measure of X has to be 0, i.e., if the measurability condition of Theorem 4.4 is redundant. We do not have an answer to this question. All we can offer is an example to show that the condition that X have *inner* measure 0 does *not* imply

$$\mathscr{B}^1(X) = \mathscr{D}'(X).$$

EXAMPLE 4.6. By [2], p. 146, there exists a set $X \subset \mathbf{R}$ such that neither X nor X^c contains an uncountable closed subset of $\mathbf{R}$. For this X, both X and X^c have inner measure 0. (Regularity of the Lebesgue measure.) Then $X_d = \mathbf{R}$. We show that X is a Baire space, so that (by Lemma 4.3) $\mathscr{B}^1(X) \neq \mathscr{D}'(X)$.

Let $U_1, U_2, \ldots$ be open subsets of $\mathbf{R}$ such that every $U_n \cap X$ is dense in X. Then each U_n is dense in $\mathbf{R}$, so $\bigcap_{n\in N} U_n$ is a dense G_δ-subset of $\mathbf{R}$. For every interval J of $\mathbf{R}$, $J \cap \bigcap_{n\in N} U_n$ is an uncountable Borel set, therefore contains an uncountable closed subset of $\mathbf{R}$ ([4], p. 151.) and therefore is not contained in X^c. It follows that $\bigcap_{n\in N} (U_n \cap X)$ is dense in X.

References

[1] C. Goffman, *Real Functions*, Rinehart & Co, Inc., New York 1957.

[2] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, Berlin-Heidelberg-New York 1965.

[3] H. Lebesgue, *Leçons sur l'intégration*, Chelsea Publ. Co., New York 1973.

[4] N. Lusin, *Leçons sur les ensembles analytiques*, Chelsea Publ. Co., New York 1972.

[5] G. Petruska and M. Laczkovich, *Baire 1 Functions, Approximately Continuous Functions and Derivatives*, Acta Math. Acad. Sci. Hung. 25 (1974), 189–212.

[6] A. C. M. van Rooij and W. H. Schikhof, *A Second Course on Real Functions*, Cambridge Univ. Press, Cambridge 1982.

[7] A. C. M. van Rooij, *On Derivatives of Funcions Defined on Disconnected Sets*, II, Fund. Math. 131 (1988), 93–102.

[8] W. H. Schikhof, *The Set of Derivatives in a non-Archimedean Field*, Math. Ann. 216 (1975), 67–70.

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On derivatives of functions defined on disconnected sets, II *

by

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Abstract. Let X be a totally disconnected subset of $\mathbf{R}$ without isolated points, let $\mathscr{B}^1(X)$ be the first class of Baire on X and let $\mathscr{D}'(X)$ be the set of all functions $X \to \mathbf{R}$ that have primitives on X. Then $\mathscr{D}'(X) \subset \mathscr{B}^1(X)$. We show that $\mathscr{D}'(X)$ is a "large" subset of $\mathscr{B}^1(X)$. Thus, $\mathscr{D}'(X)$ contains all approximately continuous functions on X; and if $f, g \in \mathscr{B}^1(X)$ are such that $f(x) < g(x)$ for all x in X, then between f and g there is an element of $\mathscr{D}'(X)$.

5. How large is $\mathscr{D}'(X)$? We now know that, at least for certain sets X with empty interiors, $\mathscr{D}'(X)$ is a proper subset of $\mathscr{B}^1(X)$. Our present purpose is to show that it has to be a large subset. (Th. 5.6, Th. 6.1.)

Let $Y \subset X \subset \mathbf{R}$. We call Y a *clopen* subset of X if it is both relatively closed and relatively open.

We will make use of the following observation. If $X \subset \mathbf{R}$, $\mathring{X} = \emptyset$ and if $\mathscr{U}$ is an open cover of X, then there exists a cover of X by countably many pairwise disjoint subsets $X_1, X_2, \ldots$, each of which is clopen in X and contained in an element of $\mathscr{U}$. (As X^c is dense in $\mathbf{R}$, $\mathscr{U}$ has a refinement $\mathscr{V}$ consisting of clopen subsets of X. By using the Lindelöf property of X one obtains a countable subcover $(V_n)_{n\in N}$ of $\mathscr{V}$. Now set $X_n = V_n \setminus (V_1 \cup \ldots \cup V_{n-1})$ $(n \in N)$.)

