 in place of q_0, and U_1 in place of E^n. We can find a domain $U_2 \subset U_1$ and an open set $V_2 \subset U_1$, with $p \in U_2$, $\mathrm{diam}(U_2) < 1/4$, $V_2 \cap U_2 = \emptyset$, and $X_1 \subset U_2 \cup V_2$. Let $X_2 = X_1 \cap U_2$ and $Y_2 = X_1 \cap V_2$. As above, we can find an arc $A_1 \subset U_1$ from q_1 to a point $q_2 \in U_2$, so that A_1 misses X.

Proceeding in this way, we can construct a sequence of arcs $A_i \subset U_i$ from q_i to q_{i+1} so that A_i misses X, $\mathrm{diam}(A_i) < 1/2^i$, and $p \in \lim A_i$. Then $\bigcup_{i=0}^{\infty} A_i \cup \{p\}$ contains an arc A from q_0 to p so that $A - \{p\} \subset U$. ■

2.6. COROLLARY. *Let U be a domain in E^n, $\mathrm{Bd}(U)$ compact, and $\{p\}$ a component of $\mathrm{Bd}(U)$. Then p is accessible from U.*

Proof. Since U is a component of $E^n - \mathrm{Bd}(U)$, no closed subset $K \subset \mathrm{Bd}(U)$ missing p separates p from some $q \in U$. ■

2.7. THEOREM (a version of the Borsuk Homotopy Extension Theorem; see p. 86 of [H–W]). *Let C be a closed subset of a normal space S. Let $f \simeq g\colon C \to Y$, where Y is an* ANR *(absolute neighborhood retract). Suppose that F is an extension of f to S and that G is an extension of g to S. Then there is a neighborhood U of C in S such that $F|U \simeq G|U$.*

3. Accessibility and homotopic maps.

In this section we characterize, for a point p of a compactum X in E^n, the accessibility of p from a complementary domain U of X. Our main theorem of this section is Theorem 3.1 which asserts that p is accessible from U iff for any point $q \in U$, $\pi_q|X-\{p\} \simeq \pi_p|X-\{p\}$.

We note that accessibility of a point $p \in X$ may depend on the embedding of X in E^n. As a Corollary of the characterization, we answer Question (1) of the Introduction by showing that the pseudo arc minus a (any) point admits an essential map to S^1.

We also construct an example of a nonseparating continuum M in E^3 with an accessible point p, so that $\pi_p|M-\{p\}$ is necessarily inessential, but p is not accessible by a *polygonal* arc.

3.1. THEOREM. *Let X be a compactum in E^n, U a complementary domain of X, q a point in U, and p a point in X. Then p is accessible from U iff $\pi_p|X-\{p\} \simeq \pi_q|X-\{p\}$.*

Proof. Suppose that p is accessible from U. Then there is an arc A from q to p such that $A-\{p\} \subset U$. We parameterize A by a map $f\colon [0,1] \twoheadrightarrow A$ such that $f(0) = q$ and $f(1) = p$. Then we define a homotopy $\{f_t\}_{0 \leqslant t \leqslant 1}$ between $\pi_q|X-\{p\}$ and $\pi_p|X-\{p\}$ by

$$f_t = \pi_{f(t)}|X-\{p\}\,.$$

Conversely, suppose that $\pi_p|X-\{p\} \simeq \pi_q|X-\{p\}$. By Theorem 2.7 applied to the closed subset $X-\{p\}$ of the space $E^n-\{p, q\}$, there is a bounded neighborhood V_0 of $X-\{p\}$ in $E^n-\{p, q\}$ such that $\pi_p|V_0 \simeq \pi_q|V_0$. By way of notation, if Y is a set in $E^n-\{p\}$, let $\overline{Y}$ denote the closure of Y in $E^n-\{p\}$.

Since $q \notin X$, we may apply normality and conclude that there is a neighborhood V of $X-\{p\}$ in $E^n-\{p\}$ such that

(1) $X-\{p\} \subset V \subset \overline{V} \subset V_0$,
(2) $q \in E^n-\mathrm{Cl}(V)$,
(3) $\mathrm{Bd}(V) \cap X \subset \{p\}$, and
(4) $\pi_p|\overline{V} \simeq \pi_q|\overline{V}$.

From (1) and (2) it also follows that there is a complementary domain W of $\mathrm{Cl}(V)$ in E^n such that

(5) $q \in W \subset U \subset E^n-X$, and
(6) $\mathrm{Bd}(W) \subset \mathrm{Bd}(V) \subset \mathrm{Cl}(V) = \overline{V} \cup \{p\}$.

Now $p \notin W$. If $p \notin \mathrm{Bd}(W)$, then p lies in a complementary domain W_0 of the compactum $\mathrm{Bd}(W)$, with W_0 distinct from W. Hence by Theorem 2.1, $\pi_p|\mathrm{Bd}(W) \not\simeq \pi_q|\mathrm{Bd}(W)$. Noting (6), this contradicts (4). Therefore, we have

(7) $p \in \mathrm{Bd}(W)$.

We now have $\mathrm{Cl}(W)$, a closed, connected subset of $\mathrm{Cl}(U)$, meeting x only in p. It follows from Theorem 2.3 that p is accessible from U. ■

3.2. COROLLARY. *Let X be a compactum in E^n, p a point of X, and U the unbounded complementary domain of X. Then p is accessible from U iff $\pi_p|X-\{p\}$ is inessential.*

Proof. Apply Theorem 3.1 and Corollary 2.2. ■

3.3. COROLLARY. *Let p be a point of a compactum X. If there is an embedding of X in E^n in such a way that p is inaccessible from the unbounded complementary domain of X, then there is an essential map of $X-\{p\}$ onto S^{n-1}.*

3.4. Remark. It is by applying Corollary 3.3 that examples of continua-minus a-point can be produced which admit essential maps to a sphere.

For example, let P denote the pseudo-arc ([Mo], [B]) and let p be any point in P. By the indecomposability of P, there are inaccessible points in any planar embedding of P. By the homogeneity of P [B], there is an embedding of P in E^2 with p inaccessible. Hence, $P-\{p\}$ admits an essential map to S^1, answering Question (1) of the Introduction in the affirmative.

The $\sin 1/x$ *continuum* $C = \mathrm{Cl}(\{(x, y) \in E^2 | \sin(1/x) = y \,\&\, x \in (0, 1]\})$ minus any point in the limit segment (the interval from $(0, 1)$ to $(0, -1)$ on the y-axis) admits an essential map to S^1; see the embedding described in Section 5.3.

In [B–M] two of the authors show that there is an embedding of the Knaster U-continuum (bucket handle) in the plane with its (unique) endpoint inaccessible. Hence, the Knaster U-continuum minus its endpoint admits an essential map to S^1.

In [M1] (or [M2]) one of the authors shows that given an indecomposable chainable continuum X and a point p in X, there is an embedding of X in E^2 with p inaccessible. The techniques used to prove that theorem can be extended to prove the following:

3.4.1. THEOREM. *If X is a chainable compactum containing a continuum of convergence X_0, then there is a point $p \in X_0$ and an embedding $e\colon X \to E^2$ such that $e(p)$ is inaccessible. Moreover, p may be taken to be any point in X_0, except for at most two (a pair of opposite endpoints of X_0).*

It follows from Theorem 3.4.1 that *every* chainable continuum *except the arc* admits an essential map to S^1 upon the removal of some point. In Section 4, we show that, for maps to S^1, inaccessibility, and even embeddability in E^2, are not necessary, however.

3.5. COROLLARY. *Let X be a compactum in E^n such that* $\dim(X) \leqslant n-2$. *Then every point of X is accessible in every embedding of X in E^n.*

Proof. Suppose that X lies in E^n so that $p \in X$ is inaccessible from the unbounded complementary domain U of X. Then $\pi_p|X-\{p\}$ is an essential map onto S^{n-1} by Corollary 3.2. But $\dim(X-\{p\}) < n-1$, so $\pi_p|X\{p\}$ must be inessential by Theorem VI. 6 of [H–W], a contradiction. ■

3.6. Remark. Theorem VI. 13 of [H–W] asserts that a compact subset C of E^n separates E^n iff there is an essential map of C onto S^{n-1}. Remark 3.4 and the examples in Section 5.3 show that the hypothesis of compact is required for the "if" part of the theorem.

