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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

On derivatives of functions defined on disconnected sets, I

by

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Abstract. Let X be a subset of $\boldsymbol{R}$ without isolated points, let $\mathscr{B}^1(X)$ be the first class of Baire on X and let $\mathscr{D}'(X)$ be the set of all functions $X\to\boldsymbol{R}$ that have primitives on X. Then $C(X)\subset\mathscr{D}'(X)\subset\mathscr{B}^1(X)$. For a Lebesgue measurable set X it is shown that $\mathscr{D}'(X)=\mathscr{B}^1(X)$ if and only if X is negligible.

1. Introduction. Let X be a subset of $\boldsymbol{R}$. Differentiation of a function $f\colon X\to\boldsymbol{R}$ usually is considered only in case X is a (closed or open) interval of $\boldsymbol{R}$. The definition, however, makes very good sense in more general situations. Whenever a is a non-isolated point of X one can define $f'(a)$ to be $\lim_{x\to a}(x-a)^{-1}(f(x)-f(a))$, if only the limit exists. Thus one obtains a function f', the "derivative" of f, defined on a subset of X.

The purpose of the present paper is to study, for a given set X, the space of all functions on X that are derivatives. In particular we investigate the relations between $\mathscr{D}'(X)$, the space of all derivatives on X, and $\mathscr{B}^1(X)$, the first class of Baire on X. (A function $X\to\boldsymbol{R}$ belongs to $\mathscr{B}^1(X)$ if it is the pointwise limit of a sequence of continuous functions $X\to\boldsymbol{R}$.) If X is an interval, it is elementary that $\mathscr{D}'(X)$ is a proper subset of $\mathscr{B}^1(X)$. It is not overly difficult to show that the inclusion $\mathscr{D}'(X)\subset\mathscr{B}^1(X)$ is valid for every set $X\subset\boldsymbol{R}$ that has no isolated points (Th. 2.1), but there are non-trivial cases in which the inclusion is not proper. (E.g., it follows from our Th. 3.4 that $\mathscr{D}'(\boldsymbol{Q})$ contains *all* functions on $\boldsymbol{Q}$). Our basic question is: *for what sets X do we have $\mathscr{D}'(X)=\mathscr{B}^1(X)$?*

If $X\subset\boldsymbol{R}$ and if a is an interior point of X, then the characteristic function of $\{a\}$ is an obvious example of a function belonging to $\mathscr{B}^1(X)$ but not to $\mathscr{D}'(X)$; as a consequence, we are mainly interested in sets X that have no interior points. For such an X our question is non-trivial. For Lebesgue measurable sets we give an answer in Th. 4.4.

Our techniques and results are similar to the ones of G. Petruska and M. Laczkovich in [5]; there, however, the basic problem is how to determine the sets X for which $\mathscr{D}'(\boldsymbol{R})$ and $\mathscr{B}^1(\boldsymbol{R})$ have the same restrictions to X.

In Part II of this paper we study the structure of $\mathscr{D}'(X)$. (See [7].)

Historical remark. It may be worth observing that our central question has been answered first in p-adic analysis. In fact, the following theorem is proved in [8]. *Let X be a subset of the field Q_p of the p-adic numbers. Suppose that X has no isolated points. Let $\mathscr{D}'(X)$ be the set of all derivative functions $X \to Q_p$ and let $\mathscr{B}^1(X)$ be the set of all pointwise limits of sequences of continuous functions $X \to Q_p$. Then $\mathscr{D}'(X) = \mathscr{B}^1(X)$.*

This result has been one of the starting points for the investigations laid down in the present paper.

Notations and elementary facts. Let $X \subset \boldsymbol{R}$. The closure, the interior and the complement of X *relative to* $\boldsymbol{R}$ are denoted $\overline{X}$, $\mathring{X}$ and X^c, respectively. (Instead of $Y \cap X^c$ we also write $Y \setminus X$.) X^* is the set of all non-isolated points of X.

$C(X)$ is the space of all continuous functions $X \to \boldsymbol{R}$, $\mathscr{B}^1(X)$ is the first class of Baire on X.

Let X be the domain of a function g; let $a \in X^*$. If

$$\lim_{\substack{x \to a \\ x \in X}} \frac{g(x)-g(a)}{x-a}$$

exists, we say that g is *differentiable at* a and denote by $g'(a)$ the value of the limit. Thus we get a function g' whose domain is a subset of X^*. If $A \subset X^*$ and if f is a function defined on a set that contains A, then by

$$g' = f \quad \text{on } A$$

we mean that $g'(a) = f(a)$ for all $a \in A$. In a similar way we use expressions like

$$|g'-f| \leqslant 1 \quad \text{on } A.$$

If $g\colon X \to \boldsymbol{R}$, $a \in X^*$ and g is differentiable at a, and if $Y \subset X$ and $a \in Y^*$, then the restriction function $g|Y$ is differentiable at a and $(g|Y)'(a) = g'(a)$. (Keep in mind the difference between $g'|Y$ and $(g|Y)'$.)