LEMMA 5.1. *Let $X \subset \mathbf{R}$, $\mathring{X} = \emptyset$. Let A be a relatively closed subset of X. Let $\psi \in C(X)$, $\psi(x) = 0$ for all $x \in A$, $\psi(x) > 0$ for all $x \in X \setminus A$. Let $h \in C(X)$. Then there exists a j in $C(X)$ with*

$$|j-h| \leqslant \psi,$$
$$j' = 0 \quad \text{on } X^* \setminus A.$$

Proof. For $x \in X \setminus A$, let $U(x) = \{y \in X \setminus A : |h(x)-h(y)| < \psi(y)\}$. By the above remark, there exist a family $(Y_n)_{n\in N}$ of pairwise disjoint clopen subsets of $X \setminus A$, covering $X \setminus A$, and a family $(x_n)_{n\in N}$ of elements of $X \setminus A$, such that $Y_n \subset U(x_n)$

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* This paper is a continuation of [7], using the same notations.

for every n. Define $j\colon X\to \boldsymbol{R}$ by

$$j(x)=\begin{cases}h(x) & \text{if } x\in A,\\ h(x_n) & \text{if } n\in \boldsymbol{N},\ x\in Y_n.\end{cases}$$

Then $|j-h|\leqslant\psi$. Each Y_n is open in X, so $j'=0$ on $X^*\setminus A$ and j is continuous at all points of $X\setminus A$. From the inequality $|j-h|\leqslant\psi$ it follows that j is also continuous at the points of A. ■

Further, we need a technical lemma:

LEMMA 5.2. *Let $A\subset \boldsymbol{R}$ be closed, $F\in C(A)$ and $S\subset\{a\in A^*\colon F$ is differentiable at $a\}$. Then F has an extension h in $C(\boldsymbol{R})$ that is differentiable at all points of S* (so that $h'=F'$ on S).

If $F\in \mathrm{Lip}(A)$, then h can be chosen in $\mathrm{Lip}(\boldsymbol{R})$.

(Theorem 5.5.3 of [5] is a special case.)

Proof. *Step* I. Assume that every element of S is a two-sided accumulation point of A.

Extend F to a function $h\colon \boldsymbol{R}\to\boldsymbol{R}$ by interpolating linearly on the bounded components of the open set A^c and making h constant on the unbounded components. It is elementary that h is continuous and that $h\in\mathrm{Lip}(\boldsymbol{R})$ in case $F\in\mathrm{Lip}(A)$. Let $s\in S$; we prove that $h'(s)$ exists.

Take $\varepsilon>0$. There exist $a,b\in\boldsymbol{R}$ such that $a<s<b$ and $|\Phi F(x,s)-F'(s)|\leqslant\varepsilon$ as soon as $x\in A\cap[a,b]\setminus\{s\}$. As s is a two-sided accumulation point of A, we can choose such a and b in A. Then the graphs of $h|_{[a,s)}$ and $h|_{(s,b]}$ are contained in the convex hulls of the graphs of $F|_{A\cap[a,s)}$ and $F|_{A\cap(s,b]}$, respectively. It follows that $|\Phi h(x,s)-F'(s)|\leqslant\varepsilon$ for all x in $[a,b]\setminus\{s\}$. Consequently, h is differentiable at s.

Step II. In order to make the above applicable we construct a closed set $B\subset\boldsymbol{R}$ and a continuous function $G\colon B\to\boldsymbol{R}$ such that

$B\supset A$; $F=G|_A$; $G\in\mathrm{Lip}(B)$ if $F\in\mathrm{Lip}(A)$;
G is differentiable at all points of S;
every point of S is a two-sided accumulation point of B.

(If this can be accomplished, the lemma is proved by applying Step I to B and G.)

Let $S^+=\{s\in S\colon s\notin \mathrm{clo}\,A\cap(s,\infty)\}$, $S^-=\{s\in S\colon s\notin\mathrm{clo}\,A\cap(-\infty,s)\}$. As $S\subset A^*$ we have $S^+\cap S^-=\emptyset$.

For $s\in S^+\cup S^-$ choose positive numbers α_s and β_s such that

if $s\in S^+$, then $(s,s+\alpha_s)\subset A^c$,
if $s\in S^-$, then $(s-\alpha_s,s)\subset A^c$,

$3\beta_s\leqslant\alpha_s$, $\beta_s\leqslant\alpha_s^3$,
if $x\in A$ and $|x-s|\leqslant\beta_s$, then $|F(x)-F(s)|\leqslant\alpha_s^2$

and define a set B_s by

$$B_s=\{2s-x\colon x\in A,\ |x-s|\leqslant\beta_s\}.$$

If $s\in S^+$, then $B_s\subset[s,s+\beta_s]\subset[s,s+\alpha_s)$; if $s\in S^-$, then $B_s\subset(s-\alpha_s,s]$. Therefore, $B_s\cap A=\{s\}$ for all $s\in S^+\cup S^-$.

By distinguishing special cases and exploiting the relation $3\beta_s\leqslant\alpha_s$ one obtains

$$\text{(1)}\quad \begin{array}{l}\text{if } s\in S^+\cup S^-,\ x\in B_s,\ a\in A, \text{ then } |x-a|\geqslant\tfrac{2}{3}|s-a|,\\ \text{if } s,t\in S^+\cup S^-,\ x\in B_s,\ y\in B_t, \text{ then } |x-y|\geqslant\tfrac{1}{3}|s-t|.\end{array}$$

Thus, the sets B_s $(s\in S^+\cup S^-)$ are pairwise disjoint.

Define

$$B=A\cup\bigcup\{B_s\colon s\in S^+\cup S^-\}.$$

One easily proves that B is closed in $\boldsymbol{R}$ and that all points of S are two-sided accumulation points of B.

