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Diamond and λ-systems

by

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Abstract. The notion of a λ-system generalizes that of a stationary set. In this paper variants of diamond for λ-systems are considered. In particular, a form of the weak diamond principle is defined and shown to be consistent with the strong negation of the continuum hypothesis. An application of these principles is given to the Whitehead problem in abelian group theory.

§ 0. Introduction. In [S2] Shelah introduced the notion of a λ-system in order to analyze exactly how a non-free abelian group (or other structure) fails to be free. A λ-system is a generalization of a stationary set. In this paper we will consider variants of $\diamondsuit$ for λ-systems. In Section 1 we will define what is meant by a λ-system and remark that $\diamondsuit$ for all λ-systems is equivalent to $\diamondsuit$ for all stationary sets. We will then introduce a new variant of $\diamondsuit$, a definable version of the weak diamond principle. (The weak diamond principle was introduced in [DS].) We will show this principle for $\lambda < 2^{\aleph_0}$ is consistent with $2^{\aleph_0} = 2^{\aleph_1}$. In fact it is true whenever we add $2^{\aleph_0}$ Cohen reals to the ground model.

In section 2 we will give an application of the definable weak diamond principle to the Whitehead problem for abelian groups. We will show that it is consistent with $2^{\aleph_0} = 2^{\aleph_1}$, that every Whitehead group is free. That this result is largely of technical interest seems in part a reflection on the psychology of mathematics. Although we are interested in knowing when statements are independent of CH, there is little interest in knowing when things are independent of $\neg$CH. Of course there is reason behind this view since CH has strong consequences while experience has shown that $\neg$CH has few consequences. However some mathematicians, including Woodin [W], have studied the independence of statements from $\neg$CH.

§ 1. λ-systems. After reading the definition of a λ-system, the reader may find it helpful to turn to Section 2 and see how λ-systems naturally arise.

Definition. Assume λ is a regular uncountable cardinal. A *λ-system* is a labeled subtree $\langle S, \langle B_\eta, \lambda_\eta \colon \eta \in S\rangle\rangle$ of ${}^{<\omega}\lambda$ satisfying:

(0) $B_{\langle\,\rangle} = 0$;

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(1) $\lambda = \lambda_{\langle\,\rangle}$;
(2) for all $\eta \in S$, λ_η is regular;
(3) $\eta \in S_f$ (the terminal nodes of S) iff $\lambda_\eta = \omega$;
(4) suppose η is not terminal then
  (a) $E_\eta = \{i: \eta^\frown\langle i\rangle \in S\}$ is stationary in λ_η;
  (b) for all $i \in E_\eta$, $\lambda_{\eta^\frown\langle i\rangle} \leqslant |B_{\eta^\frown\langle i\rangle}| < \lambda_\eta$;
  (c) if $i < j \in E_\eta$, then $B_{\eta^\frown\langle i\rangle} \subseteq B_{\eta^\frown\langle j\rangle}$;
  (d) if $j \in E_\eta$ and j is a limit point of E_η, then

$$B_{\eta^\frown\langle j\rangle} = \bigcup \{B_{\eta^\frown\langle i\rangle}: i<j,\ i \in E_\eta\}.$$

A set S satisfying (1)–(4a) above is called a *λ-set*.

To simplify notation we let $\bar{B}_\eta = \bigcup B_{\eta\restriction l}$ $(1 \leqslant l \leqslant l(\eta))$. Here $l(\eta)$ denotes the length of η.

Remark. The notion of a λ-system was introduced in [S2]. Here we have used the definition from the appendix of that paper which differs inessentially from the definition in the main body. If we assume that the B_η's are disjoint except as demanded in the definition above, then any λ-system is essentially determined by the associated λ-set.

Although our main interest in this paper is in λ-systems as we have defined them, we can replace $\aleph_0$ by any regular cardinal μ and define for regular cardinals $\lambda > \mu$ a *(λ, μ)-system*. In the definition above the following changes must be made. First λ is required to be a regular cardinal $> \mu$. In clause (2), each λ_η is required to be a regular cardinal $\geqslant \mu$. Finally in clause (3), replace ω by μ. In particular, a $(\lambda, \aleph_0)$-system is just a λ-system. The treatment of (λ, μ)-systems is parallel to that of λ-systems.

