On diamond sequences

by

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Abstract. It is shown that the principle $\diamondsuit_\varkappa$ is equivalent to the existence of a size $2^\varkappa$ family F of partitions of $\varkappa$ into $\varkappa$ many pieces such that whenever A_α, $\alpha < \varkappa$, are pieces chosen from distinct members of F, the diagonal intersection of the A_α's is stationary. Various strengthenings of $\diamondsuit_\varkappa$ are considered. We also investigate reflection properties of diamond sequences.

0. Our set-theoretic notation is standard. Let us recall some definitions.

Let μ be an infinite limit ordinal, and let $S \subseteq \mu$. We say that $s_\alpha \subseteq \alpha$, $\alpha < \mu$, is a $\diamondsuit_\mu(S)$-sequence if for every $A \subseteq \mu$, the set $\{\alpha \in S\colon s_\alpha = A \cap \alpha\}$ is stationary in μ. The principle $\diamondsuit_\mu(S)$ asserts the existence of such a sequence. $\diamondsuit_\mu(\mu)$ is abbreviated as $\diamondsuit_\mu$.

If $\diamondsuit_\mu$ holds, then μ is an uncountable cardinal satisfying $\mu^{<\mu} = \mu$.

Given cardinals $\lambda \geqslant \mu$ and a set $S \subseteq [\lambda]^{<\mu}$, $\diamondsuit_{\mu,\lambda}(S)$ asserts the existence of a sequence $s_a \subseteq a$, $a \in [\lambda]^{<\mu}$, such that for any $A \subseteq \lambda$, the set $\{a \in S\colon A \cap a = s_a\}$ is stationary in $[\lambda]^{<\mu}$. Such a sequence is called a $\diamondsuit_{\mu,\lambda}(S)$-sequence.

If μ is regular and $S \subseteq \mu$, then $\diamondsuit_\mu(S)$ and $\diamondsuit_{\mu,\mu}(S)$ are easily seen to be equivalent.

For the duration of this paper, $\varkappa$ will denote a fixed uncountable cardinal. We let $(\varkappa)^\varkappa$ denote the collection of all partitions X of $\varkappa$ into $\varkappa$ many pieces $X(\alpha)$, $\alpha < \varkappa$.

A collection I of subsets of $\varkappa$ is said to be an ideal over $\varkappa$ if $E \in I$ whenever $E \subseteq A \cup B$ for $A, B \in I$. We set $I^+ = \{A \subseteq \varkappa\colon A \notin I\}$. I is $\varkappa$-complete if I is closed under unions of fewer than $\varkappa$ of its elements. I is normal if every regressive function $f\colon A \to \varkappa$, $A \in I^+$, is constant on some $B \in I^+$. Given $A \in I^+$, we set

$$I|A = \{B \subseteq \varkappa\colon A \cap B \in I\}.$$

$NS_\varkappa$ denotes the ideal of non-stationary subsets of $\varkappa$.

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1. In this section we investigate the properties of some families of partitions which are associated with diamond sequences.

Recall that by Ketonen's lemma (see, e.g., Proposition 1 in [M1]), $\varkappa^{<\varkappa} = \varkappa$ holds iff there are $X_\alpha \in (\varkappa)^\varkappa$, $\alpha < 2^\varkappa$, such that for each $h\colon 2^\varkappa \to \varkappa$, there is a uniform $\varkappa$-complete filter F_h over $\varkappa$ with each $X_\alpha(h(\alpha))$ in F_h. We shall show that if we further require that each F_h be normal, then the corresponding statement is equivalent to $\diamondsuit_\varkappa$.

The following is a joint result with R. B. Jensen.

PROPOSITION 1.1 (Jensen, Matet). *Given $S \subseteq \varkappa$, the following are equivalent*:

(i) *$\diamondsuit_\varkappa(S)$ holds.*

(ii) *There are $X_\alpha \in (\varkappa)^\varkappa$, $\alpha < 2^\varkappa$, such that the set*

$$S \cap \Delta\{X_{f(\alpha)}(g(\alpha))\colon \alpha < \varkappa\}$$

is stationary in $\varkappa$ whenever $f\colon \varkappa \to 2^\varkappa$ is one-one and $g\colon \varkappa \to \varkappa$.

Proof. (i) → (ii): Let $s_\alpha \subseteq \alpha$, $\alpha < \varkappa$, be a fixed $\diamondsuit_\varkappa(S)$-sequence, and select a bijection v from $\varkappa$ to $\varkappa\times\varkappa\times\varkappa\times\varkappa$. For each $\alpha \in \varkappa$, choose a function $j_\alpha\colon 2^\alpha \to \alpha$ with the following property: in case $v[s_\alpha]$ consists of all $(\beta, \gamma, \delta, \nu)$ in $\alpha\times\alpha\times\alpha\times\alpha$ such that $f_\beta(\gamma) = \delta$ and $g(\beta) = \nu$, where $g \in \alpha^\alpha$, $f_\beta \in 2^\alpha$ for every $\beta \in \alpha$, and $f_\beta \neq f_\gamma$ whenever $\beta \neq \gamma$, then for every $\beta \in \alpha$, $j_\alpha(f_\beta) = g(\beta)$. Now define $H\colon 2^\varkappa \times \varkappa \to \varkappa$ by letting $H(F, \alpha) = j_\alpha(F \restriction \alpha)$. Pick $G \in \varkappa^\varkappa$, and $F_\alpha \in 2^\varkappa$, $\alpha < \varkappa$, such that $F_\alpha \neq F_\beta$ for $\alpha \neq \beta$. Denote by K the collection of all $(\beta, \gamma, \delta, \nu)$ in $\varkappa\times\varkappa\times\varkappa\times\varkappa$ such that $F_\beta(\gamma) = \delta$ and $G(\beta) = \nu$. Let C denote the set of all $\alpha \in \varkappa$ such that $\mathrm{ran}(G \restriction \alpha) \subseteq \alpha$, $v[\alpha] = \alpha\times\alpha\times\alpha\times\alpha$, and $F_\beta \restriction \alpha \neq F_\gamma \restriction \alpha$ for all $\beta, \gamma \in \alpha$ with $\beta \neq \gamma$. Since C is closed and unbounded, the set T of all $\alpha \in C \cap S$ with $v^{-1}[K] \cap \alpha = s_\alpha$ is stationary in $\varkappa$. Clearly, $H(F_\beta, \alpha) = G(\beta)$ whenever $\alpha \in T$ and $\beta \in \alpha$. The desired partitions are then easily defined from H.

