Diamond and \( \lambda \)-systems

by

Alan H. Mekler \(^1\) (Burnaby) and Saharon Shelah \(^2\) (Jerusalem)

Abstract. The notion of a \( \lambda \)-system generalizes that of a stationary set. In this paper variants of diamond for \( \lambda \)-systems are considered. In particular, a form of the weak diamond principle is defined and shown to be consistent with the strong negation of the continuum hypothesis. An application of these principles is given to the Whitehead problem in abelian group theory.

§ 0. Introduction. In [S2] Shelah introduced the notion of a \( \lambda \)-system in order to analyze exactly how a non-free abelian group (or other structure) fails to be free. A \( \lambda \)-system is a generalization of a stationary set. In this paper we will consider variants of \( \diamondsuit \) for \( \lambda \)-systems. In Section 1 we will define what is meant by a \( \lambda \)-system and remark that \( \diamondsuit \) for all \( \lambda \)-systems is equivalent to \( \diamondsuit \) for all stationary sets. We will then introduce a new variant of \( \diamondsuit \), a definable version of the weak diamond principle. (The weak diamond principle was introduced in [DS].) We will show this principle for \( \lambda < 2^{2^{\aleph_0}} \) is consistent with \( 2^{\aleph_0} = 2^{\aleph_1} \). In fact it is true whenever we add \( 2^{\aleph_0} \) Cohen reals to the ground model.

In section 2 we will give an application of the definable weak diamond principle to the Whitehead problem for abelian groups. We will show that it is consistent with \( 2^{\aleph_0} = 2^{\aleph_1} \) that every Whitehead group is free. That this result is largely of technical interest seems in part a reflection on the psychology of mathematics. Although we are interested in knowing when statements are independent of CH, there is little interest in knowing when things are independent of \( \neg \text{CH} \). Of course there is no reason behind this view since CH has strong consequences while experience has shown that \( \neg \text{CH} \) has few consequences. However some mathematicians, including Woodin [W], have studied the independence of statements from \( \neg \text{CH} \).

§ 1. \( \lambda \)-systems. After reading the definition of a \( \lambda \)-system, the reader may find it helpful to turn to Section 2 and see how \( \lambda \)-systems naturally arise.

Definition. Assume \( \lambda \) is a regular uncountable cardinal. A \( \lambda \)-system is a labeled subtree \( \langle S, \langle B_\alpha \rangle \lambda; \eta @ S \rangle \) of \( ^{<\lambda}\lambda \) satisfying:

\[ (0) \ B_\gamma \chi = 0 ; \]

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1.2. Theorem. It is consistent with $2^\omega = 2^\omega$ that for every regular uncountable \( \lambda < 2^\omega \) and \( \lambda \)-system \( \mathcal{F} \), there is a \( \lambda \)-system \( \Phi(\mathcal{F}) \) holds. Further this combinatorial principle is true if $2^\omega$ Cohen reals are added to the ground model.

Proof. Assume that we have added $2^\omega$ Cohen reals to \( V \) in order to obtain \( V[G] \). Working now in \( V[G] \), we consider \( A \) and \( \mathcal{F} \) as above. By considering the forcing as a two stage iteration we can assume that \( A \) and \( \mathcal{F} \) \( \in V \). Choose \( |S| \) Cohen reals and index them as \( r_\eta \) for \( \eta \in S \). Define \( \sigma(\eta) = 0 \) if \( \eta = 0 \) \( r_\eta \), then \( r_\eta \in I \), index the Cohen reals where \( I \subset S \).

Suppose \( \psi : \{ \eta \in \mathcal{B}_\eta \mid \eta \leq \eta_\eta \} \). We now claim that there is a \( \lambda \)-set \( T \subset S \) such that for all \( \eta \in T \), there is a set \( l_\eta \subseteq I \) with \( r_\eta \neq I \), but \( \psi \mid \mathcal{B}_\eta \in V[\{ \eta : \eta \leq l_\eta \}] \). (The set \( I \) is the ground model, so there is no hidden subtlety in this intermediate model.) To define the \( I_\eta \) first assign \( (\eta) \) to each \( \eta \in \mathcal{B}_\eta \). We define \( T \) to be

\[ \{ \eta \in S : \exists \eta_\eta \leq \eta \in \mathcal{B}_\eta \text{ such that the value of } \psi(\eta) \text{ is determined by } (\eta) \} \]. Note that \( J_0 \) exists since we are using a c.c. forcing. Then define \( I_0 \) to be \( \cup J_0 (\eta \in \mathcal{B}_\eta) \). We define \( T \) to be

\[ \{ \eta \in S : \exists \eta_\eta \leq \eta \in \mathcal{B}_\eta \} \].

