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Solution to a compactification problem of Sklyarenko

by

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Dedicated to Professor Yukihiro Kodama
On his 60th birthday

Abstract. Concerning a function Skl originally introduced by Sklyarenko to study compactness deficiency def, we establish a theorem that $\operatorname{Skl} X = \operatorname{def} X$ for every separable metrizable space X. This answers a problem of Sklyarenko affirmatively.

1. Introduction. All spaces considered in this paper are assumed to be separable and metrizable. By a compactification of a space X, we mean a compact metrizable space containing X as a dense subspace. For undefined notion see [3] and [5].

The *compactness deficiency* $\operatorname{def} X$ of a space X is the least integer n for which X has a compactification αX with $\dim(\alpha X - X) = n$.

J. de Groot [4] proved that a space X has a compactification αX with $\dim(\alpha X - X) \leqslant 0$ if and only if X is rim-compact. Motivated by this result, to study further def he introduced the *small* (resp. *large*) *inductive compactness degree* $\operatorname{cmp} X$ (resp. $\operatorname{Cmp} X$) of a space X. In general, the inequality $\operatorname{cmp} X \leqslant \operatorname{Cmp} X \leqslant \operatorname{def} X$ holds [5]. The well-known conjecture of de Groot that $\operatorname{cmp} X = \operatorname{def} X$ has been negatively solved by R. Pol [9]; the space X of Pol's example has $\operatorname{cmp} X = 1$ and $\operatorname{Cmp} X = \operatorname{def} X = 2$. It is unknown whether there is a space X with $\operatorname{Cmp} X < \operatorname{def} X$. [1]

Another condition to study def is due to E. Sklyarenko [10], [11], which is denoted by $\operatorname{Skl} X \leqslant n$ as in Isbell's book [6]; a space X has $\operatorname{Skl} X \leqslant n$ if X has a base $\mathscr{B}$ such that $\operatorname{Bd} B_0 \cap \operatorname{Bd} B_1 \cap \ldots \cap \operatorname{Bd} B_n$ is compact for any $n+1$ distinct members of $\mathscr{B}$. Sklyarenko proved that $\operatorname{Skl} X \leqslant \operatorname{def} X$ [10] and asked whether $\operatorname{Skl} X = \operatorname{def} X$ for every space X [11]. Recently, J. M. Aarts, J. Bruijning and J. van Mill [2] proved that $\operatorname{Cmp} X \leqslant \operatorname{Skl} X$. In this paper we give an affirmative answer to Sklyarenko's problem above. Namely, we shall establish a theorem that $\operatorname{Skl} X = \operatorname{def} X$ for every space X. As an application it will be shown that a non-compact space X has a compactification αX with $\dim(\alpha X - X) = n$ if and only if $\operatorname{Skl} X \leqslant n$.

2. Preliminaries and lemmas. Let $\mathscr{S}$ be a collection of subsets of a space X. We shall write $[\mathscr{S}]^n$ for $\{\mathscr{T}\colon \mathscr{T}$ is a subcollection of $\mathscr{S}$ with $|\mathscr{T}| = n\}$, $\bigcap \mathscr{S}$

[1] Added in proof. Recently the author has constructed such a space.

for $\bigcap\{S\colon S\in\mathscr{S}\}$, $\bigcup\mathscr{S}$ for $\bigcup\{S\colon S\in\mathscr{S}\}$, $\operatorname{Bd}\mathscr{S}$ for $\{\operatorname{Bd}S\colon S\in\mathscr{S}\}$ and $\operatorname{Cl}\mathscr{S}$ for $\{\operatorname{Cl}S\colon S\in\mathscr{S}\}$.

2.1. Definition. Let $\mathscr{A}$ be a collection of closed subsets of a space X. The *k-order* of $\mathscr{A}$ is defined as follows: $k\text{-ord}\,\mathscr{A}\leqslant n$ if $\bigcap\mathscr{A}'$ is compact for every $\mathscr{A}'\in[\mathscr{A}]^{n+1}$.

With this terminology, $\operatorname{Skl}X\leqslant n$ if and only if X has a base $\mathscr{B}$ such that $k\text{-ord}\operatorname{Bd}\mathscr{B}\leqslant n$. Skl is a general notion of the covering dimension dim, indeed $\operatorname{Skl}X\leqslant\dim X$ for every space X, since $\dim X\leqslant n$ if and only if X has a base $\mathscr{B}$ such that $\operatorname{ord}\operatorname{Bd}\mathscr{B}\leqslant n$ [8, Theorem 12.13] and in general $k\text{-ord}\operatorname{Bd}\mathscr{B}\leqslant\operatorname{ord}\operatorname{Bd}\mathscr{B}$.

