Flows on one-dimensional spaces

by

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Abstract. A separable and metrizable space X is called a *matchbox manifold* if each point x of X has an open neighborhood which is homeomorphic to $S_x \times \boldsymbol{R}$ for some zero-dimensional space S_x. The following special cases have been studied before. In [4] for each x the space S_x is the Cantor discontinuum and in [1] for each x the space S_x is a copy of the rationals $\boldsymbol{Q}$. Each path component of a matchbox space admits a parametrization by the reals $\boldsymbol{R}$ in a natural way. This is the main tool in defining the orientability of matchbox spaces. It is shown that a one-dimensional, separable and metrizable space X is an orientable matchbox space if and only if X is the phase space of some flow without rest points. The continuous dynamical systems without rest points on one-dimensional spaces coincide with the suspensions $\Sigma(S,f)$ of homeomorphisms $f\colon\ S \to S$ for some zero-dimensional space S.

All spaces under consideration are separable and metrizable.

0. Introduction. The central question discussed in this paper is: on what one-dimensional spaces can there be defined a flow? Of course, we must say *flow without rest points*. Indeed, on any space there can be defined the trivial flow with no motion at all.

Typical examples of one-dimensional flows without rest points are suspensions $\Sigma(S,f)$ of homeomorphisms $f\colon S \to S$ on zero-dimensional spaces ([7]). It is the main result of this paper that the converse is also true.

STRUCTURE THEOREM. *Let $\pi\colon\ X \times \boldsymbol{R} \to X$ be a dynamical system without rest points on a one-dimensional space X. Then there exists a zero-dimensional space S and a homeomorphism $f\colon\ S \to S$ such that $\pi\colon\ X \times \boldsymbol{R} \to X$ is topologically equivalent to the suspension $\Sigma(S,f)$.*

The theorem is proved in Section 5.2. For compact spaces the result can be found in [8]. In [1] the theorem has been proved for the case where X is a single orbit. The structure theorem is proved by extension and generalization of the methods of the latter paper. Our paper can indeed be seen as a continuation of [1]. The main idea is to recover the phase space of a flow from its local structure. In view of the section theorem, first proved by Whitney [12], the local structure of a one-dimensional flow without rest points is rather simple. In such a system each point x has a neighbourhood which is homeomorphic to $S_x \times \boldsymbol{R}$ for some zero-dimensional space S_x. It must be emphasized that S_x may vary with x. Spaces in which each point x has a neighbourhood of the form $S_x \times \boldsymbol{R}$ for some zero-dimensional space S_x are called *matchbox manifolds* in this paper. Apart from [1] they have also been studied in [4] and [5] in the setting of compact spaces.

This paper is organised as follows. We first develop some basic geometry of matchbox manifolds in Section 1. In Section 2 we prove the existence of so-called long and wide matchboxes. After discussing the orientability of matchbox manifolds in Section 3 we examine the return map in Section 4. Finally, in Section 5 the structure theorem is proved.

The proof of the structure theorem can as well be given within the framework of flows. In that way a somewhat shorter proof can be obtained, mainly because the discussion of orientability becomes redundant. However, by the approach made in this paper, a better insight in de geometrical structure of one-dimensional flows and their sections can be obtained. It should be observed that facts about sections and mappings between them, e.g. the continuity of the first return map, are by no means easy to prove in the absence of an implicit function theorem. As a by-product of our approach we get the following theorem.

CHARACTERIZATION THEOREM. *A space X is the phase space of some one-dimensional flow without rest points if and only if X is an orientable matchbox manifold.*

This theorem is proved in Section 5.1.

Standing notation. Throughout the paper the following notation is used. S denotes any zero-dimensional subspace of $\boldsymbol{R}$;

$$F_S = \{(x, y) \in \boldsymbol{R}^2 \mid x \in S, \ -1 \leqslant y \leqslant 1\},$$

$$E_S = \{(x, y) \in \boldsymbol{R}^2 \mid x \in S, \ -1 < y < 1\}.$$

The set F_S is called a *standard matchbox*. For each $x \in S$ the set $\{x\} \times [-1,1]$ is called a *match in F*.

If no confusion is likely to arise we simply write F and E instead of F_S and E_S. The natural projections of the space F_S onto S and $[-1,1]$ are denoted by pr_1 and pr_2 respectively. Both pr_1 and pr_2 are open. As $[-1,1]$ is compact, pr_1 is closed as well.

1. Geometry of matchbox manifolds.

1.1. DEFINITION. A space X is called a *matchbox manifold* if for each $x \in X$ there is a zero-dimensional space S_x such that $S_x \times \boldsymbol{R}$ is homeomorphic to an open neighborhood of x.

THEOREM. *Let $\pi\colon X \times \boldsymbol{R} \to X$ be any flow on a one-dimensional space X. If π has no rest points, then X is a matchbox manifold.*

Proof. Because there are no rest points, for each $x \in X$ there is a closed subset S_x of X — called a *local section at x* — and a real number $\eta > 0$ such that

$$x \in S_x, \quad \pi\colon S_x \times [-\eta, \eta] \to \pi(S_x \times [-\eta, \eta]) \text{ is a homeomorphism}$$

and

$$\pi(S_x \times [-\eta, \eta]) \text{ is a neighborhood of } x.$$

See [6] for a proof.

Now let N be an open neighborhood of $\pi(x, 0) = x$ in S_x such that for some ε with $0 < \varepsilon < \eta$ we have $\pi(N \times (-\varepsilon, \varepsilon)) \subset \operatorname{int} \pi(S \times [-\eta, \eta])$. Then $\pi\colon N \times (-\varepsilon, \varepsilon) \to \pi(N \times (-\varepsilon, \varepsilon))$ is a homeomorphism and $\pi(N \times (-\varepsilon, \varepsilon))$ is open in $\operatorname{int} \pi(S \times [-\eta, \eta])$, whence in X. It follows from [2], 1.9.E, that $\dim S_x$ as well as $\dim N$ is zero.