(ii) → (i): Choose Z_α, $\alpha < \varkappa$, such that each Z_α is a partition of $\varkappa$ into two pieces $Z_\alpha(0)$, $Z_\alpha(1)$, and that $\Delta\{Z_\alpha(g(\alpha))\colon \alpha < \varkappa\} \notin \mathrm{NS}_\varkappa|S$ for all $g \in 2^\varkappa$. We shall actually define $2^\varkappa$ many $\diamondsuit_\varkappa(S)$-sequences. Let $h \in 2^\varkappa$ be given, and set $s_\alpha^h = \{\beta < \alpha\colon \alpha \in Z_\beta(h(\beta))\}$ for all $\alpha < \varkappa$. Then fix $A \subseteq \varkappa$, and define $g \in 2^\varkappa$ by letting $g(\alpha) = h(\alpha)$ iff $\alpha \in A$. It is easily verified that for every $\gamma < \varkappa$, we have $s_\gamma^h = A \cap \gamma$ iff $\gamma \in \Delta\{Z_\alpha(g(\alpha))\colon \alpha < \varkappa\}$. Hence s_α^h, $\alpha < \varkappa$, is a $\diamondsuit_\varkappa(S)$-sequence. Now let $h, k \in 2^\varkappa$ be given with $h \neq k$. If $\beta < \varkappa$ is such that $h(\beta) \neq k(\beta)$, then clearly $s_\alpha^h \neq s_\alpha^k$ for all $\alpha > \beta$.

We shall prove below a more general two-cardinal version of that result (see Proposition 1.6). Before we do that, we shall present some consequences of Proposition 1.1.

We start by stating an easy corollary.

COROLLARY 1.2. *Given $S \subseteq \varkappa$, the following are equivalent*:

(i) $\diamondsuit_\varkappa(S)$.

(ii) *There are $f_\alpha\colon 2^\varkappa \to \varkappa$, $\alpha < \varkappa$, such that for every $h\colon 2^\varkappa \to \varkappa$ and every $g\colon \varkappa \to 2^\varkappa$, the set of all $\alpha \in S$ with $h \circ g \restriction \alpha = f_\alpha \circ g \restriction \alpha$ is stationary in $\varkappa$.*

(iii) *There is a function H from $2^\varkappa \times 2^\varkappa$ to the power set of $\varkappa$ such that $|H(\alpha, \beta) \cap H(\alpha, \gamma)| < \varkappa$ for all $\alpha, \beta, \gamma < 2^\varkappa$, and that $\Delta\{H(f(\alpha), g(\alpha))\colon \alpha < \varkappa\} \notin \mathrm{NS}_\varkappa|S$ whenever $f\colon \varkappa \to 2^\varkappa$ is one-one and $g\colon \varkappa \to 2^\varkappa$.*

We denote by $Y_\varkappa$ the collection of all those families $X_\alpha \in (\varkappa)^\varkappa$, $\alpha < 2^\varkappa$, such that $\Delta\{X_{f(\alpha)}(g(\alpha))\colon \alpha < \varkappa\} \notin \mathrm{NS}_\varkappa$ for all $g\colon \varkappa \to \varkappa$ and one-one $f\colon \varkappa \to 2^\varkappa$. Let X_α, $\alpha < 2^\varkappa$, be such a family, and let $h\colon 2^\varkappa \to \varkappa$. We define a collection F_h of subsets of $\varkappa$ by letting $A \in F_h$ iff there are $f\colon \varkappa \to 2^\varkappa$ and a closed unbounded subset C of $\varkappa$ such that

$$C \cap \Delta\{X_{f(\alpha)}(h(f(\alpha)))\colon \alpha < \varkappa\} \subseteq A.$$

It is not difficult to see that F_h is a $\varkappa$-complete normal filter over $\varkappa$. Note that $X_\alpha(h(\alpha)) \in F_h$ for all $\alpha < 2^\varkappa$.

We let $\mathrm{ND}_\varkappa$ denote the set of all those $S \subseteq \varkappa$ such that $\diamondsuit_\varkappa(S)$ does not hold. Devlin [D] observed that $\mathrm{ND}_\varkappa$ is a $\varkappa$-complete ideal.

PROPOSITION 1.3. *Let X_α, $\alpha < 2^\varkappa$, be a family in $Y_\varkappa$, and let $h\colon 2^\varkappa \to \varkappa$. Then $F_h \cap \mathrm{ND}_\varkappa = \emptyset$.*

Proof. Let $f\colon \varkappa \to 2^\varkappa$ be given, and let S denote the set $\Delta\{X_{f(\alpha)}(h(f(\alpha)))\colon \alpha < \varkappa\}$. Select one-one functions g, l, from $\varkappa$ to $2^\varkappa$, such that the range of l is the disjoint union of the ranges of f and g. Denote by C the collection of all $\alpha < \varkappa$ such that both $\mathrm{ran}(f \restriction \alpha)$ and $\mathrm{ran}(g \restriction \alpha)$ are included in $\mathrm{ran}(l \restriction \alpha)$. Note that C is closed and unbounded. For each $\alpha < \varkappa$, set $s_\alpha = \{\beta < \alpha\colon \alpha \in X_{g(\beta)}(h(g(\beta)))\}$. Given $A \subseteq \varkappa$, choose $m \in \varkappa^\varkappa$ so that $m(\alpha) \neq h(l(\alpha))$ iff $l(\alpha) \in g[\varkappa - A]$. It is easy to verify that $C \cap \Delta\{X_{l(\alpha)}(m(\alpha))\colon \alpha < \varkappa\}$ is included in the set $\{\alpha \in S\colon s_\alpha = A \cap \alpha\}$. Hence s_α, $\alpha < \varkappa$, is a $\diamondsuit_\varkappa(S)$-sequence.