For $x\in B$ we can define $\bar{x}\in A$ by

$\bar{x}=x$ if $x\in A$,
$\bar{x}=s$ if $s\in S^+\cup S^-$, $x\in B_s$.

($\bar{x}$ is the point of A that is closest to x.) Formula (1) yields

$$\text{(2)}\qquad |\bar{x}-\bar{y}|\leqslant 3|x-y|\quad\text{for all } x,y\in B.$$

If $x\in B$, then both $\bar{x}$ and $2\bar{x}-x$ lie in A. Thus, we can define a function $G\colon B\to\boldsymbol{R}$ by

$$G(x)=2F(\bar{x})-F(2\bar{x}-x)\quad(x\in B).$$

Then G is an extension of F. It follows from (2) that G is continuous and that $G\in\mathrm{Lip}(B)$ in case $F\in\mathrm{Lip}(A)$. It remains to prove the differentiability of G at the points of S.

Take $c\in S$. We show that $\lim\limits_{\substack{x\downarrow c\\ x\in B}}\Phi G(x,c)=F'(c)$ leaving $\lim\limits_{x\uparrow c}$ to the reader.

If $c\in S^+$, then B_c is a right neighborhood of c in B, and for all x in B_c we have $G(x)=2F(c)-F(2c-x)$; then certainly $\lim\limits_{\substack{x\downarrow c\\ x\in B}}\Phi G(x,c)=F'(c)$. Thus, we may assume $c\notin S^+$.

Let $\varepsilon>0$. There exists a positive δ such that

$$\text{(3)}\quad \text{if } x\in A,\ x\neq c,\ |x-c|\leqslant\delta, \text{ then } |F(x)-F(c)|\leqslant 1 \text{ and } |\Phi F(x,c)-F'(c)|\leqslant\varepsilon.$$

As $c\notin S^+$ we can choose δ such that $c+\delta\in A$.

Let $x\in B\cap(c,c+\delta]$; we are done if we can prove

$$(*)\qquad |\Phi G(x,c)-F'(c)|\leqslant 3\varepsilon+8\delta.$$

Without restriction, let $x\notin A$. (See (3).) Then there is a unique s in $S^+\cup S^-$ with $x\in B_s\setminus\{s\}$ (viz. $s=\bar{x}$). We distinguish two cases, $s\in S^-$ and $s\in S^+$.

Case 1. $s \in S^-$. We have $x < s$, $(x, s) \subset (s-\beta_s, s) \subset (s-\alpha_s, s) \subset A^c$, $c+\delta \in A$ and $c+\delta \geqslant x$; it follows that $c+\delta \geqslant s > x > c$. Then according to (3),

$$(4) \qquad |F(s)-F(c)| \leqslant 1 \quad \text{and} \quad |\Phi F(s, c)-F'(c)| \leqslant \varepsilon.$$

From this we deduce (*), observing that $|x-s|$ and $|G(x)-G(s)|$ are very small compared to $s-c$. One easily verifies

$$\begin{aligned}(x-c)(s-c)|\Phi G(x, c)-\Phi F(s, c)| &= |(s-c)(G(x)-G(s))+(s-x)(G(s)-G(c))| \\ &\leqslant (s-c)|G(x)-G(s)|+(s-x)|G(s)-G(c)| \\ &= (s-c)|F(s)-F(2s-x)|+(s-x)|F(s)-F(c)| \\ &\leqslant (s-c)|F(s)-F(2s-x)|+(s-x).\end{aligned}$$

Now $|x-s| \leqslant \beta_s$, whence, by our choice of β_s, $|F(s)-F(2s-x)| \leqslant \alpha_s^2$. Furthermore, as $c < s$ and $c \in A$, we have $s-c \geqslant \alpha_s$ and $x-c = (s-c)-(s-x) \geqslant \alpha_s-\beta_s \geqslant \frac{1}{2}\alpha_s$. Thus,

$$\begin{aligned}|\Phi G(x, c)-\Phi F(s, c)| &\leqslant (x-c)^{-1}\cdot\alpha_s^2+(x-c)^{-1}(s-c)^{-1}(s-x) \\ &\leqslant 2\alpha_s^{-1}\cdot\alpha_s^2+2\alpha_s^{-1}\cdot\alpha_s^{-1}\cdot\beta_s \leqslant 4\alpha_s \leqslant 4\cdot 2(x-c) \leqslant 8\delta.\end{aligned}$$

Now (*) follows from (4).

Case 2. $s \in S^+$.

We have $x \in (s, s+\beta_s) \subset (s, s+\alpha_s) \subset A^c$, $c < x \leqslant c+\delta$, $c \in A$ and $c+\delta \in A$, so $c \leqslant s$ and $s+\beta_s \leqslant c+\delta$. But $c \neq s$, since $s \in S^+$ and $c \notin S^+$. Thus, we have $c < s < x < s+\beta_s \leqslant c+\delta$.