If X is the domain of g and g is differentiable at all points of X^*, then g is continuous.

By $\mathscr{D}'(X)$ we denote the set of all functions $f\colon X \to \boldsymbol{R}$ for which there exists a $g\colon X \to \boldsymbol{R}$ with $g' = f$ on X^*. Thus, if X is discrete, then *every* function on X belongs to $\mathscr{D}'(X)$.

For $f\colon X \to \boldsymbol{R}$ we define a function Φf on $\{(x, y) \in X \times X\colon x \neq y\}$ by:

$$\Phi f(x) = \frac{f(y)-f(x)}{y-x}.$$

f is a *Lipschitz function* if Φf is bounded. The Lipschitz functions on X form a set called $\mathrm{Lip}(X)$.

For $f\colon X \to \boldsymbol{R}$ we put $\|f\| = \sup\{|f(x)|\colon x \in X\}$. (Possibly, $\|f\| = \infty$.) Similarly, $\|\Phi f\| := \sup\{|\Phi f(x, y)|\colon x, y \in X,\ x \neq y\}$.

The characteristic function of a set A is denoted 1_A. We will not keep track of the domain of definition of such a function: $1_{[0,1]}$ may have as its domain $\boldsymbol{R}$ or $[0, \infty)$ or, indeed, any subset of $\boldsymbol{R}$.

Our measure-theoretic terms (such as "negligible", "outer measure") always refer to the Lebesgue measure, which is called λ.

2. We prove $C(X) \subset \mathscr{D}'(X) \subset \mathscr{B}^1(X)$ for all $X \subset \boldsymbol{R}$.

THEOREM 2.1. *Let $X \subset \boldsymbol{R}$. Then $\mathscr{D}'(X) \subset \mathscr{B}^1(X)$.*

Proof. (The theorem is an elementary extension of Lemma 5.5.1 of [5], which is the special case $X \subset X^* \subset [0, 1]$.)

Without restriction, let $X \subset [0, 1]$. There exists a countable $Y \subset X$ such that $X \setminus Y$ is closed in X and dense in itself. Let $Y = \{y_1, y_2, \ldots\}$. By [5], Lemma 5.5.1, the restriction of f to $X \setminus Y$ is the pointwise limit of a sequence $g_1, g_2, \ldots$ in $C(X \setminus Y)$. By the Tietze Extension Theorem each g_n has an extension f_n in $C(X)$ with $f_n = f$ on $\{y_1, \ldots, y_n\}$. Then $f = \lim_{n\to\infty} f_n$ pointwise on X. ■

THEOREM 2.2. *Let $X \subset \boldsymbol{R}$, let $f \in C(X)$ be bounded. Then there exists a $g \in C(\boldsymbol{R})$ such that $g' = f$ on X and $\|\Phi g\| = \|f\|$.*

Proof. For $a \in \overline{X}$ define

$$h(a) = \inf_{\varepsilon>0} \sup\{f(x)\colon x \in X \cap (a-\varepsilon, a+\varepsilon)\}.$$

Then $h\colon \overline{X} \to \boldsymbol{R}$ is upper semicontinuous. Let $J_1, J_2, \ldots$ be the components of $\boldsymbol{R} \setminus \overline{X}$. Extend h to a function $j\colon \boldsymbol{R} \to \boldsymbol{R}$ by interpolating linearly on each bounded $\overline{J}_n$ and making j constant on $\overline{J}_n$ if J_n is unbounded. Then j is upper semicontinuous and bounded, hence locally Lebesgue integrable. Let $g\colon \boldsymbol{R} \to \boldsymbol{R}$ be an infinite Lebesgue integral of j. Then $\|\Phi g\| = \|j\| = \|h\| = \|f\|$. If $a \in X$, then h and j are continuous at a, so g is differentiable at a and $g'(a) = j(a) = h(a) = f(a)$. ■

THEOREM 2.3. *Let $X \subset \boldsymbol{R}$. Then $C(X) \subset \mathscr{D}'(X)$.*

Proof. Let $f \in C(X)$. Put $U = \{x \in \boldsymbol{R}\colon$ there exists a positive δ such that $X \cap (x-\delta, x+\delta) \neq \emptyset$ and f is bounded on $X \cap (x-\delta, x+\delta)\}$. This U is an open subset of $\boldsymbol{R}$ containing X.

Let J be a component of U. Choose a family $(a_n)_{n\in\boldsymbol{Z}}$ of elements of J such that $a_n < a_{n+1}$ $(n \in \boldsymbol{Z})$ and $J = \bigcup_{n\in\boldsymbol{Z}} [a_n, a_{n+1}]$. For each n, the restriction of f to $[a_n, a_{n+1}]$ is bounded. By applying the previous theorem to each of these restrictions one easily finds a function $g_J\colon J \to \boldsymbol{R}$ with $g_J' = f$ on $J \cap X$.