2.2 Lemma. *Let $\mathscr{A}$ be a locally finite collection of open subsets of a space X, $\mathscr{B}$ a base for X with $k\text{-ord}\operatorname{Bd}(\mathscr{A}\cup\mathscr{B})\leqslant n$, F a closed subset of X and U an open subset of X containing F. Then there are a locally finite collection $\mathscr{U}$ of open subsets of X and a base $\mathscr{B}'$ for X such that $F\subset\bigcup\mathscr{U}\subset\operatorname{Cl}(\bigcup\mathscr{U})\subset U$ and $k\text{-ord}\operatorname{Bd}(\mathscr{A}\cup\mathscr{B}'\cup\mathscr{U})\leqslant n$.*

Proof. We may assume that $\mathscr{B}$ is countable. Enumerate $\mathscr{B}$ as $\mathscr{B}=\{B_i\colon i\in N\}$. Consider the collection $\{(C_i,D_i)\colon i\in N\}$ of all pairs of elements of $\mathscr{B}$ such that $\operatorname{Cl}C_i\subset D_i$ and either $\operatorname{Cl}D_i\subset U$ or $D_i\cap F=\emptyset$. Let us set

$$V_i=D_i-\bigcup\{\operatorname{Cl}C_j\colon j<i\}\quad\text{for every } i\in N,$$
$$M=\{i\in N\colon \operatorname{Cl}V_i\subset U\},\qquad \mathscr{V}=\{V_i\colon i\in M\}.$$

Then $\mathscr{V}$ is a locally finite collection of open subsets of X and we have $F\subset\bigcup\mathscr{V}\subset\operatorname{Cl}(\bigcup\mathscr{V})\subset U$. Let $\mathscr{F}=\{F_i\colon i\in M\}$ be a closed cover of F such that $F_i\subset V_i$ for every $i\in M$. Since $\{C_1,\ldots,C_{i-1},D_i\}\subset\mathscr{B}$, we can express

$$\{C_1,\ldots,C_{i-1},D_i\}=\{B_{i(1)},\ldots,B_{i(i-1)},B_{i(i)}\}$$

by using the original indexing of $\mathscr{B}$. From $\operatorname{Bd}V_i\subset\bigcup\{\operatorname{Bd}B_{i(j)}\colon 1\leqslant j\leqslant i\}$, we can take a closed subset E^i_j of $\operatorname{Bd}B_{i(j)}$ for each j, $1\leqslant j\leqslant i$ such that $\operatorname{Bd}V_i=\bigcup\{E^i_j\colon 1\leqslant j\leqslant i\}$. Let us set

$$\mathscr{E}_i=\{E^i_j\colon 1\leqslant j\leqslant i\},\qquad M_i=\{i(j)\colon 1\leqslant j\leqslant i\},$$
$$E_k=\bigcup\{E^i_j\colon i(j)=k,\ 1\leqslant j\leqslant i \text{ and } i\in M\}\quad\text{for every } k\in N.$$

Then we have

(a) E_k is closed in $\operatorname{Bd}B_k$ for every $k\in N$ and $V_i\cap\operatorname{Bd}B_k=\emptyset$ for every $k\in M_i$.

Since $\mathscr{V}$ is locally finite, so is $\{E_k\colon k\in N\}$. If $E_k\neq\emptyset$, then pick up a point $x_k\in E_k$ and set $J_0=\{x_k\colon k\in N \text{ with } E_k\neq\emptyset\}$. Then J_0 is discrete and closed in X, because $\{E_k\colon k\in N\}$ is locally finite. Thus $\mathscr{B}_0=\{B\in\mathscr{B}\colon \operatorname{Bd}B\cap J_0=\emptyset\}$ is a base for X. For every $i\in M$, inductively, we shall construct $\mathscr{B}_i$, U_i, $\mathscr{K}_i$ and J_i satisfying the following conditions (1) to (5) below;

(1) U_i is open in X such that $F_i\subset U_i\subset\operatorname{Cl}U_i\subset V_i$,

(2) $\mathscr{K}_i$ is a locally finite collection of closed subsets of X such that $\operatorname{Bd}U_i=\bigcup\mathscr{K}_i$,

(3) J_i is discrete and closed in X such that $J_i\subset V_i$,

(4) $\mathscr{B}_i$ is a base for X such that $\mathscr{B}_i=\{B\in\mathscr{B}_p\colon \operatorname{Bd}B\cap J_i=\emptyset$ and $\{K\in\mathscr{K}_i\colon \operatorname{Cl}B\cap K\neq\emptyset\}$ is finite$\}$, where $p=\max\{j\in M\colon j<i\}$,

(5) $k\text{-ord}(\operatorname{Bd}(\mathscr{A}\cup\mathscr{B}_i)\cup\{\operatorname{Bd}U_j\colon j\in M \text{ with } j\leqslant i\}\cup\{E_k\colon k\in N\})\leqslant n$.

Suppose that $m\in M$. Assume that we can construct $\mathscr{B}_i$, U_i, $\mathscr{K}_i$ and J_i satisfying conditions (1) to (5) for each $i\in M$ with $i<m$. Let $p=\max\{i\in M\colon i<m\}$. By the induction hypothesis, in particular, the following property (b) is satisfied.

(b) $k\text{-ord}(\operatorname{Bd}(\mathscr{A}\cup\mathscr{B}_p)\cup\{\operatorname{Bd}U_j\colon j\in M \text{ with } j\leqslant p\}\cup\{E_k\colon k\in N\})\leqslant n$.