3.7. Example. In E^3, let S^2 be the unit sphere, and let p be the origin. Let A be a Fox–Artin arc [F–A, p. 983] from some point q in S^2 to p, with p as the "bad" point. We can fatten A into a tapering solid cone T, by fattening less and less as we approach p. Thus T is a closed ball, shaped like a solid cone, knotted in Fox–Artin fashion, with p the apex of T. Let D be the closed disk containing q in S^2 wherein T intersects S^2, and let $\mathrm{Int}(D)$ denote the open disk $D-\mathrm{Bd}(D)$ in S^2. See Figure 1.

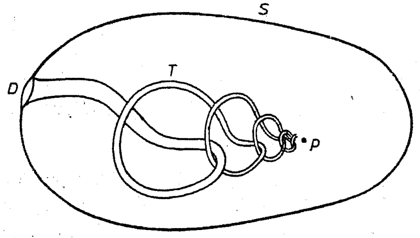

Fig. 1

Let $C = (S^2 \cup \mathrm{Bd}(T)) - \mathrm{Int}(D)$. Then C is homeomorphic to S^2, though not ambiently so, since C contains a wild arc lying in $\mathrm{Bd}(T)$. Note that $C \cup (\mathrm{Ext}(C) \cup \infty)$ is homeomorphic to B^3. However, $C \cup \mathrm{Int}(C) = M$ is *not* homeomorphic to B^3, for if it were, then C would be collared on each side, and would therefore be ambiently homeomorphic to a standard S^2 [Bro].

Now p is accessible from both $\mathrm{Ext}(C)$ and $\mathrm{Int}(C)$; however, p is accessible from $\mathrm{Ext}(C)$ only by a *nonpolygonal* arc following the "Fox–Artin channel" of $\mathrm{Int}(T)$. Nevertheless, both $\pi_p|M-\{p\}$ and $\pi_p|C-\{p\}$ are inessential.

4. Mappings to S^1. One might suspect that in Theorem 3.1 the condition that X be embeddable in E^n is much too strong. At least for the case of mappings to the circle, we are able to eliminate the condition of embeddability in the plane.

4.1. Theorem. *Let X be a compactum which is not locally connected at point $p \in X$. Suppose that U is a closed neighborhood of p in X, $\{K_j\}$ is a sequence of distinct components of U, each K_j isolated in the sequence, $\{U_j\}$ is a sequence of open sets such that $U_j \subset \mathrm{Int}(U)$ and $\lim(U_j) = \{p\}$, and $\{f_j\}$ is a sequence of mappings $f_j\colon K_j \to S^1$ such that $f_j(K_j - U_j) = 1$, and f_j is essential modulo the boundary of U. Then there exists an essential mapping $f\colon X-\{p\} \to S^1$.*

Proof. Extend f_j to a map $g_j\colon (X-U_j) \cup K_j \to S^1$ by

$$g_j(x) = \begin{cases} 1, & \text{if } x \notin U_j, \\ f_j(x), & \text{if } x \in K_j. \end{cases}$$

Since S^1 is an ANR, there exists a neighborhood V_j of $K_j \cup (X-U_j)$ such that g_j extends to a mapping $h_j\colon V_j \to S^1$. Let W_j be an open and closed neighborhood of K_j in U such that $W_j \subset V_j - \{p\}$. We may suppose that $W_j \cap W_k = \varnothing$ for $k \neq j$.

If some f_j is essential, let f be the extension of h_j to $X-\{p\}$ which is constant off W_j. We may suppose, therefore, that each f_j is inessential. Define $f\colon X-\{p\} \to S^1$ by

$$f(x) = \begin{cases} 1, & \text{if } x \notin \bigcup_{j=1}^{\infty} W_j, \\ (h_j(x))^j, & \text{if } x \in W_j. \end{cases}$$

Since the sets $\{W_j\}$ are pairwise disjoint, f is well-defined. Since $\lim(U_j) = \{p\}$ and f is constant on $X-(\bigcup_{j=1}^{\infty} U_j \cup \{p\})$, f is continuous. Note that $(h_j(x))^j$ denotes the jth power of $h_j(x)$ as an element of the multiplicative group S^1.

Since f_j is essential modulo the boundary of U, it follows that if $\varphi_j\colon K_j \to R$ were a lifting of g_j (i.e. $g_j(x) = \exp(2\pi i \varphi_j(x))$, for each $x \in (X-U_j) \cup K_j$ and φ_j is continuous), then $\mathrm{diam}(\varphi_j(K_j \cap \mathrm{Bd}(U))) \geqslant 1$. Then $\psi_j\colon K_j \to R$ defined by $\psi_j(x) = j(\varphi_j(x))$ would be a lifting of $(f_j)^j$, and $\mathrm{diam}(\psi_j(\mathrm{Bd}(U) \cap K_j)) \geqslant j$. Since $\mathrm{Bd}(U)$ is compact, and any lifting $\varphi\colon X-\{p\} \to R$ of f would have $\mathrm{diam}(\varphi(\mathrm{Bd}(U))) \geqslant \mathrm{diam}(\psi_j(\mathrm{Bd}(U) \cap K_j)) \geqslant j$, for each positive integer j, it follows that no such lifting of f exists. Hence, f is essential. ■

4.2. Remark. Theorem 3.1 guarantees that an essential map of $X-\{p\} \subset E^2$ onto S^1 exists provided that p is inaccessible from some complementary domain of X in E^2. The essential map of Theorem 4.1 does not depend upon the embeddability of X in E^2 with p inaccessible, but rather upon the structure of X near p.

Thus, for example, the wedge X of two $\sin 1/x$ continua at a point p of the limit segment has no embedding in E^2 with p inaccessible, but Theorem 4.1 guarantees an essential map of $X-\{p\}$ onto S^1, since it is easy to find the components K_j and maps f_j required in the hypothesis.

Of course, Theorem 4.1 also applies to continua that are not embeddable in E^2 at all. For example, let X be a ray spiralling clockwise to a simple triod with a "sticker" attached to the junction point of the triod. Let p be any point of the triod.

The reader should note that the non-local-connectivity of X at p is a necessary, but not sufficient, condition for satisfying the hypothesis of Theorem 4.1. For example, consider the *comb space* $X = ((\{0\} \cup \{1/n\}_{n=1}^{\infty}) \times [0,1]) \cup ([0,1] \times \{0\})$ in E^2. Let p be the point $(0, 1)$ on the limit segment $\{0\} \times [0,1]$ of X.

5. Connections with shape theory. Using the results of either Section 3 or Section 4, there are one-dimensional nonseparating plane continua, which, upon the removal of a point, admit essential maps to S^1. Examples include the $\sin 1/x$ continuum and the pseudo-arc. In Section 5.3, we construct, for each $n \geqslant 1$, an n-dimensional continuum $X_n \subset E^{n+1}$, which is the intersection of a nested sequence of $(n+1)$-balls, so that for a certain point $p_n \in X_n$ (at which X_n is not locally connected), $X_n - \{p_n\}$ admits an essential map to S^n.

One might suppose that if $X_n - \{p_n\}$ admits an essential map to S^n, then $X_n - \{p_n\}$ shape dominates S^n. In Section 5.4, we show that $X_1 - \{p_1\}$ is shape incomparable to S^1, and we conjecture that for $n > 1$, $X_n - \{p_n\}$ is shape incomparable to S^n.

5.1. Shape theories for metric spaces. Shape theory, developed by Borsuk [Bo1] for compact metric spaces (compacta), has several inequivalent extensions to wider classes of topological spaces, including, but not limited to, metric spaces, which nevertheless agree on compacta. Among these, the extensions due to Fox [F] and Mardešić and Segal [M–S], for arbitrary topological spaces, agree on metric spaces. Borsuk extended his theory to metric spaces in [Bo3]. Therefore, we shall confine our attention to the weak and strong shape theories of Borsuk, as presented in [Bo4], and the shape theory of Fox.

Godlewski and Nowak [G–N] show the interrelationship between the strong shape theory of Borsuk and that of Fox. In the process, they implicitly suggest a definition of shape intermediate to the strong shape theory of Borsuk and the shape theory of Fox. We make their definition explicit below.

5.1.1. Fundamental sequences and homotopies. Let X and Y be closed subsets of AR (absolute retract) spaces P and Q, respectively. Let $\{f_k\colon P \to Q\}$ be a sequence of maps. Consider the following conditions:

F: For every neighborhood V of Y (in Q), there is a neighborhood U of X (in P), such that $f_k|U \simeq f_{k+1}|U$ in V, for almost all k.