Further properties of $\mathrm{ND}_\varkappa$ can be derived from Proposition 1.3. We first extend a result of Devlin [D].

COROLLARY 1.4. *Given subsets $S_\alpha \notin \mathrm{ND}_\varkappa$, $\alpha < \varkappa$, of $\varkappa$, there are pairwise disjoint sets $T_\alpha \subseteq S_\alpha$, $\alpha < \varkappa$, with $T_\alpha \notin \mathrm{ND}_\varkappa$.*

Proof. First make the following observation. Let $S \subseteq \varkappa$ with $S \notin \mathrm{ND}_\varkappa$. Then by Proposition 1.1 and Proposition 1.3, there are pairwise almost disjoint $E_\alpha \subseteq S$, $\alpha < 2^\varkappa$, with $E_\alpha \notin \mathrm{ND}_\varkappa$. The result now follows from Theorem 2.1 of [BHM].

Jensen [J] showed that $\mathrm{NS}_\varkappa = \mathrm{ND}_\varkappa$ holds in L whenever $\varkappa$ is regular. This property can be restated as follows.

COROLLARY 1.5. *$\mathrm{NS}_\varkappa = \mathrm{ND}_\varkappa$ iff every partition X of $\varkappa$ into $\varkappa$ many stationary sets is a member of some $K \in Y_\varkappa$.*

Proof. By Proposition 1.3 and the proof of Proposition 1.1.

We conclude this section by giving the proof of the two-cardinal version of Proposition 1.1.

PROPOSITION 1.6 (Jensen, Matet). *Assume $\varkappa$ is regular, let $\lambda \geqslant \varkappa$ be any cardinal, and let $S \subseteq [\lambda]^{<\varkappa}$. Then the following are equivalent*:

(i) $\diamondsuit_{\varkappa,\lambda}(S)$.

(ii) *There exist X_α, $\alpha < 2^\lambda$, such that each X_α is a partition of $[\lambda]^{<\varkappa}$ into $\lambda^{<\varkappa}$ many pieces $X_\alpha(\beta)$, $\beta < \lambda^{<\varkappa}$, and that the set $S \cap \Delta\{X_{k(\alpha)}(g(\alpha))\colon \alpha < \lambda\}$ is stationary in $[\lambda]^{<\varkappa}$ whenever $k\colon \lambda \to 2^\lambda$ is one-one and $g\colon \lambda \to \lambda^{<\varkappa}$.*

Proof. (i) → (ii): Let $s_a \subseteq a$, $a \in [\lambda]^{<\varkappa}$, be a fixed $\diamondsuit_{\varkappa,\lambda}(S)$-sequence, and select a bijection v from λ to $\lambda \times \lambda \times \lambda \times \lambda$. For each $a \in [\lambda]^{<\varkappa}$, choose a function j_a from 2^a to the power set of a with the following property: in case $v[s_a]$ consists of all $(\alpha, \beta, \gamma, \delta)$ in $a \times a \times a \times a$ such that $f_\alpha(\beta) = \gamma$ and $\delta \in g(\alpha)$, where g is a function from a to the power set of a, $f_\alpha \in 2^a$ for every $\alpha \in a$, and $f_\alpha \neq f_\beta$ whenever $\alpha \neq \beta$, then for every $\alpha \in a$, $j_a(f_\alpha) = g(\alpha)$. Now define a function H from $2^\lambda \times [\lambda]^{<\varkappa}$ to $[\lambda]^{<\varkappa}$ by letting $H(F, a) = j_a(F \restriction a)$. From here on, proceed as in the proof of Proposition 1.1.

(ii) → (i): Select Z_α, $\alpha < \lambda$, such that each Z_α is a partition of $[\lambda]^{<\varkappa}$ into two pieces $Z_\alpha(0)$, $Z_\alpha(1)$, and that the set $S \cap \Delta\{Z_\alpha(g(\alpha))\colon \alpha < \lambda\}$ is stationary for all $g \in 2^\lambda$. For each $h \in 2^\lambda$, w, define a $\diamondsuit_{\varkappa,\lambda}(S)$-sequence s_a^h, $a \in [\lambda]^{<\varkappa}$, by letting $s_a^h = \{\alpha \in a\colon a \in Z_\alpha(h(\alpha))\}$. Note that if $h, k \in 2^\lambda$ and $\alpha < \lambda$ are such that $h(\alpha) \neq k(\alpha)$, then $s_a^h \neq s_a^k$ whenever $\alpha \in a$.

As regards Proposition 1.6 (ii), there may be room for improvement, since in case $2^{\lambda^{<\varkappa}} > 2^\lambda$, the length of the sequence whose existence is asserted is not maximal.

2. We now concern ourselves with some combinatorial principles that strengthen $\diamondsuit_\varkappa$.

First, it is natural to ask whether other ideals can be substituted for $NS_\varkappa$ in the statement of Proposition 1.1. So let I be an ideal over $\varkappa$ with $\varkappa \notin I$. Putting I in the place of $NS_\varkappa$ in the formulation of two versions of diamond, we obtain the following principles.

$\diamondsuit_\varkappa[I]$ asserts the existence of a sequence $s_\alpha \subseteq \alpha$, $\alpha < \varkappa$, such that $\{\alpha\colon s_\alpha = A \cap \alpha\} \notin I$ for every $A \subseteq \varkappa$.

$\diamondsuit_\varkappa^*[I]$ is said to hold if there are P_α, $\alpha < \varkappa$, such that each P_α is a collection of subsets of α with $|P_\alpha| \leqslant \alpha$, and that $\{\alpha < \varkappa\colon A \cap \alpha \notin P_\alpha\} \in I$ for all $A \subseteq \varkappa$.