To construct $\mathscr{B}_m$, U_m, $\mathscr{K}_m$ and J_m we only need the property (b) and that $\mathscr{B}_p$ is a base for X. Thus in case $m=\min M$, let $p=0$. Since $\mathscr{B}_0\cap\{B_k\colon k\in N \text{ with } E_k\neq\emptyset\}=\emptyset$, by (a), the property (b) is satisfied. Let us set

$$\mathscr{H}=\{\textstyle\bigcap\mathscr{H}'\colon \mathscr{H}'\in[\operatorname{Bd}\mathscr{A}\cup\{\operatorname{Bd}U_j\colon j\in M \text{ with } j\leqslant p\}\cup\cup\{E_k\colon k\in N-M_m\}]^n\}.$$

Since $\mathscr{A}$ is locally finite, $\mathscr{A}$ is countable. Thus $\mathscr{H}$ is also countable. Enumerate $\mathscr{H}$ as $\mathscr{H}=\{H_i\colon i\in N\}$. Since $\mathscr{A}$ and $\{E_k\colon k\in N\}$ are locally finite, so is $\mathscr{H}$. Since $\operatorname{Bd}V_m=\bigcup\mathscr{E}_m$, by (b), $H\cap\operatorname{Bd}V_m$ is compact for every $H\in\mathscr{H}$. Thus there is a finite subcollection $\mathscr{B}^i$ of $\mathscr{B}_p$ for every $i\in N$ such that

(c) $H_i\cap\operatorname{Bd}V_m\subset\bigcup\mathscr{B}^i$,

(d) $\{\bigcup\mathscr{B}^i\colon i\in N\}$ is locally finite,

(e) $\operatorname{Cl}(\bigcup\mathscr{B}^i)\cap F_m=\emptyset$, and

(f) $\operatorname{Cl}(\bigcup\mathscr{B}^i)\cap(H_j-\bigcup\mathscr{B}^j)=\emptyset$ for every $j<i$.

Let $F'_m=F_m\cup\bigcup\{(H_i\cap\operatorname{Cl}V_m)-\bigcup\mathscr{B}^i\colon i\in N\}$. Then, by (c), we have $F_m\subset F'_m\subset V_m$. Since $\mathscr{H}$ is locally finite, F'_m is closed in X. Consider the collection $\{(C'_i,D'_i)\colon i\in N\}$ of all pairs of elements of $\mathscr{B}_p$ such that $\operatorname{Cl}C'_i\subset D'_i$ and either $\operatorname{Cl}D'_i\subset V_m$ or $\operatorname{Cl}D'_i\cap F'_m=\emptyset$. Let us set

$$O_i=D'_i-\bigcup\{\operatorname{Cl}C'_j\colon j<i\}\quad\text{for every } i\in N,$$
$$L=\{i\in N\colon \operatorname{Cl}O_i\subset V_m\},\quad\text{and}\quad \mathscr{O}=\{O_i\colon i\in L\}.$$

Then $\mathscr{O}$ is locally finite and $F'_m\subset O\subset\operatorname{Cl}O\subset V_m$, where $O=\bigcup\mathscr{O}$. Let us set

$$U_m=O-\bigcup\{\operatorname{Cl}(\textstyle\bigcup\mathscr{B}^i)\colon i\in N\}.$$

By (d) and (e), U_m is open in X and $F_m\subset U_m\subset\operatorname{Cl}U_m\subset V_m$. Thus condition (1) is satisfied. By the construction of O_i, we take a finite subcollection $\mathscr{C}_i$ of $\mathscr{B}_p$ such that $\operatorname{Bd}O_i\subset\bigcup(\operatorname{Bd}\mathscr{C}_i)$. Let us set

$$K_i(B)=\begin{cases}\operatorname{Bd}O_i\cap\operatorname{Bd}B & \text{if } B\in\mathscr{C}_i\\ \emptyset & \text{otherwise,}\end{cases}\quad\text{and}$$
$$K'(B)=\bigcup\{K_i(B)\colon i\in L\}\quad\text{for every } B\in\mathscr{B}_p.$$

Since $\mathcal{O}$ is locally finite, so is $\{K'(B)\colon B\in\mathcal{B}_p\}$. By (d), $\{\operatorname{Bd}B\colon B\in\mathcal{B}''\}$ is locally finite, where $\mathcal{B}''=\bigcup\{\mathcal{B}^i\colon i\in N\}$. Let us set

$$K(B)=\begin{cases}\operatorname{Bd}U_m\cap\operatorname{Bd}B & \text{if } B\in\mathcal{B}''\\ \operatorname{Bd}U_m\cap K'(B) & \text{otherwise,}\end{cases}\quad\text{and}$$
$$\mathcal{K}_m=\{K(B)\colon B\in\mathcal{B}_p\}.$$