Here the \( \leq \) denotes the lexicographic order and \( \leq \) the tree order. Now let \( T \) be the subtree of \( S \) generated by those \( \eta \in T \cap S \) such that for all \( \eta \leq \eta \eta \in \mathcal{B}_\eta \). We must check that \( T \) is a \( \lambda \)-system.

To complete the proof, we have to appeal to a consequence of the weak homogeneity of Cohen Forcing (compare, for example, Exercise 25.9, p. 271 [J]).

If we add Cohen reals then any statement about elements of the ground model is decided by the trivial condition. That is, either it or its negation is forced by the empty function. Let \( \psi \) be the name for \( \psi \) which was used implicitly in the paragraph above. Consider any condition, \( p \). We need to establish that the set of conditions, \( \lim \geq p \) which force for some \( \eta \) that \( \sigma(\eta) = \psi(\eta) \), is dense below \( p \). Obviously it is enough to show that one such condition exists. But this is easy: we can first choose \( \eta \in T \) such that \( \sigma(\eta) \) contains no information about \( r_\eta \). Then, by adding to \( p \) information about some \( r_\eta \) where \( \eta \in I_\eta \), we can get a condition \( p' \) such that \( p' \) decides \( \psi(\eta) \).

Finally we choose \( q \) so that \( q \leftrightarrow 0 \) \( r_\eta \), \( q' \) is \( \psi(\eta) \) and \( \psi(\eta) = 0 \). Obviously.

Although the above principle is consistent with $2^\omega = 2^\omega$, it is not a theorem of ZFC. The argument in [DS] that the uniformization principles true under MA-\( \pm \Gamma \) CH contradicts \( \Phi(\omega) \) also shows MA-\( \pm \Gamma \) CH contradicts \( \Phi(\omega) \). In any case we will see in the next section that \( \Phi(\omega) \) implies that groups, which assuming MA-\( \pm \Gamma \) CH would be Whitehead groups, are not.
1.3. COROLLARY. It is consistent with $2^\omega = 2^n$ that for all $\lambda < 2^\omega$ and $\lambda$-system $\mathcal{S}$, \( \Phi_\omega(\mathcal{S}) \) holds and that for all regular $\kappa \geq 2^\omega$ and $E$ a stationary subset of $\kappa$ \( \Diamond(E) \) holds.

Proof. If we begin in the proof of Theorem 1.2 with a model where \( \Diamond(\omega) \) holds for all regular uncountable cardinals then it will still hold in the final model for all $\kappa \geq 2^\omega$.

For $(\lambda, \mu)$-systems we modify the definition of $\Phi_\omega(\mathcal{S})$ by allowing $A$ to have any cardinality $< 2^n$. With this modified definition, we can prove an analogue of Theorem 1.2 for $(\lambda, \mu)$-systems.

1.4. Theorem. Suppose $\chi < \kappa < \chi$, $\kappa^{\omega^*} = \mu$ and $\kappa^{\omega^*} = \chi$. There is a notion of forcing $\mathcal{P}$ such that $|\mathcal{P}| = \chi$ and $\mathbb{P}^* \text{MA}(\mu)$ and $\Phi_\omega(\mathcal{S})$ for any $(\lambda, \mu)$-system of cardinality $< \kappa$, where $\Phi_\omega$ is redefined as above.

Proof. First force to make $2^\omega = \mu$ and MA($\mu$) hold. Now if we add $\chi$ Cohen generic subsets of $\mu$, $\Phi_\omega(\mathcal{S})$ holds for all $(\lambda, \mu)$-systems of cardinality $< \kappa$. The verification of this is just as in the proof of Theorem 1.2. Since we have added no subsets of $\mu$ of cardinality $< \kappa$, MA($\mu$) still holds.

Suppose we want to improve the theorem above and have $2^\omega = \chi$. It seems as though we have to sacrifice some of the power of $\Phi_\omega(\mathcal{S})$. If we weaken the definition of $\Phi_\omega(\mathcal{S})$ to restrict to $\mathcal{P}$ which are not only definable but absolute, then we can have $2^\omega = \chi$. Rather than stating a theorem we will only indicate the changes in the proof.

The forcing is just doing a finite support iteration of c.c.c. posets of cardinality $\mu$. The rest of the proof is much the same.

§ 2. Whitehead groups and $\neg$CH. In this section we will explain how the principle defined in the previous section can be used to prove that every Whitehead group of power $< 2^\omega$ is free. We are glad to have the chance to again explain the connection between $\lambda$-systems and non-free abelian groups. We will use the word “group” to mean “abelian group”.

We first recall a definition and some of the basic facts about $\omega$-free groups.