DEFINITION. Suppose $\mathscr{S} = \langle S, \langle B_\eta, \lambda_\eta: \eta \in S\rangle\rangle$ is a λ-system; by $\diamondsuit(\mathscr{S})$ we denote the following principle:

There is a collection $\langle C_\eta: \eta \in S\rangle$ such that for any $X \subseteq \bigcup_{\eta \in S} B_\eta$ there is a λ-set $T \subseteq S$ satisfying, for all $\eta \in T$, $X \cap \bar{B}_\eta = C_\eta$.

1.1. PROPOSITION. *For any regular cardinal λ, $\diamondsuit(\mathscr{S})$ holds for every λ-system $\mathscr{S}$ iff for every regular cardinal $\varkappa \leqslant \lambda$ and E a stationary subset of $\varkappa$ $\diamondsuit(E)$ holds.*

In order to avoid the apparent contradiction of having the weak diamond principle together with $2^{\aleph_0} = 2^{\aleph_1}$ we will use definable maps.

DEFINITION. Suppose $\mathscr{S} = \langle S, \langle B_\eta, \lambda_\eta: \eta \in S\rangle\rangle$ is a λ-system, and let $\Phi_d(\mathscr{S})$, definable weak diamond, denote the following principle:

Suppose A is a set, χ is a regular uncountable cardinal, and $|A| < 2^{\aleph_0}$. If Ψ: $\{g: g: \bar{B}_\eta \to \chi, \eta \in S_f\} \to \{0, 1\}$ is definable from A and $\mathscr{S}$ and ordinal parameters then there is a $\sigma: S_f \to \{0, 1\}$ such that for every $\psi: \bigcup_{\eta \in S} B_\eta \to \chi$ there is $\eta \in S_f$ such that $\Psi(\psi \restriction \bar{B}_\eta) = \sigma(\eta)$.

1.2. THEOREM. *It is consistent with $2^{\aleph_0} = 2^{\aleph_1}$ that for every regular uncountable $\lambda < 2^{\aleph_0}$ and λ-system $\mathscr{S}$, $\Phi_d(\mathscr{S})$ holds. Further this combinatorial principle is true if $2^{\aleph_0}$ Cohen reals are added to the ground model.*

Proof. Assume that we have added $2^{\aleph_1}$ Cohen reals to V in order to obtain $V[G]$. Working now in $V[G]$, we consider A and $\mathscr{S}$ as above. By considering the forcing as a two stage iteration we can assume that A and $\mathscr{S} \in V$. Choose $|S_f|$ Cohen reals and index them as r_η for $\eta \in S_f$. Define σ by $\sigma(\eta) = 0$ iff $0 \in r_\eta$. Let r_i, $i \in I$, index the Cohen reals where $I \supseteq S_f$.

Suppose $\psi: \bigcup_{\eta \in S} B_\eta \to \chi$. We now claim that there is a λ-set $T \subseteq S$ such that for all $\eta \in T_f$ there is a set $I_\eta \subseteq I$ with $r_\eta \notin I_\eta$ but $\psi \restriction \bar{B}_\eta \in V[\{r_i: i \in I_\eta\}]$. (The set I_η will be in the ground model, so there is no hidden subtlety in this intermediate model.) To define the I_η, first assign (in V) to each $b \in \bigcup_{\eta \in S} B_\eta$ a countable index set $J_b \subseteq I$ such that the value of $\psi(b)$ is determined by $\{r_j: j \in J_b\}$. Note that J_b exists since we are using a c.c.c. forcing. We then define I_η to be $\bigcup J_b$ $(b \in \bar{B}_\eta)$. We define $\hat{T}$ to be

$$\{\eta \in S\colon \text{if } \upsilon \leqslant_l \eta,\ b \in B_\upsilon \text{ and } \eta \leqslant \eta' \in S_f, \text{ then } \eta' \notin J_b\}.$$