Then we have $K(B)\subset\operatorname{Bd}B$ and $\mathcal{K}_m$ is a locally finite collection of closed subsets of X. Obviously, we have $\bigcup\mathcal{K}_m\subset\operatorname{Bd}U_m$. Since

$$\begin{aligned}\operatorname{Bd}U_m&\subset\operatorname{Bd}O\cup\operatorname{Bd}(\bigcup\{\operatorname{Cl}(\bigcup\mathcal{B}^i)\colon i\in N\})\\ &=\bigcup\{\operatorname{Bd}O_i\colon i\in L\}\cup\bigcup\{\operatorname{Bd}B\colon B\in\mathcal{B}''\}\\ &=\bigcup\{\bigcup\{K_i(B)\colon B\in\mathcal{B}_p\}\colon i\in L\}\cup\bigcup\{\operatorname{Bd}B\colon B\in\mathcal{B}''\}\\ &=\bigcup\{K'(B)\colon B\in\mathcal{B}_p\}\cup\bigcup\{\operatorname{Bd}B\colon B\in\mathcal{B}''\},\end{aligned}$$

we have $\operatorname{Bd}U_m\subset\bigcup\mathcal{K}_m$. Thus condition (2) is satisfied. For each $K\in\mathcal{K}_m$ with $K\neq\emptyset$ select a point $x_K\in K$. Then the set

$$J_m=\{x_K\colon K\in\mathcal{K}_m \text{ with } K\neq\emptyset\}$$

is discrete and closed in X, because $\mathcal{K}_m$ is locally finite. Since $x_K\in K\subset\operatorname{Bd}U_m\subset V_m$, we have $J_m\subset V_m$. Thus condition (3) is satisfied. Let us set

$$\mathcal{B}_m=\{B\in\mathcal{B}_p\colon \operatorname{Bd}B\cap J_m=\emptyset \text{ and } \{K\in\mathcal{K}_m\colon \operatorname{Cl}B\cap K\neq\emptyset\} \text{ is finite}\}.$$

Then, by (2) and (3), $\mathcal{B}_m$ is a base for X. Thus condition (4) is satisfied.

Next, we shall prove that condition (5) is satisfied. To this end, let $\mathcal{C}\in[\operatorname{Bd}(\mathcal{A}\cup\mathcal{B}_m)\cup\{\operatorname{Bd}U_j\colon j\in M \text{ with } j\leqslant m\}\cup\{E_k\colon k\in N\}]^{n+1}$. We shall show that $\bigcap\mathcal{C}$ is compact. We distinguish four cases.

Case 1. $\operatorname{Bd}U_m\notin\mathcal{C}$.

In this case we have $\mathcal{C}\in[\operatorname{Bd}(\mathcal{A}\cup\mathcal{B}_p)\cup\{\operatorname{Bd}U_j\colon j\in M \text{ with } j\leqslant p\}\cup\{E_k\colon k\in N\}]^{n+1}$. Thus, by (b), $\bigcap\mathcal{C}$ is compact.

Case 2. $\operatorname{Bd}U_m\in\mathcal{C}$ and $E_k\in\mathcal{C}$ for some $k\in M_m$.

By (a), we have

$$\bigcap\mathcal{C}\subset\operatorname{Bd}U_m\cap E_k\subset V_m\cap\operatorname{Bd}B_k=\emptyset.$$

Thus $\bigcap\mathcal{C}$ is compact.

Case 3. $\operatorname{Bd}U_m\in\mathcal{C}$, $E_k\notin\mathcal{C}$ for any $k\in M_m$ and $\mathcal{C}\cap\operatorname{Bd}\mathcal{B}_m=\emptyset$.

In this case we have

$$\mathcal{C}-\{\operatorname{Bd}U_m\}\in[\operatorname{Bd}\mathcal{A}\cup\{\operatorname{Bd}U_j\colon j\in M \text{ with } j\leqslant p\}\cup\{E_k\colon k\in N-M_m\}]^n,$$

Thus we have $\bigcap(\mathcal{C}-\{\operatorname{Bd}U_m\})=H_i\in\mathcal{H}$ for some $i\in N$. Since $(H_i\cap\operatorname{Cl}V_m)-\bigcup\mathcal{B}^i\subset F'_m\subset O\subset\operatorname{Cl}O\subset V_m$, we have

$$\begin{aligned}H_i\cap\operatorname{Bd}O&=H_i\cap(\operatorname{Cl}O-O)\\ &\subset H_i\cap(V_m-((H_i\cap\operatorname{Cl}V_m)-\bigcup\mathcal{B}^i))\\ &=(H_i\cap V_m\cap\bigcup\mathcal{B}^i)\cup((H_i\cap V_m)-(H_i\cap\operatorname{Cl}V_m))\\ &\subset\bigcup\mathcal{B}^i.\end{aligned}$$

On the other hand, by (f), we have

$$H_i\cap\operatorname{Bd}(\bigcup\mathcal{B}^j)\subset\operatorname{Cl}(\bigcup\mathcal{B}^i)\quad\text{for each } j\geqslant i.$$

Hence we have

$$\begin{aligned}\bigcap\mathcal{C}&=H_i\cap\operatorname{Bd}U_m\\ &\subset(H_i\cap(\operatorname{Bd}O\cup\bigcup\{\operatorname{Bd}(\bigcup\mathcal{B}^j)\colon j\in N\}))-\bigcup\mathcal{B}^i\\ &=H_i\cap\bigcup\{\operatorname{Bd}(\bigcup\mathcal{B}^j)\colon j\leqslant i\}\\ &\subset H_i\cap\bigcup\{\operatorname{Bd}B\colon B\in\bigcup\{\mathcal{B}^j\colon j\leqslant i\}\}.\end{aligned}$$

Since $\mathcal{C}\cap\operatorname{Bd}\mathcal{B}_m=\emptyset$, by (b), $H_i\cap\operatorname{Bd}B$ is compact for every $B\in\mathcal{B}_p$. Hence $\bigcap\mathcal{C}$ is compact.