FC: For every compact $A \subset X$, there is a compact $B \subset Y$, such that for every neighborhood V of B (in Q), there is a neighborhood U of A (in P), such that $f_k|U \simeq f_{k+1}|U$ in V, for almost all k.

The triple $\underline{f} = \{f_k\colon P \to Q, X, Y\}$ is called a G (*weak*) [*strong*] *fundamental sequence* iff $\underline{f}$ satisfies condition[s] F (FC) [F and FC]. We abbreviate "G (weak) [strong] fundamental sequence" by *G-sequence* (*W-sequence*) [*S-sequence*]. In the discussion to follow, we shall speak of R-sequence, where R is any one of G, W, or S.

Let $\underline{f} = \{f_k\colon P \to Q, X, Y\}$ and $\underline{g} = \{g_k\colon P \to Q, X, Y\}$ be R-sequences. Consider the following conditions:

H: For every neighborhood V of Y (in Q), there is a neighborhood U of X (in P), such that $f_k|U \simeq g_k|U$ in V, for almost all k.

HC: For every compact $A \subset X$, there is a compact $B \subset Y$, such that for every neighborhood V of B (in Q), there is a neighborhood U of A (in P), such that $f_k|U \simeq g_k|U$ in V, for almost all k.

We say that $\underline{f}$ and $\underline{g}$ are G (*weakly*) [*strongly*] *homotopic*, abbreviated *G-homotopic* (*W-homotopic*) [*S-homotopic*], denoted $\underline{f} \underset{G}{\simeq} \underline{g}$ $(\underline{f} \underset{W}{\simeq} \underline{g})$ $[\underline{f} \underset{S}{\simeq} \underline{g}]$, iff $\underline{f}$ and $\underline{g}$ satisfy condition[s] H (HC) [H and HC].

Note that two S-sequences can be W- or G-homotopic. An S-sequence is clearly both a W-sequence and a G-sequence, though not conversely, as examples in [Bo4] show. Similarly, two R-sequences which are S-homotopic are both W- and G-homotopic. For compacta, the three types of sequence are equivalent, as are the three notions of homotopy. Composition of R-sequences is defined in the natural way, and can be shown to be an R-sequence.

The S-sequence $\underline{i}_X = \{f_k\colon P \to P, X, X\}$, where $f_k = i_P\colon P \to P$, the identity map on P, for all k, is called the *identity sequence*. For point $y \in Y$, the S-sequence $\underline{y} = \{f_k\colon P \to Q, X, Y\}$, where f_k is the constant map to the point y, for all k, is called a *constant sequence*. If we do not specify the point $y \in Y$, we denote a constant sequence by $\underline{0}$. Note that it is not clear whether an R-sequence of constant maps is R-homotopic to a constant sequence, unless space Y is connected.

If the R-sequence $\underline{f}$ is R-homotopic to some constant sequence, then we say that $\underline{f}$ is a *trivial* R-sequence. It is not generally true that two trivial R-sequences from X to Y are R-homotopic, for suppose that Y has two components.

The following elementary properties of R-sequences and R-homotopies may be easily established:

(1) R-homotopy is reflexive, symmetric, and transitive.

(2) If $\underline{f}$, $\underline{g}$, $\underline{h}$, and $\underline{j}$ are R-sequences, $\underline{f} \underset{R}{\simeq} \underline{g}$, $\underline{h} \underset{R}{\simeq} \underline{j}$, and the compositions $\underline{fh}$ and $\underline{gj}$ are defined, then $\underline{fh} \underset{R}{\simeq} \underline{gj}$.

(3) If $\underline{f} = \{f_k\colon P \to Q, X, Y\}$ is an R-sequence, then $\underline{i}_Y\underline{f} = \underline{f}\underline{i}_X = \underline{f}$.

(4) If $\underline{f}$ is an R-sequence and $\underline{0f}$ is defined, then $\underline{0f} \underset{R}{\simeq} \underline{0}$.

(5) If $\underline{f}$ is an R-sequence, $\underline{f0}$ is defined, and Y is connected, then $\underline{f0} \underset{R}{\simeq} \underline{0}$.

5.1.2. Mutations and homotopies. (As presented in [G–N], but due to Fox [F].) Let X and Y be closed subsets of ANR (absolute neighborhood retract) spaces P and Q, respectively. The family $U(X, P)$ of all neighborhoods of X in P is called the *complete neighborhood system* of X in P. Define $V(Y, Q)$ similarly. Let $U(X, P)$ and $V(Y, Q)$ be the complete neighborhood systems for X in P and Y in Q, respectively. A *mutation* $\bar{f}\colon U(X, P) \to V(Y, Q)$ is a collection of maps $f\colon U \to V$, $U \in U(X, P)$, $V \in V(Y, Q)$, satisfying the conditions:

M1: If $f \in \bar{f}$, $f\colon U \to V$, $U' \subset U$, $U' \in U(X, P)$, $V \subset V'$, $V' \in V(Y, Q)$, and $f'\colon U' \to V'$ is defined by $f'(x) = f(x)$, then $f' \in \bar{f}$.

M2: Every $V \in V(Y, Q)$ is the range of some $f \in \bar{f}$. (Note that V need not be the image of f.)

M3: If $f_1, f_2 \in \bar{f}$ and $f_1, f_2\colon U \to V$, then there is a $U' \in U(X, P)$ such that $U' \subset U$ and $f_1|U' \simeq f_2|U'$ (in V). Two mutations $\bar{f}, \bar{g}\colon U(X,P) \to V(Y, Q)$ are *homotopic*, abbreviated *F-homotopic*, denoted $\bar{f} \underset{F}{\simeq} \bar{g}$, iff $\bar{f}$ and $\bar{g}$ satisfy the condition.

HM: For every $f \in \bar{f}$ and $g \in \bar{g}$ such that $f, g\colon U \to V$, there is a $U' \in U(X, P)$ such that $U' \subset U$ and $f|U' \simeq g|U'$ (in V).

Condition HM is equivalent to the condition that $\bar{f} \cup \bar{g}$ be a mutation. Composition of mutations is defined in the natural way.

Godlewski and Nowak [G–N] show that given a G-sequence

$$\underline{f} = \{f_k\colon P \to Q, X, Y\},$$

the collection of maps $\bar{f}\colon U(X,P) \to V(Y, Q)$ defined by $f \simeq f_k|U$ in V for almost all k, is a mutation, and say that $\bar{f}$ is *associated to* $\underline{f}$. They show that this association preserves homotopy and composition. That is, $\underline{f} \underset{G}{\simeq} \underline{g}$ iff $\bar{f} \underset{F}{\simeq} \bar{g}$, for $\bar{f}$ associated to $\underline{f}$ and $\bar{g}$ associated to $\underline{g}$.

The mutation $\bar{u}\colon U(X,P) \to U(X,P)$ consisting of all inclusions $i_U\colon U \to V$, where $U, V \in U(X, P)$ and $U \subset V$, is called the *identity mutation*. It is associated to the identity sequence $\underline{i}_X$. A mutation $\bar{y}\colon U(X,P) \to V(Y, Q)$ consisting of all constant maps $g\colon U \to V$ such that $g(x) = y$, for all $x \in U$, for a fixed $y \in Y$, is called a *constant mutation*. It is associated to the constant sequence $\underline{y}$. We may denote a constant mutation without specifying the point y by $\bar{0}$. A mutation which is F-homotopic to a constant mutation is called a *trivial mutation*. Two trivial mutations are not necessarily F-homotopic if Y is not connected.

Now let R denote any member of $\{G, F, W, S\}$, and let "F-sequence" mean "mutation." With appropriate alterations in notation, properties 5.1.1 (1)–(5) apply to mutations and F-homotopies.

5.1.3. Shape domination and shape equivalence. Let X and Y be closed subsets of AR (ANR) spaces P and Q, respectively. We say that X *R-shape dominates* Y, denoted $X \underset{R}{\geqslant} Y$, iff there are two R-sequences $\underline{f} = \{f_k\colon P \to Q, X, Y\}$ and $\underline{g} = \{g_k\colon Q \to P, Y, X\}$ (mutations $\bar{f}\colon U(X,P) \to V(Y, Q)$ and $\bar{g}\colon V(Y, Q) \to$ $\to U(X,P)$) such that $\underline{fg} \underset{R}{\simeq} \underline{i}_Y$ $(\bar{f}\bar{g} \underset{F}{\simeq} \bar{v})$. If also, $\underline{fg} \underset{R}{\simeq} \underline{i}_X$ $(\bar{g}\bar{f} \underset{F}{\simeq} \bar{u})$, we say that X and Y are *R-shape equivalent*, denoted $X \underset{R}{=} Y$. Thus we obtain R-shape theory for metric spaces. When we are referring to compacta, where the theories all agree, or when a statement is true in all four theories, we may drop the prefixed letter.