In particular, $c < s \leqslant c+\delta$. Then (3) implies

$$(5) \qquad |\Phi F(s, c)-F'(c)| \leqslant \varepsilon.$$

But also $|(2s-x)-c| = |(s-x)+(s-c)| \leqslant |s-x|+|s-c| = x-c \leqslant \delta$, so that, again by (3),

$$|\Phi F(2s-x, c)-F'(c)| \leqslant \varepsilon \quad \text{if } 2s-x \neq c.$$

If $2s-x \neq c$, then by substitution one finds

$$\Phi G(x, c)-F'(c) = 2\frac{s-c}{x-c}\big(\Phi F(s, c)-F'(c)\big)-\frac{2s-x-c}{x-c}\big(\Phi F(2s-x, c)-F'(c)\big),$$

whence, recalling $c < s < x$ and $|2s-x-c| \leqslant x-c$,

$$|\Phi G(x, c)-F'(c)| \leqslant 2\cdot|\Phi F(s, c)-F'(c)|+|\Phi F(2s-x, c)-F'(c)| \leqslant 3\varepsilon.$$

If $2s-x = c$, then it turns out that $\Phi G(x, c) = \Phi F(s, c)$, so that $|G(x, c)-F'(c)| \leqslant \varepsilon$. In any case, we have (*). ∎

If $f\colon X \to \boldsymbol{R}$ and $\mathscr{F}$ is a set of functions, then by "f has a primitive in $\mathscr{F}$" we mean that there exists a g in $\mathscr{F}$ such that $g'(x) = f(x)$ for all $x \in X$.

The restriction of a function f to a subset Y of its domain is denoted by: $f\,|\,Y$.

LEMMA 5.3. *Let $X \subset \boldsymbol{R}$, $f \in \mathscr{D}'(X)$. Then there exists a covering of X by countably many relatively closed subsets $X_1, X_2, \ldots$ such that for each n, $f\,|\,X_n$ has a primitive in* $\mathrm{Lip}(\boldsymbol{R})$.

Proof. Choose $F \in C(X)$ with $F' = f$ in X^*.

For $n \in N$, let $Z_n = \{a \in X\colon$ if $x \in X \cap (a, a+n^{-1})$, then $|\Phi F(x, a)| \leqslant n\}$. Then $(Z_n)_{n\in N}$ is a covering of X by relatively closed subsets. For each n write Z_n as a union of countably many relatively closed subsets $Z_{n1}, Z_{n2}, \ldots$ with diameters smaller than n^{-1}. Then $F|Z_{ni} \in \mathrm{Lip}(Z_{ni})$ for every n and i.

We see from this that *there is a covering $(X_n)_{n\in N}$ of X by bounded relatively closed subsets such that $F|X_n \in \mathrm{Lip}(X_n)$ for all n*. Furthermore, each X_n is a union of a countable set and a relatively closed subset without isolated points. From all this, it follows that we may assume that $F \in \mathrm{Lip}(X)$ and that either X is a singleton set or $X = X^*$. The first case being trivial, let us suppose $X = X^*$. By uniform continuity, F extends to a function $\bar{F}$ in $\mathrm{Lip}(\bar{X})$. The graph of $\bar{F}$ is the closure of the graph of F, so that $\bar{F}$ is differentiable at all points of X. Now apply Lemma 5.2. ∎

This brings us to the central theorem of the present section.

THEOREM 5.4. *Let $X \subset \boldsymbol{R}$ have empty interior; let $f\colon X \to \boldsymbol{R}$. Suppose there is a covering $(X_n)_{n\in N}$ of X by relatively closed subsets such that $f\,|\,X_n \in \mathscr{D}'(X_n)$ for every n. Then $f \in \mathscr{D}'(X)$.* (The condition that X have empty interior is necessary.)

Proof. By Lemma 5.3 we may assume that there exist $h_1, h_2, \ldots$ in $\mathrm{Lip}(\boldsymbol{R})$ such that $h_n' = f$ on X_n $(n \in N)$. Furthermore, suppose $X_1 \neq \emptyset$.

For $n \in N$, let $\psi_n(x) = \mathrm{dist}(\{x\}, X_1 \cup \ldots \cup X_n)$ $(x \in X)$. By Lemma 5.1 we can find $j_1, j_2, \ldots \in C(X)$ with, for every n,

$$\begin{aligned}&j_1 = 0, \\ &|j_{n+1}-(h_n+j_n-h_{n+1})| \leqslant 2^{-n}\psi_n^2, \\ &j_{n+1}' = 0 \text{ on } X^*\setminus(X_1 \cup \ldots \cup X_n).\end{aligned}$$

For $n \in N$, let $g_n = h_n+j_n$. Then $g_n \in C(X)$, $g_n' = h_n' = f$ on

$$X^* \cap X_n\setminus(X_1 \cup \ldots \cup X_{n-1})$$

and $|g_{n+1}-g_n| \leqslant 2^{-n}\psi_n^2$. In particular, $g_{n+1} = g_n$ on $X_1 \cup \ldots \cup X_n$, so there is a $g\colon X \to \boldsymbol{R}$ with

$$g = g_n \quad \text{on } X_1 \cup \ldots \cup X_n \quad (n \in N).$$

Actually, $g = \lim\limits_{n\to\infty} g_n$ pointwise on X, so that for every n,

$$|g-g_n| \leqslant \sum_{k=n}^{\infty}|g_{k+1}-g_k| \leqslant \sum_{k=n}^{\infty}2^{-k}\psi_k^2 \leqslant \sum_{k=n}^{\infty}2^{-k}\psi_n^2 \leqslant \psi_n^2.$$

From this, we prove that $g' = f$ on X^*.