Case 4. $\operatorname{Bd}U_m\in\mathcal{C}$, $E_k\notin\mathcal{C}$ for any $k\in M_m$ and $\mathcal{C}\cap\operatorname{Bd}\mathcal{B}_m\neq\emptyset$.

Let $\operatorname{Bd}B\in\mathcal{C}\cap\operatorname{Bd}\mathcal{B}_m$. By the construction of $\mathcal{B}_m$,

$$\mathcal{K}'_m=\{K\in\mathcal{K}_m\colon \operatorname{Cl}B\cap K\neq\emptyset\}$$

is finite. Let $\mathcal{K}'_m=\{K_i\colon i\leqslant q\}$, where $K_i=K(B_{n_i})$ for some $B_{n_i}\in\mathcal{B}_p$. Then we have

$$\begin{aligned}\operatorname{Bd}U_m\cap\operatorname{Bd}B&=\bigcup\mathcal{K}_m\cap\operatorname{Bd}B\\ &=\bigcup\{K_i\colon i\leqslant q\}\cap\operatorname{Bd}B\\ &\subset\bigcup\{\operatorname{Bd}B_{n_i}\colon i\leqslant q\}\cap\operatorname{Bd}B.\end{aligned}$$

Since $x_{K_i}\in K_i\subset\operatorname{Bd}B_{n_i}$, we have $J_m\cap\operatorname{Bd}B_{n_i}\neq\emptyset$. Thus $B_{n_i}\notin\mathcal{B}_m$ for each $i\leqslant q$, therefore

$$(\mathcal{C}-\{\operatorname{Bd}U_m\})\cup\{\operatorname{Bd}B_{n_i}\}\in[\operatorname{Bd}(\mathcal{A}\cup\mathcal{B}_p)\cup\{\operatorname{Bd}U_j\colon j\in M \text{ with } j\leqslant p\}\cup\cup\{E_k\colon k\in N\}]^{n+1}.$$

By (b), $\bigcap(\mathcal{C}-\{\operatorname{Bd}U_m\})\cap\operatorname{Bd}B_{n_i}$ is compact for each $i\leqslant q$. Since

$$\begin{aligned}\bigcap\mathcal{C}&=\bigcap(\mathcal{C}-\{\operatorname{Bd}U_m,\ \operatorname{Bd}B\})\cap\operatorname{Bd}U_m\cap\operatorname{Bd}B\\ &\subset\bigcap(\mathcal{C}-\{\operatorname{Bd}U_m\})\cap\bigcup\{\operatorname{Bd}B_{n_i}\colon i\leqslant q\},\end{aligned}$$

$\bigcap\mathcal{C}$ is compact.

Hence in any case condition (5) is satisfied.

Let us set

$$\mathscr{U} = \{U_i\colon\ i \in M\}.$$

Then we have $F \subset \bigcup \mathscr{U} \subset \mathrm{Cl}(\bigcup \mathscr{U}) \subset U$, because $F_i \subset U_i \subset \mathrm{Cl}\, U_i \subset V_i$ for every $i \in M$. Since $\mathscr{V}$ is locally finite, so is $\mathscr{U}$. Let $J = \bigcup \{J_i\colon\ i \in M\}$. Then J is discrete and closed in X, because each J_i is closed and discrete in X, $J_i \subset V_i$ and $\mathscr{V}$ is locally finite. Let $\mathscr{K} = \bigcup \{\mathscr{K}_i\colon\ i \in M\}$. Then $\mathscr{K}$ is locally finite, because each $\mathscr{K}_i$ is locally finite, $\bigcup \mathscr{K}_i = \mathrm{Bd}\, U_i \subset V_i$ and $\mathscr{V}$ is locally finite. Thus the collection

$$\mathscr{B}' = \{B \in \mathscr{B}_0\colon\ \mathrm{Bd}\, B \cap J = \emptyset \text{ and } \{K \in \mathscr{K}\colon\ \mathrm{Cl}\, B \cap K \neq \emptyset\} \text{ is finite}\}$$

is a base for X. By the construction of $\mathscr{B}_i$, we have $\mathscr{B}' \subset \bigcap \{\mathscr{B}_i\colon\ i \in M\}$. Hence, by (5), $k\text{-ord}\,\mathrm{Bd}(\mathscr{A} \cup \mathscr{B}' \cup \mathscr{U}) \leqslant n$.