1.2. DEFINITIONS. Let X be a topological space. Suppose that $h\colon F_S \to X$ is a topological embedding such that $h(F_S)$ is closed and $h(E_S)$ is open in X. Then the set $V = h(F_S)$ is called a *matchbox in X*. The induced map $h\colon F_S \to V$ is called a *parametrization of V*. The sets $h(\{x\} \times [-1, 1])$, $x \in S$, are called *matches of V*. We also say that V is a *matchbox neighborhood of* $h(x, 0)$, $x \in S$.

In [1] various properties of matchboxes have been discussed for the special case that each section is homeomorphic to $\boldsymbol{Q}$. Many of these results can be generalized in a straightforward manner and are only briefly discussed here.

PROPOSITION. *Suppose that W is a neighborhood of x in a matchbox manifold X. Then there is a matchbox neighborhood V of x such that $x \in V \subset W$.*

Proof. Let $h\colon S_x \times \boldsymbol{R} \to W_0$ be a homeomorphism onto an open neighborhood W_0 of x. As $\boldsymbol{R}$ is homogeneous, we may assume that $x = h(y, 0)$ for some $y \in S_x$. The rest of the proof is as in [1], 2.3.

Remark. It is very useful to note that if W is a matchbox neighborhood of $x = h(0, 0)$ in X with parametrization $h\colon F_S \to X$, then for all clopen subsets T of S with $0 \in T$ and for all ε with $0 < \varepsilon \leqslant 1$ also $h(T \times [-\varepsilon, \varepsilon])$ is a matchbox neighborhood of x.

1.3. Here we present two examples of matchbox manifolds which might clarify the discussion.

(i) X is the subset of $\boldsymbol{R}^2$ defined by

$$X = \left\{(x, y) \mid y = \sin\frac{1}{x}, \ x > 0\right\} \cup \{(0, y) \mid -1 < y < 1\}.$$

(ii) X is the subset of $\boldsymbol{R}^2$ which in polar coordinates is given by

$$X = \{(1, \varphi) \mid 0 \leqslant \varphi < 2\pi\} \cup \{(r, \varphi) \mid r = 1 + \exp\varphi, \ \varphi \in \boldsymbol{R}\}.$$

From both examples it is clear that the homeomorphism type of the matchbox neighborhood may vary with x.

1.4. Proposition. *Suppose that $\{V_\alpha|\ \alpha\in A\}$ is a discrete collection of matchboxes in a matchbox manifold X. Then $\bigcup\{V_\alpha|\ \alpha\in A\}$ is a matchbox in X.*

Proof. Let $h_\alpha\colon F_{S_\alpha}\to V_\alpha$ be a parametrization, $\alpha\in A$. As $\{h_\alpha(E_{S_\alpha})|\ \alpha\in A\}$ is a disjoint collection of open sets in the separable space X, the index set A is countable. We then may assume without loss of generality that the collection $\{S_\alpha|\ \alpha\in A\}$ is discrete. In an obvious way the map $h\colon\bigcup\{F_{S_\alpha}|\ \alpha\in A\}\to\bigcup\{V_\alpha|\ \alpha\in A\}$ is defined, which for each α on F_{S_α} agrees with h_α. Because $\{V_\alpha|\alpha\in A\}$ is discrete, it follows that h is a topological embedding. The rest of the proof is obvious.

The proofs of the following two propositions are omitted as they are easy.

Proposition. *A matchbox manifold is atriodic.*

Proposition. *An open subset of a matchbox manifold is a matchbox manifold.*

Proposition. *Suppose that X is a matchbox manifold and that Y is an arc component of X. Then Y is also a matchbox manifold.*

Proof. Let $y\in Y$ and let V be a matchbox neighborhood in X with parametrization $h\colon F\to V$. Write $C=\mathrm{pr}_1(h^{-1}(V\cap Y))$ and $F_C=C\times[-1,1]$, a standard matchbox. Because Y is maximal with respect to arcwise connectedness,

$$F_C=h^{-1}(V\cap Y).$$

It follows that y has a matchbox neighborhood in Y.

1.5. Lemma. *Let X be a matchbox manifold. Suppose that V is a matchbox in X and that J is an arc in X. Then the intersection $V\cap J$ consists of finitely many arcs only. Each such arc with the exception of at most two is a match of V.*

Proof. Cf. Lemma 2.4 of [1].

Definition. Let F_S be any standard matchbox. Let K be a clopen subset of S and let $t,b\colon K\to[-1,1]$ be continuous functions such that $b<t$. The set $W=\{(x,u)|\ x\in K,\ b(x)\leqslant u\leqslant t(x)\}$ is called a *simple matchbox in F_S with base K.*

Observe that W is a matchbox in F_S indeed.

Lemma. *Let there be given a standard matchbox F_S. Suppose that V is a matchbox in F_S with parametrization $h\colon F_T\to V$. Suppose that for some $x\in T$ we have*

$$h(\{x\}\times[-1,1])=V\cap\mathrm{pr}_1^{-1}(\mathrm{pr}_1(h(x))).$$

Then there is a clopen neighborhood K of x in S and a simple matchbox W in F_S with basa K such that $W=V\cap\mathrm{pr}_1^{-1}(K)$.

Proof. Cf. [1], Lemma 2.5.

1.6. Definition. Let X be a matchbox manifold. Suppose that J is an arc with parametrization $g\colon[0,1]\to X$ and that V is a matchbox in X with parametrization $h\colon F_S\to V$. Then V is called a *matchbox along J* if for some $x\in S$

(i) $J\cap V=h(\{x\}\times[-1,1])$, and

(ii) the map $t\rightsquigarrow g^{-1}(h(x,t))$ is increasing.

As in [1], 3.4, it can be shown that if J is an arc in a matchbox manifold and if $x\in J$, but x is not an endpoint of J, then x has arbitrarily small neighborhoods V such that V is a box along J.