W-shape and S-shape are, respectively, the weak and strong shape theories of Borsuk [Bo4]. G-shape is the theory implicit in [G–N], and furnishes therein the bridge to F-shape, which is the shape theory of Fox [F], restricted to metric spaces.

By the Kuratowski–Wojdysławski Theorem [Bo2, p. 79], any metric space can be embedded in some normed linear space as a closed subset of its convex hull. Since a convex subset of a normed linear space is an AR, it follows that the above definitions apply to all metric spaces. Theorem III. 3.3.4 of [Bo4] and Theorem 3.2 of [F] show that the choice of AR's (ANR's) and closed embeddings does not alter the relationship (if any) of shape domination or equivalence between X and Y; hence, the definitions of shape domination and equivalence are unambiguous.

5.1.4. Some relationships among shape theories. As before, let R denote any one of G, W, S, or F. Let $\underline{1}$ denote a one-point space, A denote an arc, C denote the $\sin 1/x$ continuum, and W denote the Warsaw circle (obtained from C by identifying the points $(0, -1)$ and $(1, \sin 1)$.) The following elementary properties are common to all four theories (X and Y are metric spaces):

(1) $X \underset{R}{=} Y$ implies $X \underset{R}{\geqslant} Y$ and $Y \underset{R}{\geqslant} X$.

(2) $X \underset{R}{\geqslant} \underline{1}$.

(3) $X \underset{R}{=} \underline{1}$ iff $\underline{i}_X \underset{R}{\simeq} \underline{0}$ $(\bar{u} \underset{F}{\simeq} \bar{0})$.

(4) $i_X \simeq 0$ implies $X \underset{R}{=} \underline{1}$.

(5) $\underline{1} \underset{R}{=} A \underset{R}{=} C$.

(6) $S^1 \underset{R}{=} W$.

If $X \underset{R}{=} \underline{1}$, we say that X has *trivial R-shape*. By (4), contractibility of X implies that X has trivial shape; however, the converse is not true, as C is not contractible.

The following relationships among the four shape theories can be established from the preceding definitions and remarks:

(7) $X \underset{S}{\geqslant} Y$ implies $X \underset{W}{\geqslant} Y$,

(8) $X \underset{S}{\geqslant} Y$ implies $X \underset{G}{\geqslant} Y$, and

(9) $X \underset{G}{\geqslant} Y$ implies $X \underset{F}{\geqslant} Y$; therefore

(10) $X \underset{S}{\geqslant} Y$ implies $X \underset{F}{\geqslant} Y$, and

(11) $X \underset{S}{\geqslant} Y$ implies $X \underset{W}{\geqslant} Y$ and $X \underset{F}{\geqslant} Y$.

(7′)–(11′) As (7)–(11), with "$=$" replacing "$\geqslant$".

The converse of Property (1) is false. Let X be a sequence of concentric circles in E^2 of decreasing diameter converging to a limit circle. Let Y be a sequence of concentric circles in E^2 converging to a point. Then $X \underset{R}{\geqslant} Y$ and $Y \underset{R}{\geqslant} X$, but $X \underset{R}{\neq} Y$. (This example appears in a preprint of Borsuk's "Lectures on the Theory of Shape" given at the University of California, Riverside in April 1974.)

Properties (7), (7′), (8), and (8′) are obvious. Properties (9) and (9′), and consequently (10) and (10′), are established in [G–N] through the process of associating a G-sequence with a mutation, as described in Section 5.1.2. Godlewski and Nowak also provide an example which they use to show that (9), (9′), (10), and (10′) are not reversible. Borsuk uses the same example to show that (7) and (7′) are not reversible. The example follows.

Let N be the space of natural numbers and let W be the Warsaw circle (as defined above). It follows from Sum Theorems 4.1 and 4.2 of [G–N] and the topological equivalence of the direct sum of N copies of S^1 (respectively, W) and $S^1 \times N$ (respectively, $W \times N$), that $S^1 \times N \underset{F}{\geqslant} W \times N$ and $S^1 \times N \underset{F}{=} W \times N$. It follows from Product Theorems 7.1 and 7.4 of [Bo4], that $S^1 \times N \underset{W}{\geqslant} W \times N$ and $S^1 \times N \underset{W}{=} W \times N$. However, it can be shown ([G–N, p. 391]; [Bo4, p. 108]) that $S^1 \times N \underset{G}{\not\geqslant} W \times N$ and $S^1 \times N \underset{G}{\neq} W \times N$. Thus also, $S^1 \times N \underset{S}{\not\geqslant} W \times N$ and $S^1 \times N \underset{S}{\neq} W \times N$.

5.2. Shape and contractibility. Property 5.1.4(4) is a special case of the fact that homotopy equivalence, and thus deformation, preserves R-shape [Bo4; F]. While contractibility implies R-shape equivalence to a point, the converse is false, as Example 5.1.4(5) shows. However, the triviality of the identity sequence (mutation) can be interpreted as a weakened form of contractibility.

5.2.1. Neighborhood contractibility. Let X be embedded as a closed subset of an ANR space P. Consider the following two conditions:

NC: For every neighborhood V of X, there is a neighborhood U of X, such that U is contractible in V.

NCC: For every compactum $A \subset X$, there is a compactum (in fact, continuum) $B \subset X$, such that for every neighborhood V of B, there is a neighborhood U of A, such that U is contractible in V.

We say that X is [*strongly*] *neighborhood contractible* (*on compact subsets*) iff X satisfies condition[s] NC (NCC) [NC and NCC].

Via the Borsuk Homotopy Extension Theorem, NC is equivalent to the statement that the inclusion map $i_X\colon X \to P$ is homotopic to a constant in every neighborhood V of X. Similarly, NCC is equivalent to the statement that for every compactum $A \subset X$, there is a continuum B, with $A \subset B \subset X$, such that the inclusion map $i_A\colon A \to P$ is homotopic to a constant in every neighborhood V of B. If NC (NCC) holds for X closed in an ANR space P, then NC (NCC) holds for X closed in any ANR space Q. Hence, *via* the Kuratowski–Wojdysławski Theorem, NC (NCC) is a topological property of a space.

For brevity, let "X is NC (NCC) [SNC]" denote that X is [strongly] neighborhood contractible (on compact subsets.) It can be shown that

(1) X is NC iff $\underline{i}_X \underset{G}{\simeq} \underline{0}$ iff $X \underset{G}{=} \underline{1}$.
(2) X is NC iff $\bar{u} \underset{F}{\simeq} \bar{0}$ iff $X \underset{F}{=} \underline{1}$.
(3) X is NCC iff $\underline{i}_X \underset{W}{\simeq} \underline{0}$ iff $X \underset{W}{=} \underline{1}$.
(4) X is SNC iff X is NC and NCC iff $\underline{i}_X \underset{S}{\simeq} \underline{1}$ iff $X \underset{S}{=} \underline{1}$.

Thus we have for trivial shape (unlike shape in general) that

(5) $X \underset{S}{=} \underline{1}$ iff $X \underset{W}{=} \underline{1}$ and $X \underset{F}{=} \underline{1}$.

Let X be a space. We say that X is *continuum-wise connected* iff given any two points p and q in X, there is a continuum $X_0 \subset X$, such that $p, q \in X_0$. It follows directly from the definitions that

(6) X is NC implies X is connected.
(7) X is NCC implies X is continuum-wise connected.

5.2.2. Essential maps and trivial shape. Let X be a closed subset of an ANR space P. We have observed that X is NC iff the inclusion map $i_X\colon X \to P$ is such that $i_X \simeq 0$ in every neighborhood V of X in P. Suppose that $f\colon X \to Q$ is an essential map of X to an ANR space Q. Then $0 \not\simeq f \simeq f i_X$. Hence $i_X \not\simeq 0$. On the other hand, if no essential map from X to any ANR exists, then the inclusion map is homotopic to a constant. Thus we have that the following are equivalent:

(1) X is NC.
(2) X admits no essential map to an ANR.
(3) $X \underset{G}{=} \underline{1}$.
(4) $X \underset{F}{=} \underline{1}$.