DEFINITION. Suppose $A$ is an $\omega$-free group of cardinality $\kappa$, i.e. every subgroup of cardinality $< \kappa$ is free. An $\omega$-filtration of $A$ is a sequence $\langle A_\alpha : \alpha < \kappa \rangle$ of pure subgroups such that: for all $\alpha$, $|A_\alpha| < \kappa$; if $\alpha < \beta$, $A_\alpha \subseteq A_\beta$; if $\beta$ is a limit ordinal, then $A_\beta = \bigcup A_\alpha$; and for all $\alpha$ if $A_\alpha/A_\alpha$ is not $\omega$-free, then $A_{\alpha+1}/A_\alpha$ is not free.

Suppose $A$ is $\omega$-free and $|A| = \kappa$. If $\kappa$ is singular then $A$ is free [S1]. Further if $\kappa$ is regular and $\langle A_\alpha : \alpha < \kappa \rangle$ is an $\omega$-filtration of $A$, then $A$ is free iff

$$E = \{ \alpha : A_{\alpha+1}/A_\alpha \text{ is not free} \}$$

is not stationary (see for example [E]).

Fix now a $\lambda$-group $A$ of cardinality $\lambda$. We will now define an augmented $\lambda$-system associated to $A$. We will define the $\lambda$-system $\langle \beta, B_\beta \rangle \in \mathcal{S}$ by induction. As we will shall choose for $\eta \in \mathcal{S}$, $A_\eta \subseteq A$ such that $|A_\eta| = \lambda$ and $A_\eta + \sum_{i < \lambda} B_{\eta+i} \subseteq \sum_{i < \lambda} B_{\eta+i}$ is $\lambda$-$\omega$-free and not free. To begin we let $\lambda_\chi = \lambda$ and $A_{\chi^2} = A$. Suppose now that $\eta \in \mathcal{S}$ and everything has been defined for $\eta$. Choose $\langle B_{\eta+i} : i < \lambda \rangle$ so that $A_\eta = \bigsqcup B_{\eta+i}$; and $A_\eta + \sum_{i < \lambda} B_{\eta+i} \subseteq \sum_{i < \lambda} B_{\eta+i}$ a $\lambda_\eta$-filtration of $A_\eta + \sum_{i < \lambda} B_{\eta+i} \subseteq \sum_{i < \lambda} B_{\eta+i}$. Let

$$E_\eta = \{ \alpha : B_{\eta+i} + \sum_{i < \lambda} B_{\eta+i} \subseteq \sum_{i < \lambda} B_{\eta+i} \text{ is not free} \}.$$ 

For $\alpha \in E_\eta$, let $B_{\eta, \alpha} = B_\alpha$. Choose $A_{\eta, \alpha} \subseteq A_{\eta+1}$ so that for some $A_{\eta, \alpha}$

$$A_{\eta, \alpha} + \sum_{i < \lambda} B_{\eta, \alpha+i} \subseteq \sum_{i < \lambda} B_{\eta, \alpha+i}$$

is $\lambda_{\eta, \alpha}$-free but not free. Here we identify $\omega$-free with torsion-free.

The following proposition will be used later.

2.1. PROPOSITION. Let $A$ be a $\lambda$-group of cardinality $\lambda$ and use the notation above. For all $\eta \in \mathcal{S}$ there is a set of free generators, $X_\eta \subseteq B_\eta$, for $\sum_{i < \lambda} B_{\eta+i}$.

Proof. We show $X_\eta$ exists by induction on $|\eta|$. For $|\eta| = 0$, $B_\eta$ is a free group, so we can choose $X_\eta$ a set of free generators of $B_\eta$. In general suppose we have chosen $\eta \subseteq B_{\eta+1}$ a set of free generators for $\sum_{i < \lambda} B_{\eta+i}$. Since $\sum_{i < \lambda} B_{\eta+i} \subseteq \sum_{i < \lambda} B_{\eta+i}$ is free, we can choose $Z \subseteq B_\eta$ whose images freely generate this group. Then we let $X_\eta = Y \cup Z$.

Our treatment of Whitehead groups will follow that of [E]. Recall that a group $A$ is a Whitehead group iff $\text{Ext}(A, Z) = 0$. There are various ways to deal with Ext we will use factor sets and transformations. A factor set from $A$ to $G$ is a function $f : A \times A \rightarrow G$ satisfying:

$$f(u, v) = f(c, A); f(u, v) + f(u + v, w) = f(u, v + w) + f(v, w);$$

and $f(u, 0) = f(0, v) = 0$ for all $u, v, w \in A$. Let $\text{Fact}(A, G)$ denote the set of factor sets from $A$ to $G$. Given $h : A \rightarrow G$ such that $h(0) = 0$, define $h : A \rightarrow G$ to be the map $h(a, b) = h(a) + h(b) - h(a + b)$. The set of such maps denoted $\text{Trans}(A, G)$ is the set of transformation sets. It is standard that

$$\text{Ext}(A, G) = \text{Fact}(A, G) / \text{Trans}(A, G).$$

Further if $\delta h = \delta g$, then $h - g$ is a homomorphism. Suppose that $A$ is a free group and $X$ is a set of free generators of $A$. Then if $f$ is a factor set there is a unique $h$, which we denote $h_{X,f}$ such that $h(x) = 0$ for all $x \in X$ and $\delta h = f$. 4
To apply the definable weak diamond principle we will need the following "killing lemma".