Carrying out this construction for every component of U one obtains a function $g_U\colon U \to \boldsymbol{R}$ with $g_U' = f$ on X. Let g be the restriction of g_U to X. Then $g' = f$ on X^*. ■

Unless X is discrete, $\mathscr{D}'(X)$ always contains discontinuous functions. The classical example for $X = \boldsymbol{R}$ is $f\colon x \to \sin x^{-1}$ $(x \neq 0)$, $f(0) = 0$. From this, by translations and restrictions one obtains examples for every X that contains an interval. For an X with empty interior we have:

THEOREM 2.4. *Let $X \subset \mathbf{R}$, $\mathring{X} = \emptyset$. Then $1_{\{a\}} \in \mathscr{D}'(X)$ for every $a \in X$.*

Proof. Take $a \in X$. For every $n \in N$ choose $p_n, q_n \in X^c$ such that

$$a-n^{-1} < p_n < a-(n+1)^{-1} \text{ and } a+(n+1)^{-1} < q_n < a+n^{-1}.$$

Define

$$g(x) = \begin{cases} a & \text{if } x \in \{a\} \cup (-\infty, p_1) \cup (q_1, \infty), \\ p_n & \text{if } n \in N,\ x \in (p_n, p_{n+1}), \\ q_n & \text{if } n \in N,\ x \in (q_{n+1}, q_n). \end{cases}$$

Then $g'(x) = 1_{\{a\}}(x)$ for every x in X. ∎

3. Our main theorem. In this section we show that $\mathscr{D}'(X) = \mathscr{B}^1(X)$ if X is Lebesgue negligible. Recall that λ is the Lebesgue measure

$$\text{For } A, B \subset \mathbf{R}, \quad \operatorname{dist}(A, B) := \inf\{|a-b|:\ a \in A,\ b \in B\}.$$

LEMMA 3.1. *Let $A \subset \mathbf{R}$ be closed, let $X \subset A^c$ be negligible and let $\varepsilon > 0$. Then there exists an open set $U \subset \mathbf{R}$ such that*

$$X \subset U \subset A^c,$$
$$\lambda(U \cap [s, t]) \leqslant \varepsilon(t-s)^2 \quad \text{if } s \in A,\ t \in [s, \infty),$$
$$\lambda(U \cap [t, s]) \leqslant \varepsilon(s-t)^2 \quad \text{if } s \in A,\ t \in (-\infty, s].$$

Proof. Choose open intervals $J_1, J_2, \ldots$ such that $\bar{J}_n \subset A^c$ for all n whereas $\bigcup_{n \in N} J_n = A^c$. For each n, $\operatorname{dist}(A, J_n) > 0$, so we can choose an open set $U_n \subset \mathbf{R}$ with

$$X \cap J_n \subset U_n \subset J_n,$$
$$\lambda(U_n) \leqslant \varepsilon \cdot 2^{-n} \cdot (\operatorname{dist}(A, J_n))^2.$$

Put $U = \bigcup_{n \in N} U_n$. Of course, U is open and $X \subset U \subset A^c$.

Take $s \in A$, $t \in [s, \infty)$. If $n \in N$ and $U_n \cap [s, t] \neq \emptyset$, then $J_n \cap [s, t] \neq \emptyset$, so $\operatorname{dist}(A, J_n) \leqslant \operatorname{dist}(\{s\}, J_n) \leqslant t-s$. Thus,

$$\begin{aligned} \lambda(U \cap [s, t]) &\leqslant \Sigma\{\lambda(U_n):\ n \in N,\ U_n \cap [s, t] \neq \emptyset\} \\ &\leqslant \Sigma\{\varepsilon \cdot 2^{-n} . (\operatorname{dist}(A, J_n))^2:\ n \in N,\ U_n \cap [s, t] \neq \emptyset\} \\ &\leqslant \Sigma\{\varepsilon \cdot 2^{-n}(t-s)^2:\ n \in N\} = \varepsilon(t-s)^2. \end{aligned}$$

In the same way one shows that $\lambda(U \cap [t, s]) \leqslant \varepsilon(s-t)^2$ for $s \in A$ and

$$t \in (-\infty, s]. \ ∎$$

LEMMA 3.2. *Let $X \subset \mathbf{R}$, $\lambda(X) = 0$. Let $A_0, A_1, \ldots$ be closed subsets of $\mathbf{R}$, $\emptyset = A_0 \subset A_1 \subset \ldots$ and let $c_1, c_2, \ldots \in \mathbf{R}$. Then there exists an absolutely continuous function $g: \mathbf{R} \to \mathbf{R}$ such that*

$$g'(x) = c_n \quad \text{if } n \in N \text{ and } x \in X \cap A_n \cap A_{n-1}^c,$$
$$\|\Phi g\| \leqslant \sup\{|c_n|:\ n \in N\}.$$

In addition, if $c_n \geqslant 0$ for every n, then for g one can choose an increasing function.

Proof. Put $\gamma_n = \sup\{|c_k|:\ k \leqslant n\}$ $(n \in N)$.