We therefore have a partial answer to the second question in the Introduction. By the above and 5.1.4(10), there is no metric space of trivial G-, F-, or S-shape which admits an essential map to S^1.

5.2.3. QUESTIONS. We raise several questions concerning the interrelationship of the notion of trivial shape in the four shape theories discussed.

(1) Is there a metric space X (necessarily noncompact) such that X is of trivial W-shape (so X is NCC), but admits an essential map to S^n (so X is not NC and $X \underset{F}{\neq} \underline{1}$)?

(2) Is there a metric space X (necessarily noncompact) which is NC (so $X \underset{F}{=} \underline{1}$), but not NCC (so $X \underset{W}{\neq} \underline{1}$)?

(3) Do the four versions of shape theory discussed agree on the class of metric spaces with trivial shape?

5.2.4. ONE-DIMENSIONAL EXAMPLES. The $\sin 1/x$ continuum minus a point in the limit segment and the pseudo-arc minus any point each map essentially to S^1; hence, by 5.2.2 and 5.1.4(10), each has nontrivial G-, F-, and S-shape. Each is continuum-wise disconnected by the removal of said point; hence, by 5.2.1(3) and (7), each is of nontrivial W-shape.

In Lemma 5.4.6, we show that a cellular continuum in E^2 minus an inaccessible point is neither NC nor NCC.

5.3. *n*-dimensional examples. Let X_1 be the $\sin 1/x$ continuum embedded in the interior of $I^2 \subset E^2$ so that the midpoint p_1 of the limit segment is $(0, 0)$, the limit segment is the interval from $(0, \frac{1}{2})$ to $(0, \frac{1}{2})$ on the y-axis, and the ray "wraps around" the limit segment. See Figure 2. It is evident that p_1 is inaccessible (from $E^2 - X_1$). Let $I^2 = I_1 \times I_2$, $I_i = [-1, 1]$, for all i, and $I^k = \prod_{i=1}^{k} I_i$.

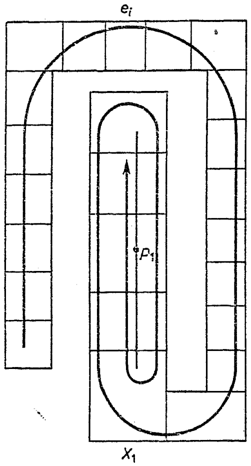

Fig. 2

For $n>1$, let $X_n = X_{n-1}\times I_{n+1}$ and $p_n = (p_{n-1}, 0)\in I^{n+1}$. Then $X_n\subset I^{n+1}\subset E^{n+1}$ as a closed subset and $p_n\in \mathrm{Int}(I^{n+1})$. We show below that p_n is inaccessible from $E^{n+1}-X_n$, and, hence, by the results of Section 3, $X_n-\{p_n\}$ admits an essential map to S^n.

5.3.1. LEMMA. *The point $p_n\in X_n$ is inaccessible from $E^{n+1}-X_n$.*

Proof. The reader may establish that p_1 is inaccessible. For $n>1$, suppose by way of contradiction that p_n is accessible. Let A be an arc of accessibility from a point $q\in E^{n+1}-I^{n+1}$ to p_n; that is, $A\cap X_n = \{p_n\}$, and p_n is an endpoint of A. Let A_0 be that component of $A\cap I^{n+1}$ which contains p_n. Since p_n is an interior point of I^{n+1}, A_0 is nondegenerate. Let $\pi\colon E^{n+1}\to E^2$ be the natural projection onto the first two coordinates.

We claim that $\pi(A_0)$ is nondegenerate. For otherwise, A_0 would be a nondegenerate subset of $I^{n+1}\cap\pi^{-1}(p_1)$. Hence, A_0 would be a nondegenerate subset of X_n, a contradiction of $A\cap X_n = \{p_n\}$. Thus $\pi(A_0)$ is a nondegenerate continuum.

If $p\in\pi(A_0)\cap X_1$, then $\pi^{-1}(p) = \{p\}\times E^{n-1}$, where E^{n-1} is the last $n-1$ coordinates of E^{n+1}. Since $A_0\subset I^{n+1}$ and $\{p\}\times I^{n-1}\subset X_n$, we have $\pi^{-1}(p)\cap A_0\subset X_n\cap A_0$. Therefore, $p = p_1$. Since $\pi(A_0)$ meets X_1 only in p_1, it follows by Theorem 2.3 that p_1 is accessible, a contradiction. ■

5.3.2. THEOREM. *For $n\geqslant 1$, X_n has the following properties:*

(1) X_n is a cellular continuum in E^{n+1}.
(2) X_n has trivial shape.

(3) X_n is not locally connected at p_n.
(4) $X_n-\{p_n\}$ admits an essential map to S^n.

Proof. In the embedding described for X_1, we may assume that X_1 is the intersection of a defining sequence $\{e_i\}_{i=1}^{\infty}$ of chains of rectangular 2-balls. The union of the links of any given chain, $\bigcup e_i$, is a long, thin 2-ball. See Figure 2. For each $\varepsilon>0$, there is an $i\geqslant 1$, such that $B^2 = \bigcup e_i$ is a 2-ball neighborhood of X_1 contained in the ε-neighborhood of X_1. Then $B^2\times I^{n-1}$ is an $(n+1)$-ball neighborhood of X_n in I^{n+1} contained in the ε-neighborhood of X_n. Hence, X_n is the intersection of a decreasing sequence of $(n+1)$-balls, and so is cellular. It can then be shown that X_n has trivial shape, using the fact that a point has a similar sequence of neighborhoods. (That is, X_n is NC.) Since p_n is inaccessible from $E^{n+1}-X_n$ by Lemma 5.3.1, it follows by Corollary 3.2 that $X_n-\{p_n\}$ admits an essential map to S^n.

Let L_1 denote the limit segment of X_1. Let $L_n = L_{n-1}\times I_{n+1}$, for all $n>1$. Then L_n is the limit hyperplane of X_n. It is not hard to see that L_n is a continuum of convergence of X_n, and so X_n is not locally connected at any point of L_n, including p_n. ■

5.3.3. THEOREM. *For $n\geqslant 1$, $X_n-\{p_n\}$ is neither* NC, *nor* NCC. *Thus, $X_n-\{p_n\}$ is of nontrivial R-shape, $R\in\{G, F, W, S\}$.*

Proof. That $X_n-\{p_n\}$ is not NC follows from the existence of an essential map to S^n and Proposition 5.2.2. Therefore, by 5.2.2 and 5.1.4(10), $X_n-\{p_n\}$ has nontrivial G-, F-, and S-shape.

To show that $X_n-\{p_n\}$ is not NCC, and thus has nontrivial W-shape, we will show that for each $n\geqslant 1$, there is a compactum $A_n\subset X_n-\{p_n\}$ for which there is no continuum B with $A_n\subset B\subset X_n-\{p_n\}$ satisfying condition NCC.

Let A_1 be the two point set consisting of the endpoints of the limit segment L_1 of X_1. For $n>1$, let $A_n = (A_{n-1}\times I_{n+1})\cup(L_{n-1}\times\{-1, 1\})$. Observe that A_n is the S^{n-1} bounding the n-ball L_n, where $L_n = L_{n-1}\times I_{n+1}$ is the limit hyperplane of X_n.

Suppose that B is a continuum in $X_n-\{p_n\}$ containing A_n. Since B is compact, there is an $(n+1)$-ball C about p_n missing B. We may assume that there is a diameter D of C whose endpoints lie in $E^{n+1}-X_n$. From the endpoints of D extend disjoint rays R_1 and R_2 in $E^{n+1}-X_n$ to ∞. Then $L = R_1\cup D\cup R_2$ is a line in $E^{n+1}-B$ such that A_n *links* L. That is, A_n is not contractible in $E^{n+1}-L$. Thus there is a neighborhood $V = E^{n+1}-L$ of B such that no neighborhood U of A_n contracts in V. Hence, $X_n-\{p_n\}$ is not NCC. ■

5.4. Shape incomparability to S^n. Our main theorem of this section is Theorem 5.4.9, which implies that each of our one-dimensional examples ($X_1-\{p_1\}$, the $\sin 1/x$ continuum minus *any* point in the limit segment, the pseudo-arc minus any point) is shape incomparable to S^1. We conjecture that for $n>1$, example $X_n-\{p_n\}$ of Section 5.3 is shape incomparable to S^n. The method of proof below, however, cannot be extended to higher dimensions.