Thus if $S \subseteq \varkappa$ is stationary, then $\diamondsuit_\varkappa^*[NS_\varkappa|S]$ is the usual $\diamondsuit_\varkappa^*(S)$.

Some brief remarks are in order.

We observe that by Proposition 1.3, $\diamondsuit_\varkappa$ implies $\diamondsuit_\varkappa[ND_\varkappa]$. It is shown in [M2] that if $\varkappa \leqslant 2^\lambda$ for some cardinal $\lambda < \varkappa$, then $\diamondsuit_\varkappa$ is equivalent to $\diamondsuit_\varkappa[\{A \subseteq \varkappa\colon A \subseteq \lambda\}]$. In that case one has an attractive reformulation of diamond in terms of partitions, i.e. $\diamondsuit_\varkappa$ holds iff there are $X_\alpha \in (\varkappa)^\varkappa$, $\alpha < \varkappa$, such that $\Delta\{X_\alpha(h(\alpha))\colon \alpha < \varkappa\} \neq \emptyset$ for all $h \in \varkappa^\varkappa$. We do not know whether that holds in general. We remark that $\diamondsuit_\varkappa[[\varkappa]^{<\varkappa}]$

holds whenever $\varkappa$ is regular and the set of all uncountable cardinals $\mu < \varkappa$ such that $\diamondsuit_\mu[[\mu]^{<\mu}]$ holds is unbounded in $\varkappa$.

A collection K of subsets of $\varkappa$ is said to be greedy if whenever $s_\alpha \subseteq \alpha$, $\alpha < \varkappa$, there exists $A \subseteq \varkappa$ such that $\{\alpha\colon s_\alpha = A \cap \alpha\} \in K$. Clearly, $\diamondsuit_\varkappa[I]$ holds iff I is not greedy. Also, note that $\diamondsuit_\varkappa^*[I]$ fails whenever I^+ is greedy.

Assume $V = L$, and suppose I extends $[\varkappa]^{<\varkappa}$ and is normal. Then by a still unpublished result of H. D. Donder and J. P. Levinski, $\diamondsuit_\varkappa[I]$ holds, and $\diamondsuit_\varkappa^*[I]$ fails iff I^+ is greedy.

Recall that an I partition of $\varkappa$ is a maximal family H of members of I^+ such that $A \cap B \in I$ for any distinct $A, B \in H$. Now assume $\diamondsuit_\varkappa[I]$ holds, and choose $s_\alpha \subseteq \alpha$, $\alpha < \varkappa$, such that $\{\alpha\colon s_\alpha = A \cap \alpha\} \in I^+$ for all $A \subseteq \varkappa$. Let G consist of all $B \subseteq \varkappa$ such that $B = \{\alpha < \varkappa\colon s_\alpha = A \cap \alpha\}$ for some $A \subseteq \varkappa$. Clearly, G is a family of pairwise almost disjoint members of I^+. We note that G is a maximal such partition in case I extends $[\varkappa]^{<\varkappa}$ and $([\varkappa]^{<\varkappa}|A)^+$ is greedy for all $A \in I^+$. Also, G is an I partition if $\varkappa \cap I^+ = \emptyset$ and $(I|A)^+$ is greedy for all $A \in I^+$.

This extends Theorem 4.5 of [Ty].

PROPOSITION 2.1. *Assume $\varkappa$ is regular, let $S \subseteq \varkappa$ be stationary, and let I be an ideal over $\varkappa$ such that $C \cap S \in I^+$ whenever C is a closed unbounded subset of $\varkappa$. Further assume that $\diamondsuit_\varkappa^*(S)$ holds, and that either $\varkappa$ is successor and I is $\varkappa$-complete, or else $\varkappa$ is limit and I is normal. Then there are $X_\alpha \in (\varkappa)^\varkappa$, $\alpha < 2^\varkappa$, such that $\Delta\{X_{f(\alpha)}(g(\alpha))\colon \alpha < \varkappa\} \notin I|S$ whenever $f\colon \varkappa \to 2^\varkappa$ is one-one and $g\colon \varkappa \to \varkappa$.*

Proof. Let J denote the collection of all those $D \subseteq \varkappa$ such that $D = A \cup B$ for some $A \in NS_\varkappa$ and $B \in I$. Then J is an ideal over $\varkappa$ with $I \cup NS_\varkappa \subseteq J$. Note that if I is $\varkappa$-complete (respectively normal), then J is also $\varkappa$-complete (resp. normal). Moreover, $C \cap S \in J^+$ for every closed unbounded $C \subseteq \varkappa$.

We first show that $\diamondsuit_\varkappa[J|S]$ holds. Pick functions H_α from $|\alpha|$ to the power set of $\alpha \times \alpha$, $\alpha < \varkappa$, such that for every $B \subseteq \varkappa \times \varkappa$, we have

$$\{\alpha < \varkappa\colon B \cap (\alpha \times \alpha) \notin \operatorname{ran}(H_\alpha)\} \in NS_\varkappa|S.$$