The preceding lemma enables us to choose open subsets $U_1, U_2, \ldots$ of $\mathbf{R}$ such that for each n

$$X \cap A_{n-1}^c \subset U_n \subset A_{n-1}^c,$$
$$2\gamma_n \cdot \lambda(U_n \cap [s, t]) \leqslant 2^{-n}(t-s)^2 \quad \text{if } s \in A_{n-1} \text{ and } t \geqslant s,$$
$$2\gamma_n \cdot \lambda(U_n \cap [t, s]) \leqslant 2^{-n}(s-t)^2 \quad \text{if } s \in A_{n-1} \text{ and } t \leqslant s.$$

As each $X \cap A_{n-1}^c$ is negligible, we can choose the U_n in such a way that

$$|c_n| \cdot \lambda(U_n) \leqslant 2^{-n} \quad (n \in N).$$

For $n \in N$, let $V_n = U_n \setminus \bigcup_{k>n} U_k$. As $\Sigma|c_n| \cdot \lambda(V_n)$ is finite, the function $\Sigma c_n 1_{V_n}$ is Lebesgue integrable, hence has an indefinite integral, g, that is absolutely continuous. If $\{c_n:\ n \in N\}$ is bounded, then g is Lipschitz and $\|\Phi g\| \leqslant \sup\{|c_n|:\ n \in N\}$. If $c_n \geqslant 0$ for each n, then g is increasing.

Take $k \in N$, $a \in X \cap A_k \cap A_{k-1}^c$: we prove $g'(a) = c_k$. We have $a \in X \cap A_{k-1}^c \subset U_k$, so there is a positive number δ with $(a-\delta, a+\delta) \subset U_k$. We are done if we can show that

$$|g(x)-g(a)-c_k(x-a)| \leqslant (x-a)^2$$

for all $x \in (a-\delta, a+\delta)$.

Thus, let $x \in [a, a+\delta)$. (The case $x \in (a-\delta, a]$ can be handled similarly.) As $[a, x] \subset [a, a+\delta) \subset U_k \subset V_n^c$ for all $n < k$, we have

$$\begin{aligned} |g(x)-g(a)-c_k(x-a)| &= \Big|\int_a^x \Big(\sum_{n=1}^{\infty} c_n 1_{V_n} - c_k\Big) d\lambda\Big| \\ &= \Big|\int_a^x (c_k 1_{V_k} - c_k) d\lambda + \int_a^x \sum_{n>k} c_n 1_{V_n} d\lambda\Big| \\ &\leqslant |c_k| \cdot \lambda(V_k^c \cap [a, x]) + \sum_{n>k} |c_n| \cdot \lambda(V_n \cap [a, x]) \\ &= |c_k| \cdot \lambda(V_k^c \cap U_k \cap [a, x]) + \sum_{n>k} |c_n| \cdot \lambda(V_n \cap [a, x]) \\ &\leqslant |c_k| \cdot \lambda\Big(\bigcup_{i>k} U_i \cap [a, x]\Big) + \sum_{n>k} |c_n| \cdot \lambda(U_n \cap [a, x]) \\ &\leqslant \sum_{n>k} (|c_k| + |c_n|) \cdot \lambda(U_n \cap [a, x]) \\ &\leqslant \sum_{n>k} 2\gamma_n \cdot \lambda(U_n \cap [a, x]) \\ &\leqslant \sum_{n>k} 2^{-n}(x-a)^2 \leqslant (x-a)^2. \ ∎ \end{aligned}$$

LEMMA 3.3. *Let $X \subset \boldsymbol{R}$ be negligible, $f \in \mathscr{B}^1(X)$, $\|f\| \leqslant 2$. Then there exists an absolutely continuous function $g\colon \boldsymbol{R} \to \boldsymbol{R}$ for which*

$$|f - g'| \leqslant 1 \quad \text{on } X,$$
$$\|\Phi g\| \leqslant 1.$$

Proof. Put $X_{-1} = \{x \in X\colon f(x) < 0\}$, $X_0 = \{x \in X\colon -1 < f(x) < 1\}$ and $X_1 = \{x \in X\colon f(x) > 0\}$. Each X_i is an F_σ-subset of X ([1], Th. 4, p. 142); hence is the intersection of X with an F_σ-subset Y_i of $\boldsymbol{R}$. Choose closed subsets $B_1, B_2, \ldots$ of $\boldsymbol{R}$ with $Y_{-1} = B_1 \cup B_4 \cap B_7 \cup \ldots$, $Y_0 = B_2 \cup B_5 \cup \ldots$ and $Y_1 = B_3 \cup B_6 \cup \ldots$ For each n, choose $c_n \in \{-1, 0, 1\}$ such that $B_n \subset Y_{c_n}$.