Let $a \in X^*$, $\varepsilon > 0$. There is a smallest n with $a \in X_n$. There is a $\delta \in (0, \frac{1}{2}\varepsilon]$ such that $|g_n(x) - g_n(a) - f(a)(x-a)| \leqslant \frac{1}{2}\varepsilon|x-a|$ as soon as x lies in $X \cap (a-\delta, a+\delta)$. Then for such x we have

$$|g(x)-g(a)-f(a)(x-a)| \leqslant |g(x)-g_n(x)| + |g_n(x)-g_n(a)-f(a)(x-a)|$$
$$\leqslant \psi_n(x)^2 + \tfrac{1}{2}\varepsilon|x-a| \leqslant |x-a|^2 + \tfrac{1}{2}\varepsilon|x-a| \leqslant (\delta + \tfrac{1}{2}\varepsilon)|x-a| \leqslant \varepsilon|x-a|.$$

Thus, $g'(a) = f(a)$. ■

For the time being we need only a very special case:

COROLLARY 5.5. *Let $X \subset \mathbf{R}$ have empty interior. Let $f\colon X \to \mathbf{R}$. Suppose we have relatively closed subsets $X_1, X_2, \ldots$ of X whose union is X and on each of which f is constant. Then $f \in \mathscr{D}'(X)$.* ■

Obviously, instead of "closed subsets" one may read "F_σ-subsets".

COROLLARY 5.6. *Let $X \subset \mathbf{R}$ have empty interior. For a subset A of X the conditions (α), (β), (γ) are equivalent.*

(α) $1_A \in \mathscr{D}'(X)$.
(β) $1_A \in \mathscr{B}^1(X)$.
(γ) *A is both F_σ and G_δ in X.*

(The conditions are satisfied if A is either closed or open in X.)

Proof. We have proved $(\alpha) \Rightarrow (\beta)$ in Theorem 2.1. For $(\beta) \Rightarrow (\gamma)$, see [1], Th. 4, p. 142. If (γ) holds, then A and $X \setminus A$ are F_σ-subsets of X and we can apply Corollary 5.5 to prove (α). ■

THEOREM 5.7. *Let $X \subset \mathbf{R}$ have empty interior. Let $f_1, f_2 \in \mathscr{B}^1(X)$, $f_1(x) < f_2(x)$ for every $x \in X$. Then there exists an f in $\mathscr{D}'(X)$ with $f_1 \leqslant f \leqslant f_2$.*

Proof. For $r \in \mathbf{Q}$, put $A(r) = \{x \in X\colon f_1(x) < r < f_2(x)\}$. Every $A(r)$ is an F_σ-subset of X. ([1], Th. 4, p. 142.) The union of the sets $A(r)$ is X. Thus, we can find a sequence $Y_1, Y_2, \ldots$ of closed subsets of X and a sequence $r_1, r_2, \ldots$ of rational numbers such that

$$X = Y_1 \cup Y_2 \cup \ldots,$$
$$Y_n \subset A(r_n) \quad (n \in N).$$

Define $f\colon X \to \mathbf{R}$ by

$$f(x) = r_n \quad \text{if } n \in N,\ x \in X_n := Y_n \setminus (Y_1 \cup \ldots \cup Y_{n-1}).$$

Then $f_1 \leqslant f \leqslant f_2$, and $f \in \mathscr{D}'(X)$ by Cor. 5.5, each X_n being an F_σ in X. ■

In particular, *if $X \subset \mathbf{R}$ has empty interior, then $\mathscr{D}'(X)$ is a uniformly dense subset of $\mathscr{B}^1(X)$.* Compare Theorem 2 in [4].

We obtain an unexpected bonus. (By an *interval of X* we mean a non-empty set that is the intersection of X with an open interval of $\mathbf{R}$.)

THEOREM 5.8. *Let $X \subset \mathbf{R}$ and assume that X is of the first category in itself* (so that $X^* = X$ and $\ddot{X} = \emptyset$.) *Then there exists an f in $C(X)$ with*

$f' = 1$ on X,
f is increasing on no interval of X.

(The converse is also true and much easier to prove.)

Proof. Cover X by countably many relatively closed subsets $X_1, X_2, \ldots$ that have empty interior relative to X. For every n, $\overline{X_n}$ has empty interior in $\overline{X}$; then so has $A := \overline{X_1} \cup \overline{X_2} \cup \ldots$ Let Y be a countable dense subset of $\overline{X} \setminus A$. Then $\overline{Y} = \overline{X}$.