2. Long and wide matchboxes.

2.1. The following theorem is crucial for the development of the theory of sections. Very roughly stated the theorem is as follows. If V_1 and V_2 are matchboxes along an arc L and if $L\cap V_1\cap V_2$ is an arc, then there is a matchbox V along L such that $V\subset V_1\cup V_2$ and $L\cap V=L\cap(V_1\cup V_2)$. By this theorem we are able to produce long matchboxes along an arc L covering subarcs of L of arbitrary length. The theorem is a generalization of the pasting theorem in [1], 3.4. In order to avoid confusion we present a precise reformulation. The proof though is an almost verbatim copy of the proof in [1] with obvious changes only.

The pasting theorem. *Let X be a matchbox manifold and let L be an arc in X. For $i=1,2$ let V_i be a matchbox along L and let $h_i\colon F_{S_i}\to V_i$ be a parametrization. Let q_i be the point in S_i such that $L\cap V_i=h_i(\{q_i\}\times[-1,1])$, $i=1,2$. Suppose that for some $s_1,s_2\in(-1,1)$,*

$$L\cap V_1\cap V_2=h_1(\{q_1\}\times[s_1,1])=h_2(\{q_2\}\times[-1,s_2]).$$

Then there are clopen neighborhoods A_i of q_i in S_i, $i=1,2$, and a matchbox V along L with parametrization $h\colon F_S\to V$ such that

$$V=h_1(A_1\times[-1,1])\cup h_2(A_2\times[-1,1]),\quad\textit{and}$$

$$h_1^{-1}h(S\times\{-1\})=A_1\times\{-1\},\quad h_2^{-1}h(S\times\{1\})=A_2\times\{1\}.$$

2.2. As a corollary we get a result about a finite sequence of matchboxes along an arc with the property that any two consecutive matchboxes intersect like V_1 and V_2 in the pasting theorem. Then there is a matchboxes V along L which covers as much of L as the union of the sequence does. See [1] for a more precise statement. We are going to use this result in the following lemma.

Lemma of the long box. *Let J be an arc in a matchbox manifold X with start x_1 and endpoint x_2. Suppose that V_1 and V_2 are disjoint matchbox neighborhoods of x_1 and x_2 respectively. For $i=1,2$ let $h_i\colon F_{S_i}\to V_i$ be a parametrization of V_i and write $h(S_i\times\{0\})=Z_i$.*

Then there is a matchbox V with parametrization $k\colon F_S\to V$ such that

$$x_1\in k(S\times\{-1\})\subset Z_1,\quad x_2\in k(S\times\{+1\})\subset Z_2,$$

$$k(S\times\{-1\})\ \textit{is clopen in}\ Z_1\quad\textit{and}\quad k(S\times\{1\})\ \textit{is clopen in}\ Z_2.$$

Stated less accurately the lemma says that there is a long box $k(F_S) = V$ with bottom in Z_1 and top in Z_2. It is to be noted that $k(S\times\{-1\})$ and $k(S\times\{1\})$ are homeomorphic.

Proof. After having replaced S_i by $S_i \cap T$ for some suitable clopen subset T of S_i, we may assume that $V_i \cap J$ is precisely one arc J_i, $i = 1, 2$. We also may assume without loss of generality that

$$J_1 = h_1(\{0\}\times[0,1]) \quad \text{and} \quad J_2 = h_2(\{0\}\times[-1,0]).$$

This can be achieved by shifting S_i and replacing $h_i(x,t)$ by $h_i(x,-t)$ if necessary.

Now we define a sequence of matchboxes along the arc

$$\tilde{J} = h_1(\{0\}\times[-1,1]) \cup J \cup h_2(\{0\}\times[-1,1]).$$

The first box of the sequence is $h_1(S_1\times[0,1])$ and the last box is $h_2(S_2\times[-1,0])$. For each point z of $J^* = \mathrm{cl}(J\setminus(J_1\cup J_2))$ a matchbox neighborhood V_z along J is selected such that $z\in V_z\subset X\setminus(Z_1\cup Z_2)$. The collection $\{\mathrm{int}\,V_z \mid z\in J^*\}$ is an open cover of J^*. Hence there is a finite subcollection $\{\mathrm{int}\,V_{z_1},\ldots,\mathrm{int}\,V_{z_n}\}$, the union of which contains J^*. We may assume that this collection is minimal. Because of minimality after possible rearrangement we get a sequence

$$h_1(S_1\times[0,1]),\ V_{z_1},\ldots,V_{z_n},\ h_2(S_2\times[-1,0]),$$

any two elements of which intersect (nicely) if and only if they are consecutive. The lemma now follows from the above-mentioned corollary.

2.3. We are now going to show that in every matchbox manifold X there exists a so-called *wide matchbox*, i.e. a matchbox which has a non-empty intersection with every arc component of X. We need the following lemma.

LEMMA. *In a standard matchbox F_S let G and U be a closed and an open subset respectively of E_S. If $G\subset U$, then there is a simple matchbox V in F_S such that $\mathrm{pr}_1 G\subset \mathrm{pr}_1 V$ and $V\subset U$.*

Proof. As pr_1 is an open as well as a closed continuous mapping, the closed set $\mathrm{pr}_1\, G$ is contained in the open set $\mathrm{pr}_1 U$. Let K be a clopen subset of S such that $\mathrm{pr}_1 G\subset K\subset \mathrm{pr}_1 U$. In each point $x\in K$ there is a clopen subset U_x of S and a closed interval $[c_x, d_x]$ such that $x\in U_x\subset K$ and $U_x\times[c_x,d_x]\subset U$.

Since the spaces under consideration are separable and metrizable, there is a sequence (x_i) in K such that $K = \bigcup\{U_{x_i} \mid i = 0,1,\ldots\}$. Write

$$K_n = U_{x_n}\setminus \bigcup\{U_{x_i} \mid i = 0,\ldots,n-1\},\quad n = 0,1,\ldots$$

It is to be observed that $\{K_n \mid n = 0,1,\ldots\}$ is a disjoint clopen cover of K. The continuous functions $b, t\colon K\to[-1,1]$ are defined by $b(x) = c_{x_n}$ and $t(x) = d_{x_n}$ for $x\in K_n$. Put $V = \{(x,u) \mid x\in K,\ b(x)\leqslant u\leqslant t(x)\}$.