2.2. Lemma. Suppose $K \subseteq F$ where $F$ is a free group and $X$ is a set of free generators for $K$. Further suppose that $0: K \to Z$ is a homomorphism which does not extend to $F$. Then for any $g: X \to Z$ either (i) for all factor sets $f \in \text{Fact}(K, Z)$, if $h_{g,f}(\equiv): F \to Z$ then for all $g \equiv g_0: F \to Z$ $\delta k \neq \delta g$ or (ii) for all $f \in \text{Fact}(K, Z)$, if $h_{g,f}(\equiv): F \to Z$ then for all $g \equiv g_0: F \to Z$ $\delta k \neq g_0$.

Proof. Suppose not. Let $g, g_0, g_1, h_0, f_0, f_1$ and $h_1$ form a counterexample. Then $h_0 - g_0 = \delta_0$ and $h_1 - g_1 = \delta_1$ are homomorphisms from $F$ to $Z$. For any $x \in X$, $0_1 - 0_0(x) = 0_1(x)$. Since $X$ generates $K$, $(0_1 - 0_0) = 0$. But $0$ does not extend to $F$, so we have a contradiction.

2.3. Theorem. It is consistent with $2^{ \aleph_0 } = 2^\omega$ that every Whitehead group is free.

Proof. We assume that the set theoretic principles of Corollary 1.3 holds. Since for all regular $\kappa \geq 2^{ \aleph_0 }$ and stationary set $E \subseteq \kappa \cap (E)$ holds, it is enough to verify that any Whitehead group of cardinality $< 2^{ \aleph_0 }$ is free. Once we have this the proof can be completed as Shelah did it originally [E]. Suppose that $|A| = \lambda$, $\lambda$ is regular, $A$ is $\omega$-free and $A$ is not free. Let $(S, \{ \lambda_\eta, A_\eta, B_\eta : \eta \in S \})$ be an augmented $\lambda$-system associated with $A$. We will define a group structure on $Z \times A$. So that the extension

$$0 \to Z \to Z \times A \to A \to 0$$

is exact, where $\lambda(\eta) = \langle \eta, 0 \rangle$ and $\lambda(\eta, 0) = \eta$. For each $\eta \in S_\eta$ choose $X_\eta \subseteq B_\eta$ a set of free generators for $\bigoplus_{i, \eta \in S_\eta} B_{\eta i}$ and $\theta_\eta : \bigoplus_{i, \eta \in S_\eta} B_{\eta i} \to Z$ a homomorphism which does not extend to $A_\eta + \bigoplus_{i, \eta \in S_\eta} B_{\eta i}$. The sequence $(X_\eta, \theta_\eta : \eta \in S_\eta)$ and the augmented $\lambda$-system are used in the definition of $\Psi$. For $\eta \in S_\eta$ and $g : B_\eta \to \chi$, let $\Psi(g) = 0$ iff (i) of Lemma 2.2 holds with respect to $X_\eta, \theta_\eta, \bigoplus_{i, \eta \in S_\eta} B_{\eta i}$ and $A_\eta + \bigoplus_{i, \eta \in S_\eta} B_{\eta i}$.

We now define $f \in \text{Fact}(A, Z)$ by induction on the lexicographic order of $S$. Before we do note that for all $\eta < \nu, A_\eta \subseteq B_\eta$. For any $\eta \in S$ let $f_\eta$ be any element of $\text{Fact}(\bigoplus_{i, \eta \in S_\eta} B_{\eta i}, Z)$ which extends $\bigcup_{\eta \neq \nu} f_\nu(0 < \eta)$. If $\eta \neq S_\eta$, then let $f_\eta = f_\nu$. Suppose $\eta \in S_\eta$ if $\sigma(\eta) = 0$ choose $h_{\eta, \eta} \subseteq h_\eta : A_\eta + \bigoplus_{i, \eta \in S_\eta} B_{\eta i} \to Z$ and if $\sigma(\eta) = 1$ choose $h_{\eta, \eta} + \theta_\eta : A_\eta + \bigoplus_{i, \eta \in S_\eta} B_{\eta i} \to Z$. The rest of the verification is exactly the same as the proof in [E].

Remark. It is also possible to give a definable weak diamond which makes no mention of $\lambda$-systems. This principle would also suffice to prove the theorem above. We have avoided doing this for two reasons. First we wanted to show how to use the definable weak diamond for $\lambda$-systems. In $L$ where we have $\Theta$ for every $\lambda$-system the method above gives a non-inductive proof that every Whitehead group is free.