Lemma 2.2 is proved.

The following lemma is needed to prove Lemmas 2.4 and 2.5 below; the proof is similar to that of Lemma 2.2.

2.3. Lemma. *Let $\mathscr{A}$ be a locally finite collection of open subsets of a space X, $\mathscr{B}$ a base for X with $k\text{-ord}\,\mathrm{Bd}(\mathscr{A} \cup \mathscr{B}) \leqslant n$, F a closed subset of X and V an intersection of finite members of $\mathscr{A} \cup \{X - \mathrm{Cl}\, A\colon\ A \in \mathscr{A}\}$ containing F. Then there are an open subset U of X, a closed discrete subset J of X and a locally finite collection $\mathscr{K}$ of closed subsets of X such that*

(1) $F \subset U \subset \mathrm{Cl}\, U \subset V$,

(2) $\mathrm{Bd}\, U = \bigcup \mathscr{K}$

(3) $J \subset V$, *and*

(4) $k\text{-ord}\,\mathrm{Bd}(\mathscr{A} \cup \mathscr{B}' \cup \{U\}) \leqslant n$, *where* $\mathscr{B}' = \{B \in \mathscr{B}\colon\ \mathrm{Bd}\, B \cap J = \emptyset$ *and* $\{K \in \mathscr{K}\colon\ \mathrm{Cl}\, B \cap K \neq \emptyset\}$ *is finite*$\}$.

Note that the above $\mathscr{B}'$ is a base for X.

2.4. Lemma. *Let $\mathscr{A}$ be a locally finite collection of open subsets of a space X, $\mathscr{B}$ a base for X with $k\text{-ord}\,\mathrm{Bd}(\mathscr{A} \cup \mathscr{B}) \leqslant n$, $\mathscr{F} = \{F_i\colon\ i \in N\}$ a collection of closed subsets of X and $\mathscr{V} = \{V_i\colon\ i \in N\}$ a subcollection of $\mathscr{A}$ with each V_i containing F_i. Then there are a collection $\mathscr{U} = \{U_i\colon\ i \in N\}$ of open subsets of X and a base $\mathscr{B}'$ for X such that $F_i \subset U_i \subset \mathrm{Cl}\, U_i \subset V_i$ for every $i \in N$ and $k\text{-ord}\,\mathrm{Bd}(\mathscr{A} \cup \mathscr{B}' \cup \mathscr{U}) \leqslant n$.*

Proof. By Lemma 2.3, inductively, we can construct $\mathscr{B}_i$, U_i, $\mathscr{K}_i$ and J_i satisfying conditions (1) to (5) in Lemma 2.2 as regarding $M = N$. Let $\mathscr{U} = \{U_i\colon\ i \in N\}$ and $\mathscr{B}' = \{B \in \mathscr{B}\colon\ \mathrm{Bd}\, B \cap J = \emptyset$ and $\{K \in \mathscr{K}\colon\ \mathrm{Cl}\, B \cap K \neq \emptyset\}$ is finite$\}$, where $J = \bigcup \{J_i\colon\ i \in N\}$ and $\mathscr{K} = \bigcup \{\mathscr{K}_i\colon\ i \in N\}$. Then, similarly to the proof of Lemma 2.2, $\mathscr{B}'$ is a base for X and $k\text{-ord}\,\mathrm{Bd}(\mathscr{A} \cup \mathscr{B}' \cup \mathscr{U}) \leqslant n$. This completes the proof of Lemma 2.4.

2.5. Lemma. *Let $\mathscr{A}$ be a locally finite collection of open subsets of a space X, $\mathscr{B}$ a base for X with $k\text{-ord}\,\mathrm{Bd}(\mathscr{A} \cup \mathscr{B}) \leqslant n$ and $\mathscr{G} = \{G_i\colon\ i \leqslant s\}$ a finite open cover of X such that each G_i is an intersection of finite members of $\mathscr{A} \cup \{X - \mathrm{Cl}\, A\colon\ A \in \mathscr{A}\}$.*

Then there are a finite open star-refinement $\mathscr{U}$ of $\mathscr{G}$ and a base $\mathscr{B}'$ for X such that $k\text{-ord}\,\mathrm{Bd}(\mathscr{A} \cup \mathscr{B}' \cup \mathscr{U}) \leqslant n$.

Proof. Let $\mathscr{F} = \{F_i\colon\ i \leqslant s\}$ be a closed shrinking of $\mathscr{G}$. By Lemma 2.3, there are a collection $\mathscr{H} = \{H_i\colon\ i \leqslant s\}$ of open subsets of X and a base $\mathscr{B}''$ for X such that $F_i \subset H_i \subset \mathrm{Cl}\, H_i \subset G_i$ for every $i \leqslant s$ and $k\text{-ord}\,\mathrm{Bd}(\mathscr{A} \cup \mathscr{B}'' \cup \mathscr{H}) \leqslant n$. Let $S = \{i\colon\ i \leqslant s\}$. Let us set

$$\begin{aligned} W_T &= \bigcap \{G_i\colon\ i \in T\} - \bigcup \{\mathrm{Cl}\, H_i\colon\ i \in S - T\} \quad \text{for each } T \subset S,\\ \mathscr{W} &= \{W_T\colon\ T \subset S\}, \quad \text{and}\\ \mathscr{O} &= \{W \cap H\colon\ W \in \mathscr{W} \text{ and } H \in \mathscr{H}\}\\ &= \{O_{Ti}\colon\ T \subset S \text{ and } i \leqslant s\}, \quad \text{where } O_{Ti} = W_T \cap H_i \end{aligned}$$