2.4. THEOREM. *In each matchbox manifold there exists a wide matchbox.*

Proof. Let X be a matchbox manifold.

In the proof the following ad hoc notation is used. For any matchbox W in X by $B(W)$ — the *bundle of* W — is denoted the union of all arc components of X which have a non-empty intersection with W. By using the lemma of the long box one can easily see that $B(W)$ is open. As X is separable and metrizable, there is a countable collection of matchboxes W_i with parametrizations $h_i\colon F_{S_i}\to W_i$, $i = 0,1,\ldots$, such that $\{h_i(E_{S_i}) \mid i = 0,1,\ldots\}$ covers X. Since X is paracompact, this open cover has a locally finite open refinement, which may be supposed to be countable, $\{U_i \mid i = 0,1,\ldots\}$. Let $\{G_i \mid i = 0,1,\ldots\}$ be a closed shrinking, that is to say, $\{G_i \mid i = 0,1,\ldots\}$ is a closed cover of X and $G_i\subset U_i$, $i = 0,1,\ldots$ Applying Lemma 2.3 we can construct a matchbox V_0 such that $V_0\subset U_0$ and $G_0\subset B(V_0)$. Inductively on n a matchbox V_n is defined in X such that

(a) $V_n\subset U_n$, $n = 0,1,\ldots$;

(b) $G_0\cup\ldots\cup G_n\subset B(V_0)\cup\ldots\cup B(V_n)$, $n = 0,1,\ldots$;

(c) $\{V_n \mid n = 0,1,\ldots\}$ is discrete.

Assuming that the V_i have been defined for $i = 0,\ldots,n-1$ satisfying (a) and (b), let

$$G_n' = G_n\setminus(B(V_0)\cup\ldots\cup B(V_{n-1})) \quad \text{and}$$
$$U_n' = U_n\setminus(V_0\cup\ldots\cup V_{n-1}).$$

Then obviously G_n' is closed and $G_n'\subset U_n'$.

Now U_n is a subset of some $h_j(E_{S_j})$. Let $G = h_j^{-1}(G_n')$ and $U = h_j^{-1}(U_n')$. By applying Lemma 2.3 a simple matchbox V^* in F_{S_j} is obtained such that $\mathrm{pr}_1 G\subset \mathrm{pr}_1 V^*$ and $V^*\subset U$. Let $V_n = h_j(V^*)$. It follows that V_n is a matchbox in X such that (a) and (b) are satisfied. Because $\{V_i \mid i = 0,\ldots,n\}$ is disjoint, $n = 0,1,\ldots$, and $\{U_n \mid n = 0,1,\ldots\}$ is locally finite, the resulting collection $\{V_i \mid i = 0,1,\ldots\}$ is discrete. By Proposition 1.4, $V = \bigcup\{V_i \mid i = 0,1,\ldots\}$ is a wide matchbox.

3. Orientation of matchbox manifolds. In this section we show how to define orientability of matchbox manifolds. We have already observed in 1.4 that an arc component of a matchbox manifold is itself also a matchbox manifold.

3.1. THEOREM. *Let X be an arcwise connected matchbox manifold. Let V be any matchbox in X with parametrization $h\colon F_S\to V$. Then S is countable.*

Proof. If X happens to be a circle, then it is easily seen that in view of Lemma 1.5 S must be finite. So we may assume that X is not a circle. As X is atriodic (1.4), it cannot contain a circle. It follows that X is uniquely arcwise connected. Let $p\in h(E_S)$. As in [1], 3.1, we write $X = R_p\cup L_p$ such that R_p and L_p are arcwise connected and $R_p\cap L_p = \{p\}$. Moreover, for $x, y\in R_p\setminus\{p\}$ (or $x, y\in L_p\setminus\{p\}$), if $x\neq y$, either $x\in \widehat{py}$ or $y\in\widehat{px}$. Here $\widehat{py}$ denotes the (unique) arc which begins in p and ends in y. We shall show now that S is countable. Write $Z = h(S\times\{0\})$ and assume that $Z \cap R_p$ is uncountable. Let D be a countable and dense subset of $Z \cap R_p$. Let $D^* = \bigcup \{\widehat{px} \mid x \in D\}$. In view of Lemma 1.5 we have $D^* \cap Z$ is countable. Let $q \in (Z \cap R_p) \setminus D^*$. As $\widehat{pq} \cap Z$ is closed in Z, there is a point $z \in D \setminus \widehat{pq}$. But then we must have $q \in \widehat{pz}$ and $q \in D^*$. This is a contradiction. It follows that $Z \cap R_p$ is countable. Similarly $Z \cap L_p$ is countable.

3.2. Arcwise connected spaces which are locally homeomorphic to $\boldsymbol{Q} \times \boldsymbol{R}$ are called *P-manifolds* in [1]. There is the following corollary to the preceding theorem.

COROLLARY. *Let X be an arcwise connected matchbox manifold. Then X is either a circle, a topological copy of $\boldsymbol{R}$ or a P-manifold.*

Proof. We continue the discussion of the proof of 3.1.

If x and y are any distinct points in Z, then by the lemma of the long box there exist homeomorphic neighborhoods of x and y in Z. It follows that Z is either discrete or dense in itself. In the former case X is a topological copy of $\boldsymbol{R}$. In the latter case Z is homeomorphic to $\boldsymbol{Q}$ ([10]). The corollary easily follows.

3.3. Let X be a matchbox manifold. As we have just seen any arc component of X is either a circle, a copy of $\boldsymbol{R}$ or a p-manifold. In each case we define the *parametrization* of the arc component C of X.