Here the $\leqslant_l$ denotes the lexicographic order and $\leqslant$ the tree order. Now let T be the subtree of S generated by those elements η of $\hat{T} \cap S_f$ such that for all $l \leqslant l(\eta)$ $\eta \restriction l \in \hat{T}$. We must check that T is a λ-system. Note that $\langle\ \rangle \in \hat{T}$. So it suffices to show that if $\eta \in \hat{T}$ then $\{\alpha: \eta^\frown\langle\alpha\rangle \in \hat{T}\}$ is stationary in λ_η. For α such that $\eta^\frown\langle\alpha\rangle \in S$ let $f(\alpha) = \sup\{\gamma$: for some $\upsilon \geqslant \eta^\frown\langle\alpha\rangle$ and $b \in B_\upsilon$ there is $\upsilon' \geqslant \eta^\frown\langle\nu\rangle$ such that $\upsilon' \in J_b$. Since λ_η is a regular cardinal, f is a function from λ_η to λ_η. For definiteness let $f(\alpha) = 0$ if it is not defined by the condition above. Let $C \subseteq \lambda_\eta$ be the closed unbounded set, $\{\beta$: for all $\alpha < \beta$, $f(\alpha) < \beta$ and β is a limit point of $E_\eta\}$. Then for any $\alpha \in C \cap E_\eta$, $\eta^\frown\langle\alpha\rangle \in \hat{T}$.

To complete the proof, we have to appeal to a consequence of the weak homogeneity of Cohen Forcing (compare, for example, Exercise 25.9, p. 271 [J]). If we add Cohen reals then any statement about elements of the ground model is decided by the trivial condition. That is, either it or its negation is forced by the empty function. Let $\tilde{\psi}$ be the name for ψ which was used implicitly in the paragraph above. Consider any condition, p. We need to establish that the set of conditions, $q \supseteq p$ which force for some η that $\sigma(\eta) = \Psi(\psi \restriction \bar{B}_\eta)$ is dense below p. Obviously it is enough to show that one such condition exists. But this is easy: we can first choose $\eta \in T_f$ such that p contains no information about r_η. Then, by adding to p information about some r_i where $i \in I_\eta$, we can get a condition p' such that p' decides $\Psi(\psi \restriction \bar{B}_\eta)$. Finally we choose q so that $q \Vdash 0 \in r_\eta$ iff $\Psi(\psi \restriction \bar{B}_\eta) = 0$.

Although the above principle is consistent with $2^{\aleph_0} = 2^{\aleph_1}$, it is not a theorem of ZFC. The argument in [DS] that the uniformization principles true under MA+ $\neg$CH contradict $\Phi(\omega_1)$ also shows MA+ $\neg$CH contradicts $\Phi_d(\omega_1)$. In any case we will see in the next section that $\Phi_d(\omega_1)$ implies that groups, which assuming MA+ $\neg$CH would be Whitehead groups, are not.

1.3. Corollary. *It is consistent with $2^{\aleph_0} = 2^{\aleph_1}$ that for all $\lambda < 2^{\aleph_0}$ and λ-system $\mathscr{S}$, $\Phi_d(\mathscr{S})$ holds and that for all regular $\varkappa \geqslant 2^{\aleph_0}$ and E a stationary subset of $\varkappa$ $\diamondsuit(E)$ holds.*

Proof. If we begin in the proof of Theorem 1.2 with a model where $\diamondsuit^{+}(\varkappa)$ holds for all regular uncountable cardinals then it will still hold in the final model for all $\varkappa \geqslant 2^{\aleph_0}$.

For (λ, μ)-systems we modify the definition of $\Phi_d(\mathscr{S})$ by allowing A to have any cardinality $< 2^{\mu}$. With this modified definition, we can prove an analogue of Theorem 1.2 for (λ, μ)-systems.