$A \cup Y$ has empty interior in $\mathbf{R}$ and both A and Y are F_σ-subsets of $A \cup Y$. By Corollary 5.5 and by the fact that $A \cup Y$ has no isolated points we have a $g \in C(A \cup Y)$ for which $g' = 1_A$ on $A \cup Y$. Let $f = g|X$. Then $f' = 1$ on X.

Suppose f is increasing on some interval $(a, b) \cap X$ of X. As g is continuous and $A \cup Y \subset \overline{X}$, g is increasing on $(a, b) \cap (A \cup Y)$. Then $g' \geqslant 0$ on $(a, b) \cap (A \cup Y)$, so $\emptyset = (a, b) \cap (A \cup Y) \cap Y = (a, b) \cap Y$. It follows that $\emptyset = (a, b) \cap \overline{Y} \supset (a, b) \cap X$. Contradiction. ■

In the same vein we have:

THEOREM 5.9. *If X is as above, then there exists an f in $\mathscr{D}'(X)$ that is unbounded on every interval of X and hence is nowhere continuous.*

(If X is not of the first category in itself, then all $\mathscr{B}^1$-functions on X have continuity points. See [2], Th. 6, p. 109.)

Proof. Cover X by relatively closed subsets $X_1, X_2, \ldots$ each of which has empty interior in X, and put $Y_n = X_n \setminus (X_1 \cup \ldots \cup X_{n-1})$ $(n \in N)$. Define

$$f(x) = n \quad \text{if } x \in Y_n \ (n \in N).$$

Each Y_n being an F_σ in X we have $f \in \mathscr{D}'(X)$ by Corollary 5.5.

No interval of X is contained in a union of finitely many of the sets X_n. Consequently, f is unbounded on every interval of X. ■

6. The structure of $\mathscr{D}'(X)$. As is known, on an interval the product of a derivative and a continuous function may fail to be a derivative. ([1], p. 17.) For sets with empty interior we prove that this is false. (Th. 6.1.)

Let $X \subset \mathbf{R}$. In the proof of Theorem 2.2 ([7]) we have seen that, if $f \in C(X)$ is bounded, then f can be extended to an upper semicontinuous function on $\mathbf{R}$ that is continuous at all points of X. It follows that any f in $C(X)$ has an upper semicontinuous extension $f_1\colon \mathbf{R} \to [-\infty, \infty]$ that is continuous at the points of X. Then, setting

$$f_2(x) = f_1(x) \quad \text{if } f_1(x) \text{ is finite},$$
$$f_2(x) = 0 \quad \text{otherwise},$$

we have a measurable function $f_2\colon \mathbf{R} \to \mathbf{R}$, extending f and continuous at the points of X.

We say that a function $f\colon X\to \mathbf{R}$ belongs to the set $\mathscr{A}(X)$ if f can be extended to a measurable function $\mathbf{R}\to\mathbf{R}$ that is approximately continuous at all points of X. (See [1], p. 18.) By the above,

$$C(X)\subset\mathscr{A}(X).$$

Furthermore,

$$\mathscr{A}(X)\subset\mathscr{B}^1(X).$$

Indeed, let $f\in\mathscr{A}(X)$. The function $\arctan\circ f$ extends to a bounded measurable function $j\colon \mathbf{R}\to\mathbf{R}$ that is approximately continuous at all points of X. If J is an indefinite integral of j, then $J'=j=\arctan\circ f$, so $\arctan\circ f\in\mathscr{B}^1(X)$ and $f\in\mathscr{B}^1(X)$.

Observe that $\mathscr{A}(X)$ is closed for uniform convergence. ([1], 5.7, p. 24.)

THEOREM 6.1. *Let $X\subset\mathbf{R}$, $\mathring{X}=\emptyset$. Then*

$$\mathscr{A}(X)\,\mathscr{D}'(X)\subset\mathscr{D}'(X).$$

In particular,

$$C(X)\,\mathscr{D}'(X)\subset\mathscr{D}'(X)$$

and

$$\mathscr{A}(X)\subset\mathscr{D}'(X).$$

(The condition $\mathring{X}=\emptyset$ is necessary: see [1], 5.5.(c), p. 21.)

Proof. Let $f\in\mathscr{A}(X)$, $g\in\mathscr{D}'(X)$. Then $f\in\mathscr{B}^1(X)$ and there exists a covering of X by countably many relatively closed subsets $X_1, X_2\dots$ such that for each n

f is bounded on X_n,

$g|X_n$ has a primitive in $\mathrm{Lip}(\mathbf{R})$. (5.3.)

By Theorem 5.4 we may assume that f is bounded on all of X and that g itself has a primitive, G, in $\mathrm{Lip}(\mathbf{R})$. Then G' is an a.e. defined measurable function and G is an indefinite integral of G'.

Extend f to a bounded measurable function $\bar f\colon \mathbf{R}\to\mathbf{R}$ that is approximately continuous at the points of X. Let H be an indefinite integral of the locally integrable function $\bar f G'$. We claim that $H'=fg$ on X.