It has been shown in [1], 3.1, that if C is a P-manifold, then there exists a continuous bijection $p\colon \boldsymbol{R} \to C$. Such a mapping is called a *parametrization*. It has been observed in [1] that the parametrizations fall into two classes, the *directions*. If C is a topological copy of $\boldsymbol{R}$, then any homeomorphism $p\colon \boldsymbol{R} \to C$ is called a *parametrization*. Also in this case in a very natural way there are two directions. As is well-known $\boldsymbol{R}$ is the universal covering space of S^1 with covering map $t \to \exp(2\pi i t)$, see e.g. [9].

In case C is a circle, let $h\colon S^1 \to C$ be any homeomorphism. The composition of the covering map and h is called a *parametrization of C*. Also in this case parametrizations fall into two classes. For any arc J in C there is a countably infinite collection of arcs which by the parametrization is mapped onto J.

3.4. Let X be a matchbox manifold. Let $\{C_\alpha \mid \alpha \in A\}$ be the collection of arc components of X. If for each $\alpha \in A$ a parametrization $p_\alpha\colon \boldsymbol{R} \to C_\alpha$ is given, we shall call the collection $\{p_\alpha \mid \alpha \in A\}$ a *parametrization of X*. Now let V be a matchbox in X with parametrization $h\colon F_S \to V$. We shall say that V is *coherently directed* by h if for each $x \in S$ and for any closed interval J in $\boldsymbol{R}$ such that

$$p_\alpha(J) = h(\{x\} \times [-1, 1])$$

for some α the composition $\mathrm{pr}_2 \circ h^{-1} \circ p_\alpha$ is increasing.

Some comment is in order. For each $x \in S$ there is a unique α such that $h(x, 0) \in C_\alpha$. In case C_α is a circle there are countably many intervals in $\boldsymbol{R}$ which by p_α are mapped onto $h(\{x\} \times [-1, 1])$. Any two such intervals however can be mapped onto each other by a translation. So for deciding whether $\mathrm{pr}_2 \circ h^{-1} \circ p_\alpha$ is increasing it is irrelevant which interval is chosen. In the other cases there is a unique interval which is mapped onto $h(\{x\} \times [-1, 1])$. See [1] for more details.

DEFINITION. A matchbox manifold X is said to be *orientable* if there is a parametrization $\{p_\alpha \mid \alpha \in A\}$ of X such that each point has a matchbox neighborhood which is *coherently directed*. In that case the parametrization is called *proper*.

3.5. EXAMPLES. In 1.3 we have presented two examples of matchbox manifolds. As may be verified (i) is not orientable, but (ii) is. In [1] 3.3 an example is presented of a P-manifold which is not orientable.

We now show that one-dimensional flows are defined on orientable matchbox manifolds.

THEOREM. *Let $\pi\colon X \times \boldsymbol{R} \to X$ be a flow without rest points on a one-dimensional space. Then X is an orientable matchbox manifold.*

Proof. The notation of 1.1 will be used. Let $\{C_\alpha \mid \alpha \in A\}$ be the set of arc components. Let $\alpha \in A$. Pick $y \in C_\alpha$. Obviously $\Gamma(y) = \pi(\{y\} \times \boldsymbol{R}) \subset C_\alpha$. The set $p_\alpha^{-1}(\Gamma(y))$ is open in $\boldsymbol{R}$. This can be seen by considering matchboxes along $\Gamma(y)$. As the orbits form a partition of the space it follows that $\Gamma(y)$ and C_α coincide. Now a parametrization for C_α is defined by $p_\alpha(t) = \pi(y, t)$. This is done for each α. We shall verify that the matchbox $\pi(N \times [-\varepsilon, \varepsilon])$ (see 1.1) is coherently directed by $\{p_\alpha \mid \alpha \in A\}$.

If $x \in N$ and $x = p_\alpha(u)$, then for any $v \in [-\varepsilon, \varepsilon]$ we have

$$p_\alpha(u+v) = \pi(y, u+v) = \pi(\pi(y, u), v) = \pi(p_\alpha(u), v) = \pi(x, v).$$

It follows that

$$\mathrm{pr}_2 \circ \pi^{-1} \circ p_\alpha(u+v) = \mathrm{pr}_2 \circ \pi^{-1} \circ \pi(x, v) = \mathrm{pr}_2(x, v) = v.$$

The matchbox $\pi(N \times [-\varepsilon, \varepsilon])$ is coherently directed by π.

3.6. Of several theorems of the previous paragraphs there are versions for the case that the matchbox manifold is orientable.

If, for example, in the pasting theorem the matchbox V_1 (or V_2) is coherently directed by h_1 (or h_2), then V is coherently directed by h. This observation is made at the end of the proof of the pasting theorem in [1], 2.1 and can with some small modifications be carried over to the general situation. Similarly there exist in an oriented matchbox manifold coherently directed long boxes. There also exist coherently directed wide matchboxes in an orientable matchbox manifold. To prove the existence one starts the proof of 2.4 with a collection of coherently directed matchboxes $h_i\colon F_{S_i} \to W_i$, $i = 0, 1, \ldots$ such that $\{h_i(E_{S_i}) \mid i = 0, 1, \ldots\}$ covers X and continus as in that same proof.

4. The return map. In this section we imitate the well-known definition of the Poincaré return map in the abstract setting of matchbox manifolds.

4.1. Standing notation. Throughout Section 4 the following notation is fixed. X is an orientable matchbox manifold. $\{C_\alpha|\alpha\in A\}$ is the collection of arc components and $\{p_\alpha|\alpha\in A\}$ is a proper parametrization; $p_\alpha\colon \boldsymbol{R}\to C_\alpha$, $\alpha\in A$. It is to be noticed that for any $x\in X$ there exists $\alpha\in A$ and $t\in\boldsymbol{R}$ such that $x = p_\alpha(t)$. V always denotes a coherently directed matchbox in X with parametrization $h\colon F_S\to V$. All matchboxes are assumed to be coherently directed. The *zero-level* of V is $Z = \{h(x,0)|\ x\in S\}$.