1.4. *Theorem. Suppose $\aleph_0 < \mu < \chi$, $\mu^{<\mu} = \mu$ and $\chi^{<\mu} = \chi$. There is a notion of forcing $\boldsymbol{P}$, such that $|\boldsymbol{P}| = \chi$ and $\Vdash_{\boldsymbol{P}}$ "MA(μ) and $\Phi_d(\mathscr{S})$ for any (λ, μ)-system of cardinality $< \chi$", where Φ_d is redefined as above.*

Proof. First force to make $2^{\aleph_0} = \mu$ and MA(μ) hold. Now if we add χ Cohen generic subsets of μ, $\Phi_d(\mathscr{S})$ holds for all (λ, μ)-systems of cardinality $< \chi$. The verification of this is just as in the proof of Theorem 1.2. Since we have added no subsets of μ of cardinality $< \mu$, MA(μ) still holds.

Suppose we want to improve the theorem above and have $2^{\aleph_0} = \chi$. It seems as though we have to sacrifice some of the power of $\Phi_d(\mathscr{S})$. If we weaken the definition of $\Phi_d(\mathscr{S})$ to restrict to Ψ which are not only definable but absolute, then we can have $2^{\aleph_0} = \chi$. Rather than stating a theorem we will only indicate the changes in the proof. The forcing is just doing a finite support iteration of c.c.c. posets of cardinality μ. The rest of the proof is much the same.

§ 2. Whitehead groups and $\neg$CH. In this section we will explain how the principle defined in the previous section can be used to prove that every Whitehead group of power $< 2^{\aleph_0}$ is free. We are glad to have the chance to again explain the connection between λ-systems and non-free abelian groups. We will use the word "group" to mean "abelian group".

We first recall a definition and some of the basic facts about $\varkappa$-free groups.

Definition. Suppose A is a $\varkappa$-free group of cardinality $\varkappa$, i.e. every subgroup of cardinality $< \varkappa$ is free. A *$\varkappa$-filtration* of A is a sequence $\langle A_\alpha\colon \alpha < \varkappa\rangle$ of pure subgroups such that: for all α, $|A_\alpha| < \varkappa$; if $\alpha < \beta$, $A_\alpha \subseteq A_\beta$; if β is a limit ordinal, then $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$; and for all α if A/A_α is not $\varkappa$-free, then $A_{\alpha+1}/A_\alpha$ is not free.

Suppose A is $\varkappa$-free and $|A| = \varkappa$. If $\varkappa$ is singular then A is free [S1]. Further if $\varkappa$ is regular and $\langle A_\alpha\colon \alpha < \varkappa\rangle$ is a $\varkappa$-filtration of A, then A is free iff

$$E = \{\alpha\colon A_{\alpha+1}/A_\alpha \text{ is not free}\}$$

is not stationary (see for example [E]).

Fix now a λ-free group A of cardinality λ. We will now define an augmented λ-system *associated* to A. We will define the λ-system $\langle S, \langle B_\eta, \lambda_\eta\colon \eta \in S\rangle\rangle$ by induction. As well we shall choose for $\eta \in S$, $A_\eta \subseteq A$ such that: $|A_\eta| = \lambda_\eta$; $A_\eta + \sum_{l\leqslant l(\eta)} B_{\eta\restriction l}/\sum_{l\leqslant l(\eta)} B_{\eta\restriction l}$ is λ_η-free and not free. To begin we let $\lambda_{\langle\,\rangle} = \lambda$, and $A_{\langle\,\rangle} = A$. Suppose now that $\eta \in S$ and everything has been defined for η. Choose $\langle B_\alpha\colon \alpha\langle\lambda_\eta\rangle$ so that $A_\eta = \bigcup B_\alpha$; and $\langle B_\alpha + \sum_{l\leqslant l(\eta)} B_{\eta\restriction l}/\sum_{l\leqslant l(\eta)} B_{\eta\restriction l}\colon \alpha < \lambda_\eta\rangle$ is a λ_η-filtration of $A_\eta + \sum_{l\leqslant l(\eta)} B_{\eta\restriction l}/\sum_{l\leqslant l(\eta)} B_{\eta\restriction l}$. Let

$$E_\eta = \{\alpha\colon B_{\alpha+1} + \sum_{l\leqslant l(\eta)} B_{\eta\restriction l}/B_\alpha + \sum_{l\leqslant l(\eta)} B_{\eta\restriction l} \text{ is not free}\}.$$