We claim that there is a $\beta < \varkappa$ with the following property: for every $A \subseteq \varkappa$, there is a $B \subseteq \varkappa \times \varkappa$ such that $A = \{\gamma < \varkappa\colon (\beta, \gamma) \in B\}$ and that the set of all $\alpha \in S$ with $|\alpha| > \beta$ and $B \cap (\alpha \times \alpha) = H_\alpha(\beta)$ belongs to J^+. Suppose otherwise. Select a counterexample A_β for each $\beta < \varkappa$, and set $B = \bigcup_{\beta < \varkappa} \{\beta\} \times A_\beta$. Pick a closed unbounded set $C \subseteq \varkappa - \{0\}$ such that $B \cap (\alpha \times \alpha) \in \operatorname{ran}(H_\alpha)$ for all $\alpha \in C \cap S$. Define $g\colon C \cap S \to \varkappa$ by letting $g(\alpha) = \beta$ iff $B \cap (\alpha \times \alpha) = H_\alpha(\beta)$. Then g is constant on some $D \in J^+$, a contradiction. So let β be as in the claim, and choose $s_\alpha \subseteq \alpha$, $\alpha < \varkappa$, so that $s_\alpha = \{\gamma\colon (\beta, \gamma) \in H_\alpha(\beta)\}$ whenever $|\alpha| > \beta$. It is easily verified that $\{\alpha \in S\colon s_\alpha = A \cap \alpha\} \in J^+$ for all $A \subseteq \varkappa$. The proof of Proposition 1.1 now goes through if we substitute J for $NS_\varkappa$. Hence there are $X_\alpha \in (\varkappa)^\varkappa$, $\alpha < 2^\varkappa$, such that $\Delta\{X_{f(\alpha)}(g(\alpha))\colon \alpha < \varkappa\} \notin J|S$ whenever $f\colon \varkappa \to 2^\varkappa$ is one-one and $g\colon \varkappa \to \varkappa$. It remains to observe that $I|S \subseteq J|S$.

Comparing Proposition 1 of [M1] with Corollary 1.2 (iii), we are led to the formulation of the following problem. Assume that $\varkappa^{<\varkappa} = \varkappa$, and let I be an ideal over $\varkappa$ which is not $2^\varkappa$-saturated. Does there then exist a function H from $2^\varkappa \times 2^\varkappa$ to the power set of $\varkappa$ such that $|H(\alpha,\beta) \cap H(\alpha,\gamma)| < \varkappa$ for all $\alpha, \beta, \gamma < 2^\varkappa$, and that $\bigcap_{\alpha \in E} H(\alpha, g(\alpha)) \in I^+$ whenever $E \in [2^\varkappa]^{<\varkappa}$ and $g: E \to 2^\varkappa$? It is not difficult to see that the answer is yes in case $I = [\varkappa]^{<\varkappa}$ and $\varkappa$ is strongly inaccessible.

Given a cardinal $\lambda \geqslant \varkappa$, $\diamondsuit_\varkappa(\lambda)$ asserts the existence of a family $X_\alpha \in (\varkappa)^\varkappa$, $\alpha < \lambda$, with the property that for every $h \in \varkappa^\lambda$, there is a stationary subset S of $\varkappa$ such that

$$\Big|S - \bigcap_{\alpha \in a} X_\alpha(h(\alpha))\Big| < \varkappa \quad \text{for all } a \in [\lambda]^{<\varkappa}.$$

This principle was introduced by Tall in [Ta] (we actually reformulated Tall's statement in terms of partitions). Note that there is no conflict of notation, as $\diamondsuit_\varkappa(\varkappa)$ is equivalent to $\diamondsuit_\varkappa$. Also $2^\lambda = 2^\varkappa$ follows from $\diamondsuit_\varkappa(\lambda)$.

Tall also considers the following assertion.

$P_S(\varkappa)$ is said to hold if whenever T_α, $\alpha < \lambda < 2^\varkappa$, are subsets of $\varkappa$ with each diagonal intersection stationary, there is a stationary $T \subseteq \varkappa$ such that $|T - T_\alpha| < \varkappa$ for every $\alpha < \lambda$.

This is immediate from Proposition 1.1.

Proposition 2.2. *Assume that $\diamondsuit_\varkappa$ and $P_S(\varkappa)$ both hold. Then $\diamondsuit_\varkappa(\lambda)$ holds for every cardinal $\lambda \geqslant \varkappa$ with $\lambda^{<\varkappa} < 2^\varkappa$.*

3. We now concern ourselves with situations when diamond sequences can be defined from others.

Let us first deal with initial segments.

Assuming $V = L$, Jensen [J] shows the existence of a universal diamond sequence. That is, there are $s_\alpha \subseteq \alpha$, α any ordinal, such that for every regular uncountable cardinal μ, s_α, $\alpha < \mu$, is a $\diamondsuit_\mu$-sequence.

We mention the following question. Given a transitive model M of set theory, can one find a generic extension $M[G]$ such that large cardinal properties are preserved, and that $M[G]$ contains a universal diamond sequence?

$A \subseteq \varkappa$ is weakly compact if for every F: $[A]^2 \to 2$, there is a $B \subseteq A$ such that either $|B| = \varkappa$ and F is constantly 0 on $[B]^2$, or else B is stationary in $\varkappa$ and F is constantly 1 on $[B]^2$. The weakly compact filter over $\varkappa$ consists of all those $A \subseteq \varkappa$ such that $\varkappa - A$ is not weakly compact.

Proposition 3.1. (i) *Let $\varkappa$ be weakly compact, let $S \subseteq \varkappa$ be stationary, and let s_α, $\alpha < \varkappa$, be a $\diamondsuit_\varkappa(S)$-sequence. Then the set of all strongly inaccessible $\mu < \varkappa$ such that s_α, $\alpha < \mu$, is a $\diamondsuit_\mu(S \cap \mu)$-sequence, lies in the weakly compact filter over $\varkappa$.*

(ii) *Assume $V = L$, and let $\varkappa$ be regular but not weakly compact. Then there is a $\diamondsuit_\varkappa$-sequence s_α, $\alpha < \varkappa$, such that for every regular cardinal $\mu < \varkappa$, s_α, $\alpha < \mu$, is not a $\diamondsuit_\mu$-sequence.*

Proof. (i) follows from the characterization of the weakly compact filter in terms of indescribability (see [B2]).

To show (ii) we use results of [J]. Pick a stationary $S \subseteq \varkappa$ such that for every regular uncountable cardinal $\mu < \varkappa$, $S \cap \mu$ is not stationary in μ. Then let s_α, $\alpha < \varkappa$, be any $\diamondsuit_\varkappa(S)$-sequence such that $s_\alpha = \emptyset$ whenever $\alpha \notin S$.