Take $c\in X$; we prove $\lim_{x\downarrow c}\Phi H(x,c)=f(c)g(c)$. Indeed, for $x>c$,

$$\begin{aligned}|\Phi H(x,c)-f(c)g(c)| &= |(x-c)^{-1}\int_c^x \bar f(t)G'(t)\,dt-f(c)g(c)|\\ &\leqslant (x-c)^{-1}|\int_c^x(\bar f(t)-f(c))G'(t)\,dt|+|f(c)|\cdot|(x-c)^{-1}\int_c^x G'(t)\,dt-g(c)|\\ &\leqslant \|G'\|\cdot(x-c)^{-1}\int_c^x|\bar f(t)-\bar f(c)|\,dt+|f(c)|\cdot|\Phi G(x,c)-g(c)|\,.\end{aligned}$$

The first term in the last line vanishes as x tends to c since $\bar f$ is bounded and approximately continuous at c. ([1], 5.5, p. 25.) ■

Let X be as above and let $f\colon X\to\mathbf{R}$. The condition $f\in\mathscr{A}(X)$ is sufficient to guarantee $f\mathscr{D}'(X)\subset\mathscr{D}'(X)$ but in general it is not necessary. In fact, it is enough to have a countable cover of X by relatively closed subsets $X_1, X_2, \dots$ such that for each n $f|X_n\in\mathscr{A}(X_n)$. Thus, f could be $1_{\{a\}}$.

It is reasonable to ask whether the condition may be weakened further to $f\in\mathscr{D}'(X)$; in other words, whether $\mathscr{D}'(X)$ is a ring. It is, of course, whenever $\mathscr{D}'(X)=\mathscr{B}^1(X)$, which is the case if X is negligible. We now show that it is *not* if the inner measure of X is positive. This leaves a gap: if the inner measure of X is 0 but the outer measure is not, then we do not know if $\mathscr{D}'(X)$ is a ring. Interestingly, this is just the situation where we do not know if $\mathscr{D}'(X)$ is the same as $\mathscr{B}^1(X)$. (See [7], 4.5.)

This leads us to considering a remarkable result of D. Preiss ([6]) who proved that $\mathscr{B}^1(\mathbf{R})=\mathscr{D}'(\mathbf{R})\mathscr{D}'(\mathbf{R})+\mathscr{D}'(\mathbf{R})$: if, more generally, the identity

$$\mathscr{B}^1(X)=\mathscr{D}'(X)\mathscr{D}'(X)+\mathscr{D}'(X)$$

is valid for *all* sets X, then the conditions "$\mathscr{D}'(X)$ is a ring" and "$\mathscr{D}'(X)=\mathscr{B}^1(X)$" are equivalent for all X.

LEMMA 6.2. *Define the function $s\colon \mathbf{R}\to\mathbf{R}$ by*

$$s(x)=\sin x^{-1}\quad \text{if } x\neq 0;\ s(0)=0\,.$$

Let $a_1, a_2, \dots\in\mathbf{R}$ be pairwise distinct; let $\varepsilon_1, \varepsilon_2, \dots$ be positive, $\sum_{n=1}^{\infty}\varepsilon_n<\infty$. Define

$$f(x)=\sum_{n=1}^{\infty}\varepsilon_n s(x-a_n)\qquad (x\in\mathbf{R})\,.$$

Then $f\in\mathscr{D}'(\mathbf{R})$. Let F be an indefinite integral of f^2. Then F is differentiable and

$$F'=f^2+\tfrac{1}{2}\sum_{n=1}^{\infty}\varepsilon_n^2 1_{\{a_n\}}\,.$$

Proof. $f\in\mathscr{D}'(\mathbf{R})$ as $s\in\mathscr{D}'(\mathbf{R})$ and $\mathscr{D}'(\mathbf{R})$ is uniformly closed. ([3], p. 92.)

f^2 is a bounded $\mathscr{B}^1$-function and therefore has indeed an indefinite integral $F\colon \mathbf{R}\to\mathbf{R}$. If $a\in\mathbf{R}\setminus\{a_1, a_2, \dots\}$, then f^2 is continuous at a so that $F'(a)=f^2(a)$. Now let $n\in\mathbf{N}$, $a=a_n$; we prove $F'(a)=f^2(a)+\frac{1}{2}\varepsilon_n^2$.

Put $g(x)=\varepsilon_n s(x-a_n)$ $(x\in\mathbf{R})$, $h=f-g$. For $x\neq a$ we have (λ being Lebesgue measure)

$$\begin{aligned}\Phi F(x,a) &= (x-a)^{-1}\int_a^x f^2\,d\lambda\\ &= (x-a)^{-1}\int_a^x\big(2h(a)g+g^2+(2g+h+h(a))(h-h(a))+h(a)^2\big)\,d\lambda\,.\end{aligned}$$

Now

$$\lim_{x\to a}(x-a)^{-1}\int_a^x g\,d\lambda = 0\,,$$

$$\lim_{x\to a}(x-a)^{-1}\int_a^x g^2 d\lambda = \tfrac{1}{2}\varepsilon_n^2\,,$$

h is continuous at a and $2g+h+h(a)$ is bounded, so

$$\lim_{x\to a}\big(2g(x)+h(x)+h(a)\big)\big(h(x)-h(a)\big) = 0\,.$$

Consequently, $\lim_{x\to a}\Phi F(x,a) = 0+\frac{1}{2}\varepsilon_n^2+0+h(a)^2 = \frac{1}{2}\varepsilon_n^2+f(a)^2$, as we wished to prove. ■

COROLLARY 6.3. *Let X be a subset of $\mathbf{R}$ such that $\mathscr{D}'(X)$ is a ring. Then the inner Lebesgue measure of X is* 0.