For $\alpha \in E_\eta$, let $B_{\eta^\frown\langle\alpha\rangle} = B_\alpha$. Choose $A_{\eta^\frown\langle\alpha\rangle} \subseteq B_{\alpha+1}$ so that for some $\lambda_{\eta^\frown\langle\alpha\rangle}$,

$$A_{\eta^\frown\langle\alpha\rangle} + B_{\eta^\frown\langle\alpha\rangle} + \sum_{l\leqslant l(\eta)} B_{\eta\restriction l}/B_{\eta^\frown\langle\alpha\rangle} + \sum_{l\leqslant l(\eta)} B_{\eta\restriction l}$$

is $\lambda_{\eta^\frown\langle\alpha\rangle}$-free but not free. Here we identify ω-free with torsion-free.

The following proposition will be used later.

2.1. Proposition. *Let A be a λ-free group of cardinality λ and use the notation above. For all $\eta \in S$ there is a set of free generators, $X_\eta \subseteq \bar{B}_\eta$, for $\sum_{l\leqslant l(\eta)} B_{\eta\restriction l}$.*

Proof. We show X_η exists by induction on $l(\eta)$. For $l(\eta) = 1$, B_η is a free group, so we can choose X_η a set of free generators of B_η. In general suppose we have chosen $Y \subseteq B_{\eta\restriction l(\eta)-1}$ a set of free generators for $\sum_{l<l(\eta)} B_{\eta\restriction l}$. Since

$$\sum_{l\leqslant l(\eta)} B_{\eta\restriction l}/\sum_{l<l(\eta)} B_{\eta\restriction l}$$

is free, we can choose $Z \subseteq B_\eta$ whose images freely generate this group. Then we let $X_\eta = Y \cup Z$.

Our treatment of Whitehead groups will follow that of [E]. Recall that a group A is a Whitehead group iff $\operatorname{Ext}(A, \boldsymbol{Z}) = 0$. There are various ways to deal with Ext we will use factor sets and transformations. A *factor set* from A to G is a function $f\colon A \times A \to G$ satisfying:

$$f(u, v) = f(v, u);\ f(u, v) + f(u+v, w) = f(u, v+w) = f(u, v+w) + f(v, w);$$

and $f(u, 0) = f(0, u) = 0$ for all $u, v, w \in A$. Let $\operatorname{Fact}(A, G)$ denote the set of factor sets from A to G. Given $h\colon A \to G$ such that $h(0) = 0$, define $\delta h\colon A + A \to G$ to be the map $\delta h(a, b) = h(a) + h(b) - h(a+b)$. The set of such maps denoted $\operatorname{Trans}(A, G)$ is the set of *transformation sets*. It is standard that

$$\operatorname{Ext}(A, G) = \operatorname{Fact}(A, G)/\operatorname{Trans}(A, G).$$

Further if $\delta h = \delta g$, then $h - g$ is a homomorphism. Suppose that A is a free group and X is a set of free generators of A. Then if f is a factor set there is a unique h, which we denote $h_{X,f}$ such that $h(x) = 0$ for all $x \in X$ and $\delta h = f$.

To apply the definable weak diamond principle we will need the following "killing lemma".

2.2. LEMMA. *Suppose $K \subseteq F$ where F is a free group and X is a set of free generators for K. Further suppose that $\theta\colon K \to \mathbf{Z}$ is a homomorphism which does not extend to F. Then for any $g\colon X \to \mathbf{Z}$ either* (i) *for all factor sets $f \in \mathrm{Fact}(K, \mathbf{Z})$, if $h_{X,f} \subseteq \bar{h}\colon F \to \mathbf{Z}$ then for all $g \subseteq \bar{g}\colon F \to \mathbf{Z}$ $\delta\bar{h} \neq \delta\bar{g}$ or* (ii) *for all $f \in \mathrm{Fact}(K, \mathbf{Z})$, if $h_{X,f}+\theta \subseteq \bar{h}\colon F \to \mathbf{Z}$ then for all $g \subseteq \bar{g}\colon F \to \mathbf{Z}$ $\delta\bar{h} \neq \delta\bar{g}$.*

Proof. Suppose not. Let $g, f_0, g_0, h_0, f_1, g_1$ and h_1 form a counterexample. Then $h_0 - g_0 = \theta_0$ and $h_1 - g_1 = \theta_1$ are homomorphisms from F to $\mathbf{Z}$. For any $x \in X$, $\theta_1 - \theta_0(x) = \theta(x)$. Since X generates K, $(\theta_1 - \theta_0) = \theta$. But θ does not extend to F, so we have a contradiction.