Then $\mathscr{O}$ covers X. Let $\mathscr{E} = \{E_{Ti}\colon\ T \subset S$ and $i \leqslant s\}$ be a closed shrinking of $\mathscr{O}$. By Lemma 2.3, we take an open cover $\mathscr{U} = \{U_{Ti}\colon\ T \subset S$ and $i \leqslant s\}$ of X and a base $\mathscr{B}'$ for X such that $E_{Ti} \subset U_{Ti} \subset \mathrm{Cl}\, U_{Ti} \subset O_{Ti}$ for every $T \subset S$ and $i \leqslant s$ and $k\text{-ord}\,\mathrm{Bd}(\mathscr{A} \cup \mathscr{B}' \cup \mathscr{U}) \leqslant n$. It suffices to prove that $\mathscr{U}$ is a star-refinement of $\mathscr{G}$. For every $U_{Ti} \in \mathscr{U}$ we have

$$\mathrm{St}(U_{Ti}, \mathscr{U}) \subset \mathrm{St}(O_{Ti}, \mathscr{O}) = \mathrm{St}(W_T \cap H_i, \mathscr{O}) \subset \mathrm{St}(\mathrm{Cl}\, H_i, \mathscr{W}).$$

Then $\mathrm{Cl}\, H_i \cap W_T \neq \emptyset$ implies that $i \in T$, therefore $W_T \subset G_i$. Hence $\mathscr{U}$ is a star-refinement of $\mathscr{G}$. This completes the proof of Lemma 2.5.

3. The equality $\mathrm{Skl}\, X = \mathrm{def}\, X$. For every space X, by a *uniformity of X agreeing with the topology*, we mean a countable collection $\Phi = \{\mathscr{G}_i\colon\ i \in N\}$ of countable open covers of X satisfying the following conditions:

(a) $\mathscr{G}_{i+1}$ is a star-refinement of $\mathscr{G}_i$ for every $i \in N$,

(b) $\{\mathrm{St}(x, \mathscr{G}_i)\colon\ i \in N\}$ is a neighborhood base at each point $x \in X$.

Let $\hat{X}$ be the completion of X with respect to Φ. Then we set

$$V^* = \hat{X} - \mathrm{Cl}_{\hat{X}}(X - V)$$

for every open subset V of X.

We need the following lemma, which was proved by K. Morita [7, Theorem 2].

3.1. Lemma. *Let X be a space with a uniformity $\Phi = \{\mathscr{G}_i\colon\ i \in N\}$ agreeing with the topology, and $\mathscr{V}$ a collection of open subsets of X. Suppose that for any $i \in N$ there is some $j \in N$ such that $\mathscr{G}_j$ is a refinement of the cover $\mathscr{W}_i$, where*

$$\mathscr{W}_i = \mathscr{V} \cup \{G \in \mathscr{G}_i\colon\ G \cap (X - \bigcup \mathscr{V}) \neq \emptyset\}.$$

Then the equality $(\bigcup \mathscr{V})^* = \bigcup \{V^*\colon\ V \in \mathscr{V}\}$ holds.

We are now in a position to prove our main theorem.

3.2. Theorem. *The equality $\mathrm{Skl}\, X = \mathrm{def}\, X$ holds for every space X.*

Proof. It suffices to prove that $\mathrm{Skl}\,X \geqslant \mathrm{def}\,X$. Suppose that $\mathrm{Skl}\,X = n$. Let $\mathscr{B}_0$ be a base for X with k-$\mathrm{ord\,Bd}\,\mathscr{B}_0 \leqslant n$, and $\{(C_i, D_i)\colon i \in N\}$ a collection of pairs of open subsets of X satisfying the following conditions:

(a) $\mathrm{Cl}\,C_i \subset D_i$ for every $i \in N$,

(b) for every $x \in X$ and for any neighborhood U of x there is $i \in N$ such that $x \in C_i$ and $D_i \subset U$.