Proof. It follows from the lemma that, if $a_1, a_2, \ldots \in \mathbf{R}$ are pairwise distinct, then

$$\sum_{n=1}^{\infty} 2^{-n} 1_{\{a_n\}} \in \mathscr{D}'(X)\,.$$

Then according to Lemma 4.3 of [7] X cannot contain a Baire space Y with $Y\subset \overline{Y_d}$. As we have seen in 4.5 and in the proof of 4.4, this implies that the inner measure of X must be 0. ■

References

[1] A. M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Mathematics 659, Springer-Verlag, Berlin–Heidelberg–New York 1978.

[2] C. Goffman, *Real Functions*, Rinehart & Co., Inc, New York 1957.

[3] H. Lebesgue, *Leçons sur l'intégration*, Chelsea Publ. Co., New York 1973.

[4] G. Petruska, *Separation and Approximation Theorems on Derivatives*, Acta Math. Acad. Sci. Hung. 25 (1974), 435–442.

[5] G. Petruska and M. Laczkovich, *Baire 1 Functions, Approximately Continuous Functions and Derivatives*, Acta Math. Acad. Sci. Hung. 25 (1974), 189–212.

[6] D. Preiss, *Algebra Generated by Derivatives*, Real Analysis Exchange 8 (1982–83), 208–216.

[7] A. C. M. van Rooij and W. H. Schikhof, *On Derivatives of Functions Defined on Disconnected Sets*, I, Fund. Math. 131 (1988), 83–92.

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Free subgroups of diffeomorphism groups

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Abstract. It is proved that the group $\mathrm{Diff}^k(X)$ of all C^k-diffeomorphisms of a given C^k-manifold X ($k = 1, 2, \ldots, \infty$) includes a non-trivial arcwise connected with respect to the Whitney C^k-topology free subgroup which consists (except for the identity) of diffeomorphisms which embed in no flow.

It is also proved that for each sequence of elements of $\mathrm{Diff}^k(X)$ there are diffeomorphisms arbitrarily close to the given ones which freely generate a subgroup in $\mathrm{Diff}^k(X)$.

0. Introduction. The fundamental concept of the Lie theory is to investigate topological groups by means of their one-parameter subgroups. For a classical (i.e. finite-dimensional) Lie group G, the set $L(G)$ of all one-parameter subgroups has a natural Lie algebra structure. Moreover, there is a one-one correspondence between arcwise connected subgroups of G and Lie subalgebras of $L(G)$.

The group $\mathrm{Diff}_c^\infty(X)$ of all compactly supported C^∞-diffeomorphisms of a C^∞-manifold X is a well-known model of an "infinite-dimensional Lie group" with the Lie algebra $\Gamma_c^\infty(X)$ of compactly supported C^∞-vector fields.

Despite of some resemblances, this group fails to have some properties of classical Lie groups, e.g. the image of the exponential map includes no neighbourhood of the identity (cf. [2], [4], [6]).

The main aim of this note is to show that the situation is even worse, namely that there are nontrivial arcwise connected free subgroups of $\mathrm{Diff}_c^\infty(X)$ consisting of diffeomorphisms (except for the identity, of course) which embed in no flow.

The main part of the proof can be found in § 3 of this note and the idea of the proof is the following (reduced to the case $X = \mathbf{R}$).

Denote by ${}^r\Phi$ a free group with r-generators. Elements of ${}^r\Phi$ can be represented by "words" $\varphi = A_{i_n}^{j_n} \ldots A_{i_1}^{j_1}$, where $i_k \in \{1, \ldots, r\}$, $j_k = \pm 1$, $j_k \neq -j_{k+1}$ providing $i_k = i_{k+1}$, and "the empty word" **1**. Set $|\varphi|$ to be the length of the "word" $\varphi \in {}^r\Phi$ and write ${}_n^r\Phi = \{\varphi \in {}^r\Phi : |\varphi| \leqslant n\}$. For a group G and $g_1, \ldots, g_r \in G$, we have the natural homomorphism ${}^r\Phi \ni \varphi \mapsto \varphi(g_1, \ldots, g_r) \in G$ given by replacing $A_{i_k}^{j_k}$ by $g_{i_k}^{j_k}$.

Let $G = \mathrm{Diff}^\infty[0, 1]$ be the group of all C^∞-diffeomorphisms of the closed interval $[0, 1]$. By a theorem of Kopell [4] there are $g \in G$ arbitrarily close to the identity in the natural C^∞-topology such that g^n embeds in no flow and has no fixed points in $(0, 1)$ for $n = 1, 2, \ldots$ We shall call them *Kopell-diffeomorphisms*.