By Lemma 3.2, applied to $A_n := B_1 \cup \ldots \cup B_n$, there exists an absolutely continuous function g on $\boldsymbol{R}$ with

$$g'(x) = c_n \quad \text{if } n \in \boldsymbol{N} \text{ and } x \in X \cap B_n \cap (B_1 \cup \ldots \cup B_{n-1})^c,$$
$$\|\Phi g\| \leqslant 1.$$

Take $x \in X$. Let $n = \min\{m \in \boldsymbol{N}\colon x \in B_m\}$. Then $g'(x) = c_n$. As

$$x \in X \cap B_n \subset X \cap Y_{c_n} = X_{c_n},$$

we have $|f(x) - c_n| \leqslant 1$, whence $|f(x) - g'(x)| \leqslant 1$. ■

Now we have enough machinery to prove our theorem.

THEOREM 3.4. *Let $X \subset \boldsymbol{R}$ be negligible Then $\mathscr{D}'(X) = \mathscr{B}^1(X)$.*

Even stronger: for every $f \in \mathscr{B}^1(X)$ there exists an absolutely continuous $g\colon \boldsymbol{R} \to \boldsymbol{R}$ with

$$g' = f \quad \text{on } X,$$
$$\|\Phi g\| = \|f\|.$$

(Partial converses are obtained in Theorems 4.2 and 4.4.)

Proof. I. We already know that $\mathscr{D}'(X) \subset \mathscr{B}^1(X)$. (Theorem 2.1.)

II. Let $f \in \mathscr{B}^1(X)$, $\|f\| \leqslant 1$. Applying Lemma 3.3 to $2f$ we obtain a $g_1 \in C(\boldsymbol{R})$ with $|f - g_1'| \leqslant \frac{1}{2}$ on X and $\|\Phi g_1\| \leqslant \frac{1}{2}$. Observe that $g_1'|_X \in \mathscr{B}^1(X)$ (not by Th. 2.1 but because $g_1'(x) = \lim_{n\to\infty} \Phi g_1(x+n^{-1}, x)$ for x in X). Thus, $f - (g_1'|_X) \in \mathscr{B}^1(X)$ and $\|f - (g_1'|_X)\| \leqslant \frac{1}{2}$, so there is a $g_2 \in C(\boldsymbol{R})$ with $|f - g_1' - g_2'| \leqslant \frac{1}{4}$ on X and $\|\Phi g_2\| \leqslant \frac{1}{4}$. Continuing in this fashion we find $g_1, g_2, \ldots \in C(\boldsymbol{R})$ such that for all n

$$\Big|f(x) - \sum_{i=1}^{n} g_i'(x)\Big| \leqslant 2^{-n} \quad (x \in X),$$
$$\|\Phi g_n\| \leqslant 2^{-n}.$$

We may assume $g_n(0) = 0$ for all n. The series $\sum g_n$ then converges locally uniformly on $\boldsymbol{R}$; let $g \in C(\boldsymbol{R})$ be its sum. Then $\|\Phi g\| \leqslant 1$. (In particular, g is absolutely continuous.)

Let $a \in X$. We prove $g'(a) = f(a)$. Take $N \in \boldsymbol{N}$. Choose $\delta > 0$ such that, if $|x-a| \leqslant \delta$, then $\sum_{n \leqslant N} |g_n(x) - g_n(a) - g_n'(a)(x-a)| \leqslant 2^{-N}|x-a|$. For $|x-a| \leqslant \delta$ we then have:

$$\begin{aligned}
&|g(x) - g(a) - f(a)(x-a)| \\
&\quad \leqslant \sum_{n \leqslant N} |g_n(x) - g_n(a) - g_n'(a)(x-a)| + \sum_{n > N} |g_n(x) - g_n(a) - g_n'(a)(x-a)| \\
&\quad \leqslant 2^{-N}|x-a| + \sum_{n > N} (2\|\Phi g_n\| \cdot |x-a|) \\
&\quad \leqslant 3 \cdot 2^{-N}|x-a|.
\end{aligned}$$

The formula $g'(a) = f(a)$ follows.

III. It is an immediate consequence that for every bounded $f \in \mathscr{B}^1(X)$ there is a $g \in C(\boldsymbol{R})$ with $g'|_X = f$ and $\|\Phi g\| = \|f\|$.

IV. Now let $f \in \mathscr{B}^1(X)$ be arbitrary. It is clear that we are done if we can find an absolutely continuous $g \in C(\boldsymbol{R})$ with $|f - g'| \leqslant 1$ on X. This is not difficult to do along the lines of the proof of Lemma 3.3:

For $k \in \boldsymbol{Z}$, put $X_k = \{x \in X\colon k-1 < f(x) < k+1\}$. Every X_k is the intersection of X with an F_σ-subset Y_k of $\boldsymbol{R}$. Choose a bijection $\omega\colon \boldsymbol{N} \to \boldsymbol{Z} \times \boldsymbol{N}$ and closed subsets $B_1, B_2, \ldots$ of $\boldsymbol{R}$ such that $Y_k = \bigcup_{i \in \boldsymbol{N}} B_{\omega(k,i)}$ for every $k \in \boldsymbol{Z}$. Let $c_{\omega(k,i)} = k$ $(k \in \boldsymbol{Z}, i \in \boldsymbol{N})$. By Lemma 3.2, applied to $A_n := B_1 \cup \ldots \cup B_n$, there is an absolutely continuous $g\colon \boldsymbol{R} \to \boldsymbol{R}$ with

$$g'(x) = c_n \quad \text{if } n \in \boldsymbol{N} \text{ and } x \in X \cap B_n \cap (B_1 \cup \ldots \cup B_{n-1})^c.$$