It follows from Propositions 5.1.4(9) and (10) that we may restrict our attention to F- and W-shape incomparability. The proof in the case of W-shape is fairly direct and elementary, and is presented in Lemmas 5.4.6 and 5.4.7 and in Theorem 5.4.8. The proof in the case of F-shape is less trivial, requiring a lemma about neighborhoods of a cellular continuum minus an inaccessible point in the punctured plane, and is presented in Lemmas 5.4.3 and 5.4.4 and in Theorem 5.4.5.

For $\varepsilon > 0$, let $S(A, \varepsilon) = \{x | d(x, A) < \varepsilon\}$ denote the open ε-neighborhood of a subset A of E^2. For C a compactum in E^2, let $\hat{C}$ denote the union of C and its bounded complementary domains, if any. We call $\hat{C}$ the *topological hull* of C. Note that we identify $E^2 \cup \{\infty\}$, the one-point compactification of E^2, with S^2.

The following two lemmas are required in the proof of Lemma 5.4.3. The proof of the first is well known.

5.4.1. Lemma. *Let A be a 0-dimensional compact subset of a domain U contained in S^2. Then there is a 2-cell D such that $A \subset \operatorname{Int}(D) \subset D \subset U$.*

5.4.2. Lemma. *Let P and Q be compact subsets of S^2 such that Q is 0-dimensional, $P \cap Q = \{p\}$ and $p \in \operatorname{Cl}(Q - \{p\})$. Let $q \in Q - \{p\}$. Then there is a 2-cell D and a sequence of mutually disjoint 2-cells $\{D_n\}_{n-1}^{\infty}$ such that*

(1) $P \cup (\bigcup_{n=1}^{\infty} D_n) \subset \operatorname{Int}(D) \subset D \subset S^2 - \{q\}$.

(2) $P \cap (\bigcup_{n=1}^{\infty} D_n) = \emptyset$,

(3) *the sequence $\{D_n\}_{n=1}^{\infty}$ converges to p (denoted $D_n \to p$), and*

(4) $D - (\bigcup_{n=1}^{\infty} \operatorname{Int}(D_n)) \subset (S^2 - Q) \cup \{p\}$.

Proof. There is a 2-cell D such that $P \subset \operatorname{Int}(D) \subset D \subset S^2 - \{q\}$ and $\operatorname{Bd}(D) \cap Q = \emptyset$. Let $\{G_n\}_{n=1}^{\infty}$ be a nested null sequence of open 2-cell neighborhoods of p in S^2 such that $Q \cap \operatorname{Bd}(G_n) = \emptyset$ for all $n \geqslant 1$. Let $U_1, U_2, \ldots$ be the components of $D - \big(P \cup \operatorname{Bd}(D) \cup (\bigcup_{n=1}^{\infty} \operatorname{Bd}(G_n))\big)$ such that $U_n \cap Q \neq \emptyset$. Then $Q_n = Q \cap U_n$ is a compact 0-dimensional subset of the domain U_n. By Lemma 5.4.1, there is a 2-cell $D_n \subset U_n$ such that $Q_n \subset \operatorname{Int}(D_n)$. Then D_n converges to p, since U_n does. The remaining properties of the 2-cells are evident from the construction. ■

5.4.3. Lemma. *Let X be a cellular continuum in E^2, p an inaccessible point of $\operatorname{Bd}(X)$, V a bounded connected open neighborhood of $X - \{p\}$ in $E^2 - \{p\}$, and Y a compact subset of V. Then there is a 2-cell D and a null sequence of mutually disjoint 2-cells $\{D_n\}_{n=1}^{\infty}$ such that*

(1) $X \cup Y \cup (\bigcup_{n=1}^{\infty} D_n) \subset \operatorname{Int}(D)$,

(2) $(X \cup Y) \cap (\bigcup_{n=1}^{\infty} D_n) = \emptyset$,

(3) $D_n \to p$, *and*

(4) $D - (\bigcup_{n=1}^{\infty} \operatorname{Int}(D_n)) \subset V \cup \{p\}$.

Proof. Let $\mathscr{C}$ be the upper-semicontinuous decomposition of E^2 into components of $E^2 - V$ and single points of V. Let $f\colon E^2 \to E^2/\mathscr{C}$ be the quotient map. By Moore's Theorem [Mr], we may assume that $E^2/\mathscr{C} = S^2$. (Because V is connected, no component of $E^2 - V$ separates E^2.) Let $A \subset E^2 - (X \cup Y)$ be a sequence of points converging to p. Then $P = f(X \cup Y)$ and $Q = f\big(A \cup (E^2 - V)\big)$ are compact subsets of S^2 such that Q is 0-dimensional and $P \cap Q = \{p'\}$, where $p' = f(p)$.

Since A converges to p and $f(A) \subset Q - \{p'\}$, we have $p' \in \operatorname{Cl}(Q - \{p'\})$. Note that $f^{-1}(p') = p$, since p is inaccessible and $f^{-1}(p')$ is a continuum in $E^2 - V$. It follows that the set-valued mapping $f^{-1}\colon S^2 \to E^2$ is continuous at p'.

Since P and Q satisfy the hypotheses of Lemma 5.4.2, with $q = f(C)$ where C is the unbounded component of $E^2 - V$, there exist a 2-cell D' and a sequence of 2-cells $\{D'_n\}_{n=1}^{\infty}$ in S^2 satisfying the conclusion of Lemma 5.4.2. Let $D = f^{-1}(D')$ and $D_n = f^{-1}(D'_n)$. Using the continuity of f^{-1} at p', and the fact that (1)–(4) of Lemma 5.4.2 are satisfied by D' and $\{D'_n\}_{n=1}^{\infty}$, it can be seen that D and $\{D_n\}_{n=1}^{\infty}$ are 2-cells satisfying conditions (1)–(4) above. ■

5.4.4. Lemma. *Let $\bar{f}\colon U(S^1, S^1) \to V(X - \{p\}, E^2 - \{p\})$ be a mutation, where $X \subset E^2$ is a cellular continuum and $p \in \operatorname{Bd}(X)$ is inaccessible. Then $\bar{f} \underset{F}{\simeq} \bar{0}$.*

Proof. Note that $U(S^1, S^1) = \{S^1\}$. Let $f \in \bar{f}$, so that $f\colon S^1 \to V$ for some $V \in V(X - \{p\}, E^2 - \{p\})$. Since p is inaccessible from $E^2 - X$

(1) $X - \{p\}$ is connected.

For, if $X - \{p\}$ were not connected, then the closure of a separator of $E^2 - \{p\}$ which separated $X - \{p\}$ would contain a nondegenerate component meeting X only in p, contradicting Theorem 2.3.

Let V_1 be the component of V containing $X - \{p\}$. We may assume that V_1 is bounded and open, and that

(2) $f(S^1) \subset V_1$.

For, if $f(S^1)$ is not contained in V_1, then, since $f(S^1)$ is connected, $f(S^1)$ is contained in some component of V distinct from V_1. Because $V_1 \subset V$ and

$$V_1 \in V(X - \{p\}, E^2 - \{p\}),$$

there is an $f_0 \in \bar{f}$, $f_0\colon S^1 \to V_1$ such that $f_0 \simeq f$ in V, which is a contradiction since $f(S^1)$ and $f_0(S^1)$ lie in different components of V.

Since each neighborhood of $X - \{p\}$ in $E^2 - \{p\}$ contains a path-connected neighborhood of $X - \{p\}$, it suffices to show that $f \simeq 0$ in V_1.

By Lemma 5.4.3, there is a 2-cell D and a null sequence $\{D_n\}_{n=1}^{\infty}$ of mutually disjoint 2-cells such that

(3) $D_n \subset \operatorname{Int}(D)-(X\cup f(S^1))$,
(4) $D_n \to p$, and
(5) $X\cup f(S^1)\subset \operatorname{Int}(D)-(\bigcup_{n=1}^{\infty} D_n)\subset V_1\cup\{p\}$.

Let $M_n = \bigcup\{D_i|\ i=1,2,\dots,n\}$ and let $N_n = \bigcup\{D_i|\ i=n,n+1,\dots,\infty\}$. Note that N_1 is the disjoint union of M_{n-1} and N_n.

By (5), it suffices to show that $f\simeq 0$ in $D-(N_1\cup\{p\})$. As a first step, we claim that

(6) $f\simeq 0$ in $D-\{p\}$.