4.2. Lemma. *For every $\alpha\in A$, $p_\alpha^{-1}(Z)$ is a countable and discrete subset of $\boldsymbol{R}$.*

Proof. In case C_α is a circle the intersection $C_\alpha\cap V$ consists of a finite and disjoint collection of arcs (Lemma 1.5). Each of these arcs is covered by a disjoint collection of arcs in $\boldsymbol{R}$ under the parametrization. So in this case the lemma is easily seen to be true. If C_α is not a circle, then p_α is bijective. In view of Theorem 3.1 $p_\alpha^{-1}(C_\alpha\cap V)$ is a countable and disjoint collection of arcs. In view of Lemma 1.5 the collection is discrete too. The lemma follows.

Definitions. A point $x = p_\alpha(t)$ from Z is called *escaping in the positive (negative) direction* if for no $s>t$ $(s<t)$ the point $p_\alpha(s)$ belongs to Z. If there exists an $s>t$ $(s<t)$ such that $p_\alpha(s)\in Z$, then $x = p_\alpha(t)$ is called *positively (negatively) returning*. In the case that $x = p_\alpha(t)$ is positively returning we write

$$r(x) = \min\{s\in\boldsymbol{R}|\,s>t \text{ and } p_\alpha(s)\in Z\}.$$

The set of positively returning points of Z is denoted by $R(Z)$. The map $p\colon R(Z)\to Z$, defined by $p(x) = p_\alpha(r(x))$ is called the *(Poincaré) return map*.

Proposition. *The set $R(Z)$ of positively returning points of Z is an open subset of Z. The map $p\colon R(Z)\to Z$ is an injective, continuous and open mapping.*

Proof. This is another application of the lemma of the long box.

4.3. The following theorem deals with the enlarging of a wide matchbox. The goal, we have in mind, is the reduction of the portion of escaping points. We first prove the next lemma.

Lemma. *Assume that the standard matchbox F_S is endowed with a totally bounded metric. Let $\varepsilon>0$ be given. Then there exists a finite partition $\{K_j|\,j = 0,\dots,n\}$ of S into clopen sets and there are continuous functions $b_j\colon K_j\to(-1,1)$, $j = 1,\dots,n$, such that each simple matchbox $V_j = \{(x,u)|\,x\in K_j,\ b_j(x)\leqslant u\leqslant 1\}$ has diameter $<\varepsilon$, $j = 1,\dots,n$.*

Proof. Let $\mathscr{V}$ be an open finite collection in F_S of mesh $<\varepsilon$, the union of which covers $S\times\{1\}$. Let $\mathscr{V}'$ be the trace of $\mathscr{V}$ on $S\times\{1\}$. Because S is zero-dimensional, there is a partition $\{K_i|\ i = 0,\dots,n\}$ of S into clopen subsets such that $\{K_i\times\{1\}|\ i = 0,\dots,n\}$ is a refinement of $\mathscr{V}'$. Now fix $j\in\{0,\dots,n\}$. For each $x\in K_j$ there is a clopen neighborhood U_x of x in K_j (whence in S) and a $c_x\in(-1,1)$ such that $U_x\times(c_x,1]$ is contained in some element of $\mathscr{V}$.

Now $\bigcup\{U_x|\ x\in K_j\} = K_j$. Let (x_i) be countable sequence in K_j such that $\bigcup\{U_{x_i}|\ i = 0,1,\dots\} = K_j$. Write $K_{jk} = U_{x_k}\setminus\bigcup\{U_{x_i}|\ i = 0,\dots,k-1\}$ and define $b_j\colon K_j\to(-1,1)$ by $b_j(x) = c_{x_k}$ for $x\in K_{jk}$.

Theorem. *Let V be a coherently directed and wide matchbox. Assume that $Z\setminus R(Z)\neq\emptyset$. Let U be an open subset of X such that $U\cap V = \emptyset$ and for any $x = p_\alpha(t)\in Z\setminus R(Z)$ for some $u>t$ we have $p_\alpha(u)\in U$. Then there is a matchbox V' with zero-level Z' such that $V'\subset U$ and $Z\subset R(Z\cup Z')$.*

Proof. Let $x = p_\alpha(t)$ be any point in $Z\setminus R(Z)$ and $p_\alpha(u)\in U$, $u>t$. We apply the lemma of the long box with $V_1 = V$, V_2 is a suitable matchbox neighborhood of $p_\alpha(u)$, which is contained in U, and J is the arc $\{p_\alpha(s)|\,t\leqslant s\leqslant u\}$. We then get a matchbox V_x with parametrization $k_x\colon F_{S_x}\to V_x$ such that

$$x = p_\alpha(t)\in k_x(S_x\times\{-1\})\subset Z, \quad\text{and}\quad p_\alpha(u)\in k_x(S_x\times\{1\})\subset U.$$

After having replaced S_x by a suitable clopen subset of itself we may assume that there exists $c_x<1$ such that $k_x(S_x\times[c_x,1])\subset U$. In this way we get a collection $\{k_x(S_x\times\{-1\})|\ x\in Z\setminus R(Z)\}$ of clopen subsets of Z, the union of which contains $Z\setminus R(Z)$. Because the spaces involved are separable and metrizable, there is a sequence (x_i) in $Z\setminus R(Z)$ such that

$$Z\setminus R(Z)\subset\bigcup\{k_{x_i}(S_{x_i}\times\{-1\})|\ i = 0,1,\dots\}.$$

We now write

$$K_n = k_{x_n}(S_{x_n}\times\{-1\})\setminus\bigcup\{k_{x_i}(S_{x_i}\times\{-1\})|\ i = 0,\dots,n-1\},$$

$$S_n = \mathrm{pr}_1 k_{x_n}^{-1}(K_n)\quad\text{and}\quad W_n = k_{x_n}(S_n\times[c_{x_n},1]),\quad n = 0,1,\dots$$

In this way we have obtained a disjoint family of matchboxes $\{W_n|\ n = 0,1,\dots\}$ in U such that for any $x = p_\alpha(t)\in Z\setminus R(Z)$ for some $u>t$ and for some n we have $p_\alpha(u)\in W_n$, $\otimes$.