For every $x \in X$ there is an $n \in \boldsymbol{N}$ with $g'(x) = c_n$ and $x \in B_n$. If k and i are such that $n = \omega(k, i)$, then $c_n = k$ and $x \in X \cap B_{\omega(k,i)} \subset X \cap Y_k \subset X_k$, so

$$|f(x) - g'(x)| < 1. \ ■$$

Comments. Let $X \in \boldsymbol{R}$ be negligible, $f \in \mathscr{B}^1(X)$. As an addendum to the previous theorem one can show: *if $f(x) \geqslant 0$ (or $f(x) > 0$) for every x in X, then f has an increasing (strictly increasing) primitive.* The proof is a refinement of the above, relying on the final part of Lemma 3.2. However, only if all bounded subsets of X are finite there exists a linear map $P\colon \mathscr{D}'(X) \to C(X)$, assigning to each $f \in \mathscr{D}'(X)$ a primitive of f and such that $P(f)$ is increasing as soon as $f \geqslant 0$.

4. The converse problem. If $X \subset \boldsymbol{R}$ and $\mathscr{D}^1(X)$, must X be Lebesgue negligible? We do not know the answer in general, but we can prove that it is positive for Lebesgue measurable sets. (Th. 4.4.)

For $X \subset \boldsymbol{R}$ we denote by X_d the set of all outer density points of X. (A point a of $\boldsymbol{R}$ is an outer density point of X if, for every positive ε, the outer measure of $X \cap (a-\delta, a+\delta)$ is at least $(1-\varepsilon) \cdot 2\delta$, provided that δ be small enough.) The Lebesgue Density Theorem ([2], Th. 18.2) says that almost every point of X is an outer density point of X. In particular, if X is not negligible, then $X\cap X_d \neq \emptyset$. Evidently, one always has $X_d \subset \overline{X}$ and $X\cap X_d \subset X^*$.

LEMMA 4.1. *Let $X\subset \boldsymbol{R}$, $f\in \mathrm{Lip}(X)$, and suppose that $f'=0$ a.e. on X. Then $f'=0$ on $X\cap X_d$.*

Proof. Take $a\in X\cap X_d$. Put $K:=\|\Phi f\|$.

f extends to a function $g\colon \boldsymbol{R}\to\boldsymbol{R}$ with $\|\Phi g\|=K$. (Extend f to $\overline{X}$ by uniform continuity; then interpolate linearly on the bounded components of $\boldsymbol{R}\setminus\overline{X}$ and make g constant on the unbounded components.) The sets $Y:=\{x\in\boldsymbol{R}\colon g$ is not differentiable at $x\}$ and $Z:=\{x\in X\colon$ either f is not differentiable at x or $f'(x)\neq 0\}$ are negligible. Define $A=Y\cup Z\cup\{x\in\boldsymbol{R}\colon g'(x)=0\}$. Then A is Lebesgue measurable, $g'=0$ a.e. on A, and $X\subset A$.

For $b\in(a,\infty)$ we have ([2], Th. 18.16):

$$|g(b)-g(a)| = \Big|\int_a^b g'\,d\lambda\Big| \leqslant \int_{[a,b]\setminus A} |g'|\,d\lambda \leqslant K\cdot\lambda([a,b]\setminus A)$$

so

$$|\Phi g(a,b)| \leqslant K\cdot(b-a)^{-1}\lambda([a,b]\setminus A).$$

Now $a\in X\cap X_d\subset A\cap A_d$; therefore, $\lim_{b\downarrow a}\Phi g(a,b)=0$.

Similarly, $\lim_{b\uparrow a}\Phi g(a,b)=0$, whence $g'(a)=0$. As $a\in X^*$, we infer that f is differentiable at a and that $f'(a)=0$. ■

This lemma yields a partial converse to Theorem 3.4.

THEOREM 4.2. *The following conditions on a set $X\subset\boldsymbol{R}$ are equivalent.*

(α) *X is negligible.*

(β) *For every bounded $f\in\mathscr{B}^1(X)$ there exists a $g\in\mathrm{Lip}(X)$ with $g'=f$ on X^*.*

Proof. For $(\alpha)\Rightarrow(\beta)$, see Theorem 3.4.