2.3. THEOREM. *It is consistent with $2^{\aleph_0} = 2^{\aleph_1}$ that every Whitehead group is free.*

Proof. We assume that the set theoretic principles of Corollary 1.3 holds. Since for all regular $\varkappa \geqslant 2^{\aleph_0}$ and stationary set $E \subseteq \varkappa$ $\diamondsuit(E)$ holds, it is enough to verify that any Whitehead group of cardinality $< 2^{\aleph_0}$ is free. Once we have this the proof can be completed as Shelah did it originally [E]. Suppose that $|A| = \lambda$, λ is regular, A is λ-free and A is not free.

Let $\langle S, \langle \lambda_\eta, A_\eta, B_\eta\colon \eta \in S\rangle\rangle$ be an augmented λ-system associated with A. We will define a group structure on $\mathbf{Z} \times A$. So that the extension

$$0 \to \mathbf{Z} \xrightarrow{\iota} \mathbf{Z} \times A \xrightarrow{\pi} A \to 0$$

is exact, where $\iota(n) = \langle n, 0\rangle$ and $\pi(\langle n, a\rangle) = a$. For each $\eta \in S_f$ choose $X_\eta \subseteq \bar{B}_\eta$ a set of free generators for $\sum_{l \leqslant l(\eta)} B_{\eta\restriction l}$ and $\theta_\eta\colon \sum_{l \leqslant l(\eta)} B_{\eta\restriction l} \to \mathbf{Z}$ a homomorphism which does not extend to $A_\eta + \sum_{l \leqslant l(\eta)} B_{\eta\restriction l}$. The sequence $\langle X_\eta, \theta_\eta\colon \eta \in S_f\rangle$ and the augmented λ-system are used in the definition of Ψ. For $\eta \in S_f$ and $g\colon \bar{B}_\eta \to \chi$, let $\Psi(g) = 0$ iff (i) of Lemma 2.2 holds with respect to X_η, θ_η, $\sum_{l \leqslant l(\eta)} B_{\eta\restriction l}$ and $A_\eta + \sum_{l \leqslant l(\eta)} B_{\eta\restriction l}$.

We now define $f \in \mathrm{Fact}(A, \mathbf{Z})$ by induction on the lexicographic order of S. Before we do note that for all $\eta <_l \upsilon$, $A_\eta \subseteq B_\upsilon$. For any $\eta \in S$ let f'_η be any element of $\mathrm{Fact}(\sum_{l \leqslant l(\eta)} B_{\eta\restriction l}, \mathbf{Z})$ which extends $\bigcup f_\upsilon(\upsilon <_l \eta)$. If $\eta \notin S_f$, then let $f_\eta = f'_\eta$. Suppose $\eta \in S_f$ if $\sigma(\eta) = 0$ choose $h_{X,f_\eta} \subseteq h\colon A_\eta + \sum_{l \leqslant l(\eta)} B_{\eta\restriction l} \to \mathbf{Z}$ and if $\sigma(\eta) = 1$ choose $h_{X,f_\eta} + \theta \subseteq h\colon A_\eta + \sum_{l \leqslant l(\eta)} B_{\eta\restriction l} \to \mathbf{Z}$. The rest of the verification is exactly the same as the proof in [E].

Remark. It is also possible to give a definable weak diamond which makes no mention of λ-systems. This principle would also suffice to prove the theorem above. We have avoided doing this for two reasons. First we wanted to show how to use the definable weak diamond for λ-systems. In L where we have $\diamondsuit$ for every λ-system the method above gives a non-inductive proof that every Whitehead group is free.

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