We may as well assume that $\lim_{n\to\infty}\operatorname{diam}(W_n) = 0$. This can be achieved as follows. First X is endowed with a totally bounded metric. Then by applying the preceding lemma each W_n can be replaced by a finite disjoint collection of matchboxes of diameter $<\dfrac{1}{n+1}$ such that $\otimes$ still holds true. From $\{W_n|\ n = 0,1,\dots\}$ we delete all W_n such that $K_n\subset R(Z)$. In this way it is garanteed that for each W_n there exists an escaping point $p_\alpha(t)\in K_n$ such that for some $u>t$, $p_\alpha(u)\in W_n$.

In view of Proposition 1.4 it only remains to show that $\{W_n|\ n = 0,1,\dots\}$ is locally finite. This is done by deriving a contradiction from the hypothesis that $\{W_n|\ n = 0,1,\dots\}$ is not locally finite at some point x. We consider two cases.

Case 1. $x = p_\beta(v)$ and for some $w>v$ we have $p_\beta(w)\in Z$. By applying the lemma of the long box it can be seen that there is a coherently directed matchbox V_1 with parametrization $h_1\colon F_{T_1}\to V_1$ such that $x\in h_1(E_{T_1})$ and $h_1(T_1\times\{1\})$ is a clopen neighborhood of $p_\beta(w)$ in Z. As $\lim_{n\to\infty} \operatorname{diam} W_n = 0$, for some n we must have $W_n \subset h_1(E_{T_1})$.

Then for some $x_1 = p_{\alpha_1}(t_1) \in Z\setminus R(Z)$ and some $u_1 > t_1$, $p_{\alpha_1}(u_1) \in W_n$. Following the matches in V_1 we find $u_2 > u_1$ such that $p_{\alpha_1}(u_2) \in Z$. A contradiction.

Case 2. $x = p_\beta(v)$ and Case 1 does not hold. As V is a wide matchbox for some $w < v$ we must have $p_\beta(w) \in Z$. Without loss of generality we may assume that w is the largest value for which this holds true. But then $p_\beta(w) \in Z\setminus R(Z)$ and $p_\beta(w) \in K_m$ for some m. As in Case 1 we get a coherently directed matchbox V_2 with parametrization $h_2: F_{T_2} \to V_2$ such that $x \in h_2(E_{T_2})$ and $h_2(T_2 \times \{-1\})$ a clopen neighborhood of $p_\beta(w)$ in K_m. As $\lim \operatorname{diam} W_n = 0$ for some W_n we must have $W_n \subset h_2(E_{T_2})$ and $n > m$. As X is atriodic, from this we get $K_n \cap K_m \neq \emptyset$. A contradiction.

4.4. We are going to show that there exists a wide matchbox V such that every point of Z is positively returning, i.e. $R(Z) = Z$. We first mimic the definition of limit sets in an abstract setting.

Definition. Suppose that the orientable matchbox manifold X is a subset of some compact space Y. Suppose that Z is a wide matchbox in X. Assume $Z\setminus R(Z) \neq \emptyset$. Let $B = \{\alpha \in A \mid p_\alpha(t) \in Z\setminus R(Z)$ for some $t \in \mathbf{R}\}$.

We define

$$\Omega(\beta) = \{x \mid x = \lim_{n\to\infty} p_\beta(t_n),\ t_n \to \infty\}, \qquad \beta \in B,$$

and

$$\Omega^* = \operatorname{cl}_Y \cup \{\Omega(\beta) \mid \beta \in B\}.$$

Lemma. $\Omega^* \cap X = \emptyset$.

Proof. We assume that $y \in \Omega^* \cap X$ for some $y \in Y$ and derive a contradiction. By applying the lemma of the long box we can find a coherently directed matchbox V in X with parametrization $h: F_S \to V$ such that $y \in h(E_S)$ and

$$h(S \times \{-1\}) \subset Z \text{ or } h(S \times \{1\}) \subset Z.$$

Let U be an open set in Y such that $U \cap X = h(E_S)$. Then there exists $\beta \in B$ and $x \in U$ such that $x \in \Omega(\beta)$. Write $x = \lim_{n\to\infty} p_\beta(t_n)$, $t_n \to \infty$. Then for some N for all $n > N$, $p_\beta(t_n) \in U$ and consequently $p_\beta(t_n) \in h(E_S)$. By applying Lemma 1.5 we easily obtain a sequence (s_n) such that $s_n \to \infty$ and $p_\beta(s_n) \in Z$. By the definition of B however there exists t_0 such that $p_\beta(t_0) \in Z\setminus R(Z)$. But then $s_n \leqslant t_0$ for all $n \in N$. This is a contradiction.

Theorem. *Let X be an orientable matchbox manifold. Then there exists a wide matchbox V such that every point of Z is positively returning and the map $p: Z \to Z$ is a homeomorphism.*

Proof. Let Y be any compactification of X. Let V_0 be any wide matchbox in X with zero-level Z_0. If $Z_0\setminus R(Z_0) \neq \emptyset$, we continue as follows. Inductively on n we shall define matchboxes V_n, $n = 0, 1, \ldots$ such that

(i) $\{V_0, V_1, \ldots, V_n\}$ is a disjoint collection, $n = 1, 2, \ldots$,
(ii) $Z_0 \cup \ldots \cup Z_{n-1} \subset R(Z_0 \cup \ldots \cup Z_n)$, $n = 1, 2, \ldots$,
(iii) $\{V_0, V_1, \ldots\}$ is locally finite.

Suppose that Ω^* is as in the above definition.