$(\beta)\Rightarrow(\alpha)$ Suppose X is not negligible. Choose a in $X\cap X_d$. By the lemma, $1_{\{a\}}$ cannot have a Lipschitz primitive. ■

If $X\subset\boldsymbol{R}$ and $\mathring{X}=\emptyset$, then for every $a\in X$ we know $1_{\{a\}}\in\mathscr{D}'(X)$ (Th. 2.4.) Thus, if $\mathring{X}=\emptyset$, then (α) and (β) are equivalent to:

(γ) *For every bounded $f\in\mathscr{D}'(X)$ there exists a $g\in\mathrm{Lip}(X)$ with $g'=f$ on X^*.*

Theorem 4.2 leaves the possibility that on a non-negligible set X every (bounded) $\mathscr{B}^1$-function has a primitive. For many sets this possibility is ruled out by the following lemma.

LEMMA 4.3. *Let $X\subset\boldsymbol{R}$. Suppose that X is a Baire space and that $X\subset\overline{X_d}$. Let $\{c_1,c_2,\ldots\}$ be a countable dense subset of $X\cap X_d$. Then*

$$\sum_{n=1}^{\infty}2^{-n}1_{\{c_n\}}\notin\mathscr{D}'(X).$$

Proof. Suppose we have an f in $C(X)$ with $f'=\sum_{n=1}^{\infty}2^{-n}1_{\{c_n\}}$ on X^*. For $p\in\boldsymbol{N}$, put

$$A_p=\{x\in X\colon \text{if } y\in X \text{ and } |x-y|<p^{-1}, \text{ then } |f(y)-f(x)|\leqslant|y-x|\}.$$

$0\leqslant f'(x)<1$ for every $x\in X^*$. Hence, the sets A_p form a countable relatively closed cover of X. Therefore, we can choose a p such that the interior of A_p in X is non-empty. There exist $a,b\in\boldsymbol{R}$ with $\emptyset\neq(a,b)\cap X\subset A_p$ and $|b-a|<p^{-1}$. Put $J=(a,b)\cap X$. If $x,y\in J$, then $|f(y)-f(x)|\leqslant|y-x|$; so $f|_J\in\mathrm{Lip}(J)$.

Furthermore, $(f|_J)'=0$ a.e. on J. By Lemma 4.1, $(f|_J)'=0$ on $J\cap J_d$, so $f'=0$ on $J\cap J_d$.

However, $(a,b)\cap X\neq\emptyset$ and $X\subset\overline{X_d}$, so $(a,b)\cap X_d\neq\emptyset$. Then $(a,b)\cap X$ is not negligible, whence $\emptyset\neq((a,b)\cap X)\cap((a,b)\cap X)_d\subset(a,b)\cap X\cap X_d$. Then $(a,b)\cap X\cap X_d$ (which is $J\cap J_d$) contains a c_n. Contradiction. ■

This gives us the converse to Theorem 3.4 for measurable sets:

THEOREM 4.4. *For a Lebesgue measurable subset X of $\boldsymbol{R}$ the conditions (α)–(γ) are equivalent.*

(α) *X is negligible.*

(β) *$\mathscr{D}'(X)=\mathscr{B}^1(X)$.*

(γ) *X has empty interior. $\mathscr{D}'(X)$ is uniformly closed (in $\boldsymbol{R}^X$).*

Proof. $(\alpha)\Rightarrow(\beta)$: Theorem 3.4.

$(\beta)\Rightarrow(\gamma)$. $\mathscr{B}^1(X)$ is uniformly closed. ([1], Th. 1, p. 138.) Further, suppose X contains an interval (a,b) of $\boldsymbol{R}$. Choose $c\in(a,b)$. Then $1_{\{c\}}\in\mathscr{B}^1(X)\setminus\mathscr{D}'(X)$.

$(\gamma)\Rightarrow(\alpha)$. Suppose X is not negligible. By the regularity of the Lebesgue measure, X contains a compact set Y of positive measure. Put $Z=\overline{Y_d}$. Then $Z\subset Y\subset X$, Z is compact (and therefore a Baire space) and $Z\subset\overline{Z_d}$. By Lemma 4.3 there exist $c_1,c_2,\ldots$ in Z such that $\sum_{n=1}^{\infty}2^{-n}1_{\{c_n\}}\notin\mathscr{D}'(Z)$. A fortiori, $\sum_{n=1}^{\infty}2^{-n}1_{\{c_n\}}\notin\mathscr{D}'(X)$.

On the other hand, by Theorem 2.4, for each $N\in\boldsymbol{N}$ we have that

$$\sum_{n=1}^{N}2^{-n}1_{\{c_n\}}\in\mathscr{D}'(X).$$

Thus, $\mathscr{D}'(X)$ is not uniformly closed. ■

Comment on Condition (γ). It is known that if X is an interval, then $\mathscr{D}'(X)$ is uniformly closed. ([3], p. 92; [6], Th. 14.2.)

Further we observe that our Theorem 4.4 closely resembles Theorem 4.19 of [5] which says (without any measurability assumptions) that $\{f|X\colon f\in\mathscr{D}'(\boldsymbol{R})\}=\{f|X\colon f\in\mathscr{B}^1(\boldsymbol{R})\}$ if and only if X is negligible.