Say that $S \subseteq \varkappa$ is subtle if for every sequence $s_\alpha \subseteq \alpha$, $\alpha < \varkappa$, and every closed unbounded set $C \subseteq \varkappa$, there are α, β in $S \cap C$ such that $\beta < \alpha$ and $s_\alpha \cap \beta = s_\beta$. The collection of all those $S \subseteq \varkappa$ such that $\varkappa - S$ is not subtle is a $\varkappa$-complete filter [B1] known as the subtle filter over $\varkappa$.

Kunen showed that $\diamondsuit_\varkappa$ holds whenever $\varkappa$ is subtle. His proof can be modified so as to yield the following.

Proposition 3.2. *Let S_δ, $\delta < \varkappa$, be subtle subsets of $\varkappa$. Then there are $s_\alpha \subseteq \alpha$, $\alpha < \varkappa$, such that for every $\delta < \varkappa$, s_α, $\alpha < \varkappa$, is a $\diamondsuit_\varkappa(S_\delta)$-sequence and the set of all $\gamma \in S_\delta$ for which s_α, $\alpha < \gamma$, is not a $\diamondsuit_\gamma(S_\delta \cap \gamma)$-sequence is not subtle.*

Proof. By induction on $\delta < \varkappa$, we define subsets N_δ, M_δ and P_δ, of $\varkappa$, and for every $\alpha \in N_\delta$, subsets s_α^δ and C_α^δ of α so that each P_δ lies in the subtle filter over $\varkappa$, N_0 consists of all infinite limit ordinals below $\varkappa$, and $N_\delta = \bigcap_{\mu < \delta} P_\mu$ for all $\delta > 0$. We define s_α^δ, C_α^δ by induction on $\alpha \in N_\delta$, as follows. If possible, pick s_α^δ, C_α^δ such that C_α^δ is a closed unbounded subset of α, and that for every $\beta \in S_\delta \cap N_\delta \cap C_\alpha^\delta$, $s_\alpha^\delta \cap \beta \neq s_\beta^\delta$. Otherwise put $s_\alpha^\delta = C_\alpha^\delta = \emptyset$. Set $M_\delta = \{\alpha \in N_\delta\colon C_\alpha^\delta = \emptyset\}$, and $P_\delta = (N_\delta - S_\delta) \cup (M_\delta \cap S_\delta)$. Note that the set $(N_\delta \cap S_\delta) - M_\delta$ is not subtle, since otherwise one could find α, β in $(N_\delta \cap S_\delta) - M_\delta$ such that $\beta < \alpha$, $s_\alpha^\delta \cap \beta = s_\beta^\delta$, and $C_\alpha^\delta \cap \beta = C_\beta^\delta$, a contradiction. Hence P_δ belongs to the subtle filter over $\varkappa$. Finally choose $s_\alpha \subseteq \alpha$, $\alpha < \varkappa$, so that $s_\alpha = s_\alpha^\delta$ whenever $\alpha \in (N_\delta \cap S_\delta) - M_\delta$. It is easily verified that s_α, $\alpha < \mu$, is a $\diamondsuit_\mu(S_\delta \cap \mu)$-sequence for all $\mu \in M_\delta$. Now fix $\delta \in \varkappa$, and let $A \subseteq \varkappa$ and a closed unbounded subset C of $\varkappa$ be given. Since M_δ is stationary, one can select a $\mu \in M_\delta$ such that $C \cap \mu$ is unbounded in μ. Then there is an $\alpha \in S_\delta \cap C \cap \mu$ with $s_\alpha = A \cap \alpha$. Thus s_α, $\alpha < \varkappa$, is a $\diamondsuit_\varkappa(S_\delta)$-sequence.

We next concern ourselves with the derivation of two-cardinal diamond sequences from a given diamond sequence.

We first introduce some notation. Let $\mu \leqslant \lambda$ be uncountable cardinals. Given a function f: $[\lambda]^{<\omega} \to \lambda$, we denote by C_f^μ the set of all nonempty $a \in [\lambda]^{<\mu}$ such that $a \cap \mu \in \mu$, and $f(d) \subseteq a$ for every nonempty finite subset d of a.

The following well-known result is essentially due to Kueker [Kue].

Proposition 3.3. *Assume $\varkappa$ is regular, and et $\lambda \geqslant \varkappa$ be any cardinal. Then the following hold:*

(i) *$C_f^\varkappa$ is a closed unbounded set for all f: $[\lambda]^{<\omega} \to \lambda$.*

(ii) *Let D be a closed unbounded subset of $[\lambda]^{<\varkappa}$. Then $C_f^\varkappa \subseteq D$ for some f: $[\lambda]^{<\omega} \to \lambda$.*

Proof. The easy proof of (i) is left to the reader.

Let us prove (ii). By induction on the size of d, define $A_d \in D$, $d \in [\lambda]^{<\omega}$, such that $d \subset A_d$, and $A_c \subseteq A_d$ whenever $c \subseteq d$. Now let f: $[\lambda]^{<\omega} \to \lambda$ satisfy the following conditions. Given $\alpha<\lambda$, $f(\{\alpha\})=\alpha+1$. Let $d\neq\emptyset$ be a fixed finite subset of λ, and let d_n, $n\leqslant p$, be the increasing enumeration of d. Then $f(d\cup\{d_p+1\})$ equals the order type of A_d; $f(d\cup\{d_p+2n+2\})=\alpha$ whenever $n\leqslant p$, $\alpha\in A_d$, and $\alpha\cap A_d$ has order type d_n; and $f(d\cup\{d_p+2n+3\})=\beta$ whenever $n\leqslant p$, $\beta\in A_{d-\{d_n\}}$, and $\beta\cap A_{d-\{d_n\}}$ has order type d_n. It is not difficult to see that $C_f^{\varkappa}\subseteq D$.

The following corollary is well known too (see, e.g., Lemma 8.2 in [B3]).