Let (U_n) be any decreasing sequence of open neighborhoods such that Ω^* is the intersection of $\{\operatorname{cl}_Y U_n \mid n = 0, 1, \ldots\}$. Let $n \in N$. Assuming that $V_0, \ldots, V_{n-1}$ have been defined satisfying (i) and (ii) we define $V^* = V_0 \cup \ldots \cup V_{n-1}$. In view of (i) and Lemma 1.4 V^* is a coherently directed and wide matchbox. We write Z^* for the zero-level of V^*. We may assume $Z^* = Z_0 \cup \ldots \cup Z_{n-1}$. Let

$$U^* = (U_n \cap X)\setminus V^*.$$

It is to be observed that $Z^*\setminus R(Z^*) \neq \emptyset$. Let $x = p_\alpha(t) \in Z^*\setminus R(Z^*)$. Because Y is compact, $\Omega(\alpha) \neq \emptyset$. As $\Omega(\alpha) \subset \Omega^*$, for some $u > t$ we must have $p_\alpha(u) \in U^*$. By Theorem 4.3 it follows that there exists a matchbox V_n with zero-level Z_n such that $R(Z^* \cup Z_n) \supset Z^*$. It follows that $\{V_0, \ldots, V_n\}$ satisfies (i) and (ii). As $\cap\{\operatorname{cl}_Y U_n \mid n = 0, 1, \ldots\} = \Omega^*$, it is easily seen that (iii) holds true also.

In this way we get a matchbox $V = \bigcup\{V_n \mid n = 0, 1, \ldots\}$ with zero-level $Z = \bigcup\{Z_n \mid n = 0, 1, \ldots\}$ such that $Z\setminus R(Z) = \emptyset$. Then in view of Proposition 4.2 $p: Z \to Z$ is a topological embedding. By a similar procedure we can extend Z to make sure that every point of Z is negatively returning also. But then p is an onto map and a homeomorphism.

5. The structure theorem.

5.1. We first prove the characterization theorem which has been stated in Section 0.

Proof. The "only if"-part is Theorem 3.5. To prove the "if"-part let Z and $p: Z \to Z$ be as in Theorem 4.4. Z is the zero-set of the coherently directed matchbox V with parametrization $h: F_S \to V$. The pattern of this part of the proof is very similar to that in [1], 4.3. Using the lemma of the long box and the fact that Z is separable, we get a countable cover $\{W_0, W_1, \ldots\}$ of Z and a collection of matchboxes $\{V_0, V_1, \ldots\}$ with parametrizations $h_i: F_{S_i} \to V_i$, $i = 0, 1, \ldots$, such that

(i) W_i is a closed subset of X and an open subset of Z,
(ii) $h_i(S_i \times \{-1\}) = h(\mathrm{pr}_1 \circ h^{-1}(W_i) \times \{\frac{1}{4}\})$ and
$h_i(S_i \times \{1\}) = h(\mathrm{pr}_1 \circ h^{-1}(p(W_i)) \times \{-\frac{1}{4}\})$, $i = 0, 1, \ldots$

It is to be noticed that S_i is homeomorphic to W_i, $i = 0, 1, \ldots$ We may assume that the cover $\{W_0, W_1, \ldots\}$ is disjoint, whence locally finite. It then follows that

$\{V_0, V_1, \ldots\}$ is a disjoint and clopen cover of $X \setminus h(S \times (-\frac{1}{4}, \frac{1}{4}))$. Also it is easily seen that $\{h(S \times [-\frac{1}{4}, \frac{1}{4}])\} \cup \{V_i|\ i = 0, 1, \ldots\}$ is a locally finite closed cover of X.

Now let $Y = Z \times [0, 1]$. The map $\pi\colon Y \to X$ is defined as follows. For $(z, t) \in Z \times [0, \frac{1}{4}]$ we let $\pi(z, t) = h(\mathrm{pr}_1 \circ h^{-1}(z), t)$ and for $(z, t) \in Z \times [\frac{3}{4}, 1]$ we let $\pi(z, t) = h(\mathrm{pr}_1 \circ h^{-1}(p(z)), t-1)$. For each $i \geq 0$ the parametrization $h_i\colon F_{S_i} \to V_i$ is used to glue $W_i \times [\frac{1}{4}, \frac{3}{4}]$ into X in such a way that $\pi(W_i \times \{\frac{1}{4}\}) = h_i(S_i \times \{-1\}$, and $\pi(W_i \times \{\frac{3}{4}\}) = h_i(S_i \times \{1\})$. This can be done in view of (ii). It is not hard to see that the map π is continuous and closed. It follows that π is a quotient map. X is homeomorphic to $Y/\sim$, where $y \sim y'$ iff $\pi(y) = \pi(y')$. This is the same as $(z, 1) \sim (p(z), 0)$ for all $z \in Z$ and the theorem easily follows.

5.2. We now prove the structure theorem. Recall that the flows $\pi\colon X \times \mathbf{R} \to X$ and $\varphi\colon Y \times \mathbf{R} \to Y$ are said to be *topologically equivalent* if there is a homeomorphism $h\colon X \to Y$ which maps each orbit of π onto an orbit of the system φ and preserves the orientation. See [7]. Now let $\pi\colon X \times \mathbf{R} \to X$ be a flow. By Theorem 3.5 X is an orientable matchbox manifold. We may assume that the proper parametrization is induced by the motion. By the preceding theorem X is the phase space of a flow $\Sigma(S, f)$. As in both flows orbits and arc components coincide, it follows that π and $\Sigma(S, f)$ are equivalent.

5.3. There is the following corollary.

COROLLARY. *Let $\pi\colon X \times \mathbf{R} \to X$ be a dynamical system without rest points on a one-dimensional space X. Then the flow π is embeddable in a flow $\tilde{\pi}\colon \tilde{X} \times \mathbf{R} \to \tilde{X}$ with $\tilde{X}$ a one-dimensional compactification of X.*

Proof. See e.g. [11] for information about extension of actions of systems. Write $(X, \pi) = \Sigma(S, f)$. It is well-known [3] that there is a zero-dimensional compactification $\tilde{S}$ and a homeomorphism $\tilde{f}\colon \tilde{S} \to \tilde{S}$ which is an extension of f. Define $\tilde{\pi}$ the suspension $\Sigma(\tilde{S}, \tilde{f})$.

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