For, suppose $f\not\simeq 0$ in $D-\{p\}$. For any nonseparating continuum $B\subset E^2$ and any map $g\colon S^1\to E^2-B$, let $\mathrm{WN}(g,B)$ denote the winding number of g about B. Then $\mathrm{WN}(f,p)\neq 0$. Since $D_n\to p$ by (4), it follows from the continuity of WN that $\mathrm{WN}(f,D_n)\neq 0$, for almost all n. Let j be such that $\mathrm{WN}(f,D_n)\neq 0$. Hence, $f\not\simeq 0$ in $D-D_n$. By (3) and since X is cellular, there is a 2-cell D' such that $X\subset D'\subset D-D_n$. Since $D'-\{p\}\in V(X-\{p\},E^2-\{p\})$, there is a map $g\colon S^1\to D'-\{p\}$ such that $f\simeq g$ in $D-(D_n\cup\{p\})\subset D-D_n$. Since D' is contractible, $g\simeq 0$ in $D-D_n$. Thus, $f\simeq 0$ in $D-D_n$, a contradiction.

Since the homotopy of (6) must miss a neighborhood of p in E^2, there is an integer $k\geqslant 1$ such that

(7) $f\simeq 0$ in $D-(N_k\cup\{p\})$.

If $k=1$, we are done: so assume that $k>1$. As in the proof of Lemma 5.4.3, there is a 2-cell D_0 such that

(8) $(X\cup N_k)\subset \operatorname{Int}(D_0)\subset D_0\subset \operatorname{Int}(D)-M_{k-1}$.

Since $D_0-(N_k\cup\{p\})\in V(X-\{p\},E^2-\{p\})$, there is a map $f_1\in\bar{f}$ such that $f_1\colon S^1\to D_0-(N_k\cup\{p\})$. By (8), $D_0-(N_k\cup\{p\})\subset D_0-(N_1\cup\{p\})\subset D-(N_1\cup\{p\})$. Thus, we have

(9) $f_1\simeq f$ in $D-(N_1\cup\{p\})$.

There is a deformation retraction of the 2-cell D onto the 2-cell D_0. Let $\varphi\colon D-(N_k\cup\{p\})\to D_0-(N_k\cup\{p\})$ be the restriction of this deformation retraction. Then φ is itself a deformation retraction by (8). Let $i\colon D_0-(N_k\cup\{p\})\to D-(N_k\cup\{p\})$ be the inclusion. It follows from (7) and (9) that

(10) $\varphi f_1\simeq \varphi f\simeq 0$ in $D_0-(N_k\cup\{p\})$.

Since $f_1(S^1)\subset D_0-(N_k\cup\{p\})$, we have that

(11) $i\varphi f_1 = f_1$.

Since by (8) the homotopy of (10) misses M_{k-1}, and since $N_1 = M_{k-1}\cup N_k$, we have that

(12) $i\varphi f_1\simeq 0$ in $D-(N_1\cup\{p\})$.

Consequently, by (9), (11), and (12), we have that

(13) $f\simeq 0$ in $D-(N_1\cup\{p\})$. ■

5.4.5. THEOREM. *Let $X\subset E^2$ be a cellular continuum and let $p\in\operatorname{Bd}(X)$ be inaccessible. Then S^1 and $X-\{p\}$ are F-shape incomparable.*

Proof. Suppose $S^1\underset{F}{\geqslant} X-\{p\}$. Then there are mutations $\bar{f}\colon U(S^1,S^1)\to V(X-\{p\},E^2-\{p\})$ and $\bar{g}\colon V(X-\{p\},E^2-\{p\})\to U(S^1,S^1)$ such that $\bar{f}\bar{g}\underset{F}{\simeq}\bar{v}$, where $\bar{v}$ is the identity mutation on $X-\{p\}$. By Lemma 5.4.4, $\bar{f}\underset{F}{\simeq}\bar{0}$. By Proposition 5.1.1(4) (and the remark at the end of Section 5.1.2), $\bar{v}\underset{F}{\simeq}\bar{f}\bar{g}\underset{F}{\simeq}\bar{0}$. So by Proposition 5.2.1(2), $X-\{p\}$ is NC. Therefore, by Proposition 5.2.2, $X-\{p\}$ admits no essential map to S^1. However, since p is inaccessible, Corollary 3.3 implies that $X-\{p\}$ does admit an essential map to S^1. In view of this contradiction, $S^1\underset{F}{\not\geqslant} X-\{p\}$.

Now suppose $X-\{p\}\underset{F}{\geqslant} S^1$. Then for mutations as defined above, we have $\bar{g}\bar{f}\underset{F}{\simeq}\bar{u}$, where $\bar{u}$ is the mutation whose only element is the identity i_{S^1} on S^1. Let $g\in\bar{g}$ and $f\in\bar{f}$ for which gf is defined. By Lemma 5.4.4, $f\simeq 0$. Hence, $i_{S^1}\simeq gf\simeq 0$, a contradiction. Thus, $X-\{p\}\underset{F}{\not\geqslant} S^1$. ■

5.4.6. LEMMA. *Let $X\subset E^2$ be a cellular continuum and $p\in\operatorname{Bd}(X)$ an inaccessible point. Then every subcontinuum of $X-\{p\}$ is contained in a cellular subcontinuum of $X-\{p\}$. However, there is a compactum in $X-\{p\}$ not contained in any subcontinuum in $X-\{p\}$; hence, $X-\{p\}$ is not* NCC.

Proof. Let K be a subcontinuum of $X-\{p\}$. Suppose K is not cellular. Then K separates E^2, because cellular continua coincide with nonseparating continua in E^2. All bounded complementary domains of K in E^2 are contained in $\operatorname{Int}(X)$, since X does not separate E^2. Since $p\notin\operatorname{Int}(X)$, $\hat{K}$, the topological hull of K, is a nonseparating, thus cellular, subcontinuum of $X-\{p\}$ containing K.

To show that $X-\{p\}$ is not NCC, it suffices to establish the following claim:

(1) There is an $\varepsilon>0$ such that $A = X\cap(E^2-S(p,\varepsilon))$ is not contained in any continuum $K\subset X-\{p\}$.

Suppose the claim is false. Then for each n, there is a real number $\varepsilon_n>0$, an open disk $D_n = S(p,\varepsilon_n)$, a compactum $A_n = X\cap(E^2-D_n)$, and a continuum K_n such that

(2) $X-\{p\}\supset\dots\supset K_n\supset A_n\supset K_{n-1}\supset A_{n-1}\supset\dots\supset K_1\supset A_1$,
(3) $K_n\cap\operatorname{Cl}(D_{n+1}) = \varnothing$, and
(4) $\{\varepsilon_n\}_{n=1}^{\infty}$ decreases to 0.

By the first paragraph of the proof, we may assume that $K_n = \hat{K}_n$. Since K_n is nonseparating, there is an arc B'_n from ∞ to p missing K_n. Let $B_n\subset B'_n$ be an arc irreducible from ∞ to $\mathrm{Bd}(D_n)$ missing X. We now inductively construct a locally connected continuum M in E^2 meeting X only in p, thus contradicting the inaccessibility of p.

Let $M_2 = B_2$. Let R be the component of $M_2 \cap D_1$ meeting $\mathrm{Bd}(D_2)$ and let S be the component of $B_3 \cap D_1$ meeting $\mathrm{Bd}(D_3)$. We claim there is an arc F (possibly degenerate, if $\mathrm{Cl}(R) \cap \mathrm{Cl}(S) \neq \emptyset$) in $D_1 - X$ from $\mathrm{Cl}(R)$ to $\mathrm{Cl}(S)$. For if not, then some component L of $\mathrm{Cl}(D_1) \cap X$ separates $\mathrm{Cl}(R)$ from $\mathrm{Cl}(S)$. Now L must meet D_2 and both components of $\mathrm{Bd}(D_1) - (\mathrm{Cl}(R) \cup \mathrm{Cl}(S))$ in points of K_1. See Figure 3. But K_1 does not meet D_2. Hence, $B_3 \cup \mathrm{Cl}(D_2) \cup M_2$ separates E^2 between points of $L \cap K_1 \cap \mathrm{Bd}(D_1)$ without meeting the continuum K_1, a contradiction.