Corollary 3.4. *Assume $\varkappa$ is regular, and let $\lambda\geqslant\varkappa$ be any regular cardinal. Let $S\subseteq\lambda$ consist of limit ordinals of cofinality $<\varkappa$. Then the following are equivalent:*

(i) *S is stationary in λ.*

(ii) *The set $\{a\in[\lambda]^{<\varkappa}\colon \bigcup a\in S\}$ is stationary in $[\lambda]^{<\varkappa}$.*

Proof. (i) → (ii): Suppose S is stationary. Given $f\colon[\lambda]^{<\omega}\to\lambda$, choose $\alpha\in S\cap C_f^{\lambda}$, and select $b\in[\alpha]^{<\varkappa}$ with $\bigcup b=\alpha$. Now define $a\in C_f^{\varkappa}$ such that $b\subseteq a\subseteq\alpha$. Then, clearly, $\cap a=\alpha$. Thus (ii) holds.

The easy proof of the converse is left to the reader.

This is immediate from Corollary 3.4.

Proposition. 3.5. *Assume $\varkappa$ is regular. Let $\lambda\geqslant\varkappa$ be a cardinal, and let s_α, $\alpha<\lambda$, be a $\diamondsuit_\lambda(S)$-sequence, where S consists of limit ordinals of cofinality $<\varkappa$. Set $t_a=a\cap s_{\cup a}$ for all $a\in[\lambda]^{<\varkappa}$, and put $T=\{a\in[\lambda]^{<\varkappa}\colon \bigcup a\in S\}$. Then t_a, $a\in[\lambda]^{<\varkappa}$, is a $\diamondsuit_{\varkappa,\lambda}(T)$-sequence.*

Let M be a transitive model of ZFC, and let λ be a regular uncountable cardinal in M. We consider the situation when a single Cohen-generic subset of λ is added to M. So let P consist of all those functions p such that $\operatorname{dom}(p)\subseteq[\lambda]^2$, $|\operatorname{dom}(p)|<\lambda$, and $\operatorname{ran}(p)\subseteq 2$. For $p,q\in P$, set $p\leqslant q$ iff $q\subseteq p$. Define $u\colon P\to\lambda$ by letting $u(p)$ be the least $\alpha<\lambda$ with $\operatorname{dom}(p)\subseteq[\alpha]^2$.

We first make the following observation, which seems to be new.

Proposition 3.6. *Let G be P-generic over M, and set*

$$s_\alpha=\{\beta<\alpha\colon (\cup G)(\beta,\alpha)=1\}$$

for every $\alpha<\lambda$. Let S be, in M, a stationary subset of λ. Then s_α, $\alpha<\lambda$, is a $\diamondsuit_\lambda(S)$-sequence.

Proof. The proof is only outlined, as we closely follow the proof of Theorem 8.3 in Chapter VII of [Kun]. Let $p\in G$ and E, C in $M[G]$ be such that p forces that $E\in 2^\lambda$ and that C is a closed unbounded subset of λ. By induction on $\alpha<\lambda$, define, in M, $p_\alpha\in P$, γ_α, x_α so that

(1) $p_0=p$, and $p_\alpha\leqslant p_\beta$ for $\beta<\alpha$;

(2) $u(p_\alpha)<\gamma_{\alpha+1}<u(p_{\alpha+1})$;

(3) $x_\alpha\colon u(p_\alpha)\to 2$;

(4) $p_{\alpha+1}$ forces that $\gamma_{\alpha+1}\in C$ and that $E\restriction u(p_\alpha)=x_\alpha$;

(5) if $\alpha=\bigcup\alpha$, then $\gamma_\alpha=\bigcup_{\beta<\alpha}\gamma_\beta$, and p_α forces that

$$(\bigcup G)(\delta,\gamma_\alpha)=E(\delta)\quad\text{for all }\delta<\gamma_\alpha.$$

Finally observe that the set $D=\{\gamma_\alpha\colon\alpha<\lambda\}$ is closed unbounded. Hence $\gamma_\alpha\in S$ for some infinite limit ordinal α.

Proposition 3.6 can be generalized as follows.

Proposition 3.7. *Let G and s_α, $\alpha<\lambda$, be as in the statement of Proposition 3.6. Let $\mu\leqslant\lambda$ be a regular uncountable cardinal, and set $t_a=a\cap s_{\cup a}$ for all $a\in[\lambda]^{<\mu}$. Let S be, in M, a stationary subset of $[\lambda]^{<\mu}$. Then t_a, $a\in[\lambda]^{<\mu}$, is a $\diamondsuit_{\mu,\lambda}(S)$-sequence.*

Proof. The proof is similar to that of Proposition 3.6. Let $p\in G$ and E,f in $M[G]$ be such that p forces that $E\in 2^\lambda$ and that $f\colon[\lambda]^{<\omega}\to\lambda$. By induction on $\alpha<\lambda$, define, in M, $p_\alpha\in P$, γ_α, x_α, y_α so that:

(1) $p_0=p$, and $p_\alpha\leqslant p_\beta$ for $\beta<\alpha$;

(2) $u(p_\alpha)<\gamma_{\alpha+1}<u(p_{\alpha+1})$;

(3) $x_\alpha\colon u(p_\alpha)\to 2$, and $y_\alpha\colon[u(p_\alpha)]^{<\omega}\to\lambda$;

(4) $p_{\alpha+1}$ forces that $E\restriction u(p_\alpha)=x_\alpha$ and that

$$f\restriction[u(p_\alpha)]^{<\omega}=y_\alpha;$$

(5) if $\alpha=\bigcup\alpha$, then $\gamma_\alpha=\bigcup_{\beta<\alpha}\gamma_\beta$, and p_α forces that

$$(\bigcup G)(\delta,\gamma_\alpha)=E(\delta)\quad\text{for all }\delta<\gamma_\alpha.$$

Finally set $D=\{\gamma_\alpha\colon\alpha<\lambda\}$ and $h=\bigcup_{\alpha<\lambda}y_\alpha$. Note that D is a closed unbounded set. Hence one can find $a\in S\cap C_h^{\mu}$ such that $\bigcup a=\gamma_\alpha$, where α is some limit ordinal.

We conclude with the following observation of H. D. Donder, which is included here with his kind permission.