4.5. The proof of the implication $(\gamma)\Rightarrow(\alpha)$ is applicable to any set $X\subset\boldsymbol{R}$ that contains a compact set of positive measure. Thus, *if $X\subset\boldsymbol{R}$ and if $\mathscr{B}^1(X)=\mathscr{D}'(X)$, then the inner measure of X is* 0. It would be more interesting to know if the *outer* measure of X has to be 0, i.e., if the measurability condition of Theorem 4.4 is redundant. We do not have an answer to this question. All we can offer is an example to show that the condition that X have *inner* measure 0 does *not* imply

$$\mathscr{B}^1(X) = \mathscr{D}'(X).$$

EXAMPLE 4.6. By [2], p. 146, there exists a set $X \subset \boldsymbol{R}$ such that neither X nor X^c contains an uncountable closed subset of $\boldsymbol{R}$. For this X, both X and X^c have inner measure 0. (Regularity of the Lebesgue measure.) Then $X_d = \boldsymbol{R}$. We show that X is a Baire space, so that (by Lemma 4.3) $\mathscr{B}^1(X) \neq \mathscr{D}'(X)$.

Let $U_1, U_2, \ldots$ be open subsets of $\boldsymbol{R}$ such that every $U_n \cap X$ is dense in X. Then each U_n is dense in $\boldsymbol{R}$, so $\bigcap_{n \in \boldsymbol{N}} U_n$ is a dense G_δ-subset of $\boldsymbol{R}$. For every interval J of $\boldsymbol{R}$, $J \cap \bigcap_{n \in \boldsymbol{N}} U_n$ is an uncountable Borel set, therefore contains an uncountable closed subset of $\boldsymbol{R}$ ([4], p. 151.) and therefore is not contained in X^c. It follows that $\bigcap_{n \in \boldsymbol{N}} (U_n \cap X)$ is dense in X.

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On derivatives of functions defined on disconnected sets, II *

by

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Abstract. Let X be a totally disconnected subset of $\boldsymbol{R}$ without isolated points, let $\mathscr{B}^1(X)$ be the first class of Baire on X and let $\mathscr{D}'(X)$ be the set of all functions $X \to \boldsymbol{R}$ that have primitives on X. Then $\mathscr{D}'(X) \subset \mathscr{B}^1(X)$. We show that $\mathscr{D}'(X)$ is a "large" subset of $\mathscr{B}^1(X)$. Thus, $\mathscr{D}'(X)$ contains all approximately continuous functions on X; and if $f, g \in \mathscr{B}^1(X)$ are such that $f(x) < g(x)$ for all x in X, then between f and g there is an element of $\mathscr{D}'(X)$.

5. How large is $\mathscr{D}'(X)$? We now know that, at least for certain sets X with empty interiors, $\mathscr{D}'(X)$ is a proper subset of $\mathscr{B}^1(X)$. Our present purpose is to show that it has to be a large subset. (Th. 5.6, Th. 6.1.)

Let $Y \subset X \subset \boldsymbol{R}$. We call Y a *clopen* subset of X if it is both relatively closed and relatively open.

We will make use of the following observation. If $X \subset \boldsymbol{R}$, $\mathring{X} = \emptyset$ and if $\mathscr{U}$ is an open cover of X, then there exists a cover of X by countably many pairwise disjoint subsets $X_1, X_2, \ldots$, each of which is clopen in X and contained in an element of $\mathscr{U}$. (As X^c is dense in $\boldsymbol{R}$, $\mathscr{U}$ has a refinement $\mathscr{V}$ consisting of clopen subsets of X. By using the Lindelöf property of X one obtains a countable subcover $(V_n)_{n \in \boldsymbol{N}}$ of $\mathscr{V}$. Now set $X_n = V_n \setminus (V_1 \cup \ldots \cup V_{n-1})$ $(n \in \boldsymbol{N})$.)

LEMMA 5.1. *Let $X \subset \boldsymbol{R}$, $\mathring{X} = \emptyset$. Let A be a relatively closed subset of X. Let $\psi \in C(X)$, $\psi(x) = 0$ for all $x \in A$, $\psi(x) > 0$ for all $x \in X \setminus A$. Let $h \in C(X)$. Then there exists a j in $C(X)$ with*

$$|j-h| \leqslant \psi,$$
$$j' = 0 \quad \text{on } X^* \setminus A.$$

Proof. For $x \in X \setminus A$, let $U(x) = \{y \in X \setminus A : |h(x)-h(y)| < \psi(y)\}$. By the above remark, there exist a family $(Y_n)_{n \in \boldsymbol{N}}$ of pairwise disjoint clopen subsets of $X \setminus A$, covering $X \setminus A$, and a family $(x_n)_{n \in \boldsymbol{N}}$ of elements of $X \setminus A$, such that $Y_n \subset U(x_n)$

* This paper is a continuation of [7], using the same notations.