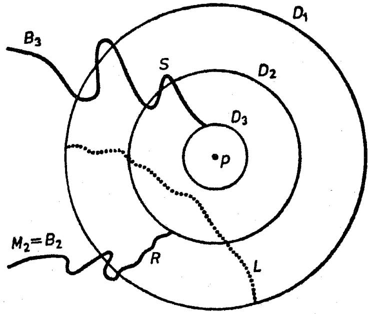

Fig. 3

Let M_3 be an arc in $M_2 \cup F \cup B_3$ from ∞ to $\mathrm{Bd}(D_3)$ which differs from M_2 only within D_1. We may repeat the above process with D_2, D_3, D_4, B_4 and M_3 replacing D_1, D_2, D_3, B_3, and M_2, respectively, to obtain an arc M_4 from ∞ to $\mathrm{Bd}(D_4)$ missing X and differing from M_3 only within D_2.

Inductively, for $n = 1, 2, \ldots$, there exists an arc M_n from ∞ to $\mathrm{Bd}(D_n)$ missing X and such that M_{n+1} differs from M_n only inside D_{n-1}. It is clear that $M = \mathrm{Cl}(\bigcup_{n=2}^{\infty} M_n) = (\bigcup_{n=2}^{\infty} M_n) \cup \{p\}$ is a locally connected continuum meeting X only at p.

5.4.7. Lemma. *Let* $\underline{f} = \{f_k\colon E^2 \to S^2 - \{p\}, S^1, X - \{p\}\}$ *be a W-sequence, where* $X \subset S^2$ *is a cellular continuum and* $p \in \mathrm{Bd}(X)$ *is inaccessible. Then* $\underline{f} \underset{W}{\simeq} \underline{0}$.

Proof. It follows from the definition of a W-sequence (5.1.1(FC)) that with $A = S^1$ there is a continuum $B \subset X - \{p\}$ such that for every neighborhood V of B, there is a neighborhood U of A, such that for almost all k, $f_k|U \simeq f_{k+1}|U$ in V. That B may be taken to be a continuum follows from the fact that S^1 is connected. By the first part of Lemma 5.4.6, we may assume that B is cellular. Thus, any neighborhood V of B contains a 2-cell neighborhood V' of B. There is a neighborhood U of S^1 such that $f_k(U) \subset V'$, for almost all k. By the contractibility of V', $f_k|U \simeq 0$ in V, for almost all k. Therefore, since B is a continuum, $\underline{f} \underset{W}{\simeq} \underline{0}$. ■

5.4.8. Theorem. *Let* $X \subset E^2$ *be a cellular continuum and* $p \in \mathrm{Bd}(X)$ *inaccessible. Then* $X - \{p\}$ *is W-shape incomparable to* S^1.

Proof. Suppose that $S^1 \underset{W}{\geqslant} X - \{p\}$. Recall that the choice of AR containing $X - \{p\}$ is immaterial. Thus we may assume that there are W-sequences

$$\underline{f} = \{f_k\colon E^2 \to S^2 - \{p\}, S^1, X - \{p\}\}$$

and

$$\underline{g} = \{g_k\colon S^2 - \{p\} \to E^2, X - \{p\}, S^1\}$$

such that $\underline{fg} \underset{W}{\simeq} \underline{i}_{X-\{p\}}$, the identity sequence on $X - \{p\}$. By Lemma 5.4.7, $\underline{f} \underset{W}{\simeq} \underline{0}$; hence, by Proposition 5.1.1(4), $\underline{i}_{X-\{p\}} \underset{W}{\simeq} \underline{fg} \underset{W}{\simeq} \underline{0}$. By Proposition 5.2.1(3), $X - \{p\}$ is NCC, which contradicts the second part of Lemma 5.4.6. Therefore, $S^1 \underset{W}{\not\geqslant} X - \{p\}$.

Now suppose that $X - \{p\} \underset{W}{\geqslant} S^1$. Then there are W-sequences as defined above for which $\underline{gf} \underset{W}{\simeq} \underline{i}_{S^1}$, the identity sequence on S^1. Since S^1 is connected, it follows from Proposition 5.1.1(5) and Lemma 5.4.7, that $\underline{i}_{S^1} \underset{W}{\simeq} \underline{gf} \underset{W}{\simeq} \underline{0}$. But clearly, $\underline{i}_{S^1} \underset{W}{\not\simeq} \underline{0}$. In view of this contradiction, $X - \{p\} \underset{W}{\not\geqslant} S^1$. ■

Combining Theorems 5.4.5 and 5.4.8, and by virtue of Propositions 5.1.4(9)–(10), we obtain the main theorem of this section.

5.4.9. Theorem. *If* $X \subset E^2$ *is a cellular continuum and* $p \in \mathrm{Bd}(X)$ *is inaccessible, then* $X - \{p\}$ *is shape incomparable to* S^1.

5.4.10. Corollary. *Each of our one-dimensional examples* ($X_1 - \{p_1\}$, *the* $\sin 1/x$ *continuum minus any point in the limit segment, the pseudo-arc minus any point*) *is shape incomparable to* S^1.

5.5. Shape incomparability to compact, noncontractible ANRs. In this Section we indicate how to extend Theorem 5.4.9 to the case where S^1 is replaced by any compact, noncontractible ANR. We obtain as our main theorem the following:

5.5.1. Theorem. *Let* $X \subset E^2$ *be a cellular continuum,* $p \in \mathrm{Bd}(X)$ *an inaccessible point, and* Z *any compact, noncontractible* ANR. *Then* $X - \{p\}$ *is shape incomparable to* Z.

Proof. We carry out the proof modulo two lemmas to be proved subsequently. The theorem follows immediately provided that Theorems 5.4.5 and 5.4.8 can be strengthened by replacing S^1 with Z. We first assume that Z is connected.

Lemma 5.4.7 can easily be extended by replacing S^1 with a connected Z and E^2 with an AR M containing Z. The strengthened version of Theorem 5.4.8 immediately follows. Thus, $X - \{p\}$ is not W-shape comparable to a connected Z.

It follows from Lemma 5.5.2, proved below, that Lemma 5.4.4 can be strengthened by replacing S^1 with a connected Z. We use Lemma 5.5.2 to find a simple closed curve in Z to which the argument of the proof of Lemma 5.4.4 is then applied. The strengthened version of Theorem 5.4.5 immediately follows. Thus, $X-\{p\}$ is F-shape incomparable to a connected Z.

Now suppose that Z is not connected. By Lemma 5.5.3, proved below, if Z shape dominates $X-\{p\}$, then some component Z_0 of Z does. Thus, this direction reduces to the first case. Since $X-\{p\}$ is connected, $X-\{p\}$ cannot shape dominate Z, which has more than one component. ∎

5.5.2. LEMMA. *If $f\colon Y\to G$ is an essential mapping of a Peano continuum Y onto a graph G, then there is a simple closed curve S in Y such that f is essential on S.*

Proof. Let T be a maximal tree in G [H, p. 117]. Let $E_1, \ldots, E_n$ be a listing of the edges of G *not* in T. Barycentrically subdivide each E_i into $E_{i,1}$ and $E_{i,2}$. For $j=1,2$, let $T_j = T\cup(\bigcup_{i=1}^{n} E_{i,j})$. Then $G = T_1\cup T_2$, with each T_j a tree. By Theorem 7 on p. 432 of [K], there exist Peano continua B_1 and B_2 in Y such that $f^{-1}(T_i)\subset B_i$ $(i=1,2)$, and f is inessential on B_i. Let $x\in B_1\cap B_2$. Let $\varphi_i\colon B_i\to J$ be liftings of $f|B_i$ to the universal covering space J of G such that $\varphi_1(x)=\varphi_2(x)$. Since f is essential and does not lift to J, there exists $y\in B_1\cap B_2$ such that $\varphi_1(y)\neq\varphi_2(y)$.

For $i=1,2$, let K_i be an arc in B_i from x to y. Let K_i have its natural order with initial point x. Let z be the first point of K_1 in $K_1\cap K_2$ such that $\varphi_1(z)\neq\varphi_2(z)$. Let L_i be the arc in K_i from x to z. Note that z is an isolated point in $L_1\cap K_2$. Let w be the last point in $L_2-\{z\}$ such that $w\in L_1$. Let M_i be the arc in K_i from w to z. Then $S=M_1\cup M_2$ is the required simple closed curve. ∎

5.5.3. LEMMA. *If the compactum X R-shape dominates the connected set Y, then some component of X R-shape dominates Y.*

Proof. In order that the maps comprising an R-sequence from Y to X be (almost all) homotopic, the fact that Y is connected implies that (almost all) the maps take Y into a neighborhood of some single component of X, no matter how small the neighborhood of that component. ∎

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