Proposition 3.8 (Donder). *Let Q consist, in M, of all functions q such that $\operatorname{dom}(q)\subseteq\delta\times\lambda$, where $\delta=(\lambda^{<\lambda})^+$, $|\operatorname{dom}(q)|<\lambda$, and $\operatorname{ran}(q)\subseteq 2$, ordered by reverse inclusion. Let G be Q-generic over M, and let $\mu\leqslant\lambda$ be a regular uncountable cardinal. Then, in $M[G]$, $\diamondsuit_{\mu,\lambda}(S)$ holds for all stationary $S\subseteq[\lambda]^{<\mu}$.*

We shall need the following lemma, which is easily obtained by modifying the proof of Proposition 3.7.

Lemma 3.9. *Let N be a transitive model of ZFC. In N, let $\mu\leqslant\lambda$ be regular uncountable cardinals, and let t_a, $a\in[\lambda]^{<\mu}$, be a $\diamondsuit_{\mu,\lambda}(S)$-sequence, where $S\subseteq[\lambda]^{<\mu}$. Suppose $(R,<)$ is, in N, a λ-closed notion of forcing, and let H be R-generic over N. Then t_a, $a\in[\lambda]^{<\mu}$, remains a $\diamondsuit_{\mu,\lambda}(S)$-sequence in $N[H]$.*

Proof of Proposition 3.8. For each $A\subseteq\delta$, let G_A consist of all $q\in G$ such that $\operatorname{dom}(q)\subseteq A\times\lambda$. Let S be, in $M[G]$, a stationary subset of $[\lambda]^{<\mu}$. Then there exists $\alpha<\delta$ with $S\in M[G_\alpha]$. By Proposition 3.7, $\diamondsuit_{\mu,\lambda}(S)$ holds in $M[G_\alpha][G_{\{\alpha\}}]$. It now follows from Lemma 3.9 that $\diamondsuit_{\mu,\lambda}(S)$ holds in $M[G]$.

References

[B1] J. E. Baumgartner, *Ineffability properties of cardinals*, I, *Infinite and Finite Sets*, Coll. Math. Soc. János Bolyai 10, Keszthely (Hungary), 1973, 109–130.

[B2] — *Ineffability properties of cardinals*, II, *Logic, Foundations of Mathematics and Computability Theory*, Butts and Hintikka (eds), Reidel, Dordrecht (Holland), 1977, 87–106.

[B3] — *Applications of the proper forcing axiom*, *Handbook of Set-Theoretic Topology*, Kunen and Vaughan (eds), North-Holland, Amsterdam 1984, 913–959.

[BHM] J. E. Baumgartner, A. Hajnal and A. Máté, *Weak saturation properties of ideals*, *Infinite and Finite Sets*, Coll. Math. Soc. János Bolyai 10, Keszthely (Hungary), 1973, 137–158.

[D] K. J. Devlin, *A note on the combinatorial principles* $\diamond(E)$, Proc. Amer. Math. Soc. 72 (1978), 163–165.

[J] R. B. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic 4 (1972), 229–308.

[Kue] D. Kueker, *Countable approximations and Löwenheim–Skolem theorems*, Ann. Math. Logic 11 (1977), 57–103.

[Kun] K. Kunen, *Set Theory, An Introduction to Independence Proofs*, North-Holland, Amsterdam 1980.

[M1] P. Matet, *Partitions and diamond*, Proc. Amer. Math. Soc. 97 (1986), 133–135.

[M2] — *Corrigendum to "Partitions and diamond"*, Proc. Amer. Math. Soc. 101 (1987), 519–520.

[Ta] F. D. Tall, *Some applications of a generalized Martin's axiom*, Trans. Amer. Math. Soc., to appear.

[Ty] A. D. Taylor, *Diamond principles, ideals and the normal Moore space problem*, Canad. J. Math. 33 (1981), 282–296.

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Diamond and λ-systems

by

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Abstract. The notion of a λ-system generalizes that of a stationary set. In this paper variants of diamond for λ-systems are considered. In particular, a form of the weak diamond principle is defined and shown to be consistent with the strong negation of the continuum hypothesis. An application of these principles is given to the Whitehead problem in abelian group theory.

§ 0. Introduction. In [S2] Shelah introduced the notion of a λ-system in order to analyze exactly how a non-free abelian group (or other structure) fails to be free. A λ-system is a generalization of a stationary set. In this paper we will consider variants of $\diamond$ for λ-systems. In Section 1 we will define what is meant by a λ-system and remark that $\diamond$ for all λ-systems is equivalent to $\diamond$ for all stationary sets. We will then introduce a new variant of $\diamond$, a definable version of the weak diamond principle. (The weak diamond principle was introduced in [DS].) We will show this principle for $\lambda < 2^{\aleph_0}$ is consistent with $2^{\aleph_0} = 2^{\aleph_1}$. In fact it is true whenever we add $2^{\aleph_0}$ Cohen reals to the ground model.

In section 2 we will give an application of the definable weak diamond principle to the Whitehead problem for abelian groups. We will show that it is consistent with $2^{\aleph_0} = 2^{\aleph_1}$, that every Whitehead group is free. That this result is largely of technical interest seems in part a reflection on the psychology of mathematics. Although we are interested in knowing when statements are independent of CH, there is little interest in knowing when things are independent of $\neg$CH. Of course there is reason behind this view since CH has strong consequences while experience has shown that $\neg$CH has few consequences. However some mathematicians, including Woodin [W], have studied the independence of statements from $\neg$CH.

§ 1. λ-systems. After reading the definition of a λ-system, the reader may find it helpful to turn to Section 2 and see how λ-systems naturally arise.

Definition. Assume λ is a regular uncountable cardinal. A *λ-system* is a labeled subtree $\langle S, \langle B_\eta, \lambda_\eta\colon \eta \in S\rangle\rangle$ of ${}^{<\omega}\lambda$ satisfying:

(0) $B_{\langle\,\rangle} = 0$;

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