By Lemmas 2.2, 2.4 and 2.5, inductively, for every $i \in N$ we can construct a base $\mathscr{B}_i$ for X, a finite open cover $\mathscr{G}_i$ of X, two locally finite collections $\mathscr{U}_i = \{U_{ij}\colon i \leqslant j\}$ and $\mathscr{V}_i = \{V_{ij}\colon i \leqslant j\}$ of open subsets of X, and a finite collection

$$\Gamma_i = \{\mathscr{W}_{ij}\colon i \leqslant j \leqslant m_i\}$$

of finite open covers of X satisfying the following conditions (1) to (6) below:

(1) $\mathrm{Cl}\,V_{ij} \subset U_{ij}$ for $i, j \in N$ with $i \leqslant j$,

(2) $\mathrm{Cl}\,C_i \subset \bigcup \mathscr{V}_i \subset \bigcup \mathscr{U}_i \subset D_i$,

(3) $\mathscr{G}_i$ is a star-refinement of $\mathscr{G}_{i-1}$,

(4) k-$\mathrm{ord\,Bd}(\mathscr{B}_i \cup \bigcup \{\mathscr{U}_j\colon j \leqslant i\} \cup \bigcup \{\mathscr{V}_j\colon j \leqslant i\} \cup \bigcup \{\mathscr{G}_j\colon j \leqslant i\}) \leqslant n$,

(5) Γ_i is the collection of all finite open covers of X of the form

$$\mathscr{W} = \{G_0, G_1, \ldots, G_n, X - \mathrm{Cl}\,G_0, X - \mathrm{Cl}\,G_1, \ldots, X - \mathrm{Cl}\,G_n, H_0, H_1, \ldots, H_m\},$$

where

$$\{G_0, G_1, \ldots, G_n\} \in [\bigcup \{\mathscr{G}_j\colon j < i\}]^{n+1}, \; H_0, H_1, \ldots, H_m \in \bigcup \{\mathscr{G}_j\colon j < i\}$$

with $\bigcap \{\mathrm{Bd}\,G_j\colon 0 \leqslant j \leqslant n\} \subset \bigcup \{H_j\colon 0 \leqslant j \leqslant m\}$ and $\mathscr{W} \notin \bigcup \{\Gamma_j\colon j < i\}$,

(6) for each $j \leqslant i$ $\mathscr{G}_i$ is a refinement of $\mathscr{W}_{ji}$ and of $\{U_{ji}, X - \mathrm{Cl}\,V_{ji}\}$.

Note that, by (b) and (2), $\bigcup \{\mathscr{U}_i\colon i \in N\}$ is a base for X. Then $\Phi = \{\mathscr{G}_i\colon i \in N\}$ is a uniformity of X agreeing with the toplogy and each $\mathscr{G}_i$ is finite. Thus the completion of X with respect to Φ is a compactification of X and it is denoted by αX. Then $\{G^*\colon G \in \mathscr{G}\}$ is a base for αX, where $\mathscr{G} = \bigcup \{\mathscr{G}_i\colon i \in N\}$. We show that $\dim(\alpha X - X) \leqslant n$. By [8, Theorem 12.13], it suffices to prove that

$$\mathrm{ord}\{\mathrm{Bd}_{\alpha X}\,G^* \cap (\alpha X - X)\colon G \in \mathscr{G}\} \leqslant n.$$

To this end, let $\{G_0, G_1 \ldots, G_n\} \in [\mathscr{G}]^{n+1}$. By (4), we have k-$\mathrm{ord\;Bd}\,\mathscr{G} \leqslant n$, therefore $\bigcap \{\mathrm{Bd}_X G_i\colon 0 \leqslant i \leqslant n\}$ is compact. Thus for every $x \in \alpha X - X$ there is a finite subcollection $\{H_0, H_1, \ldots, H_m\}$ of $\mathscr{G}$ such that $\bigcap \{\mathrm{Bd}_X G_i\colon 0 \leqslant i \leqslant n\} \subset \bigcup \{H_i^*\colon 0 \leqslant i \leqslant m\}$ and $x \notin H_i^*$ for any i, $0 \leqslant i \leqslant m$. Thus, by (5),

$$\mathscr{W} = \{G_0, G_1, \ldots, G_n, X - \mathrm{Cl}\,G_0, X - \mathrm{Cl}\,G_1, \ldots, X - \mathrm{Cl}\,G_{n0}, H_0, H_1, \ldots, H_m\} \in \Gamma_i$$

for some $i \in N$. So $\mathscr{W} = \mathscr{W}_{ij}$ for some $i \leqslant j \leqslant m_i$. By (6), $\mathscr{G}_j$ refines $\mathscr{W}$. Thus, by Lemma 3.1, we have $\bigcup \{W^*\colon W \in \mathscr{W}\} = (\bigcup \mathscr{W})^* = X^* = \alpha X$. Since $x \notin H_i^*$ for any $0 \leqslant i \leqslant m$, we have $x \in G_i^*$ or $x \in (X - \mathrm{Cl}\,G_i)^* = \alpha X - \mathrm{Cl}_{\alpha X} G_i^*$ for some $0 \leqslant i \leqslant n$. This implies that $x \notin \bigcap \{\mathrm{Bd}_{\alpha X} G_i^*\colon 0 \leqslant i \leqslant n\}$, therefore

$$\bigcap \{\mathrm{Bd}_{\alpha X} G_i^*\colon 0 \leqslant i \leqslant n\} \subset X.$$

Thus $\mathrm{ord}\{\mathrm{Bd}_{\alpha X} G^* \cap (\alpha X - X)\colon G \in \mathscr{G}\} \leqslant n$. Hence $\dim(\alpha X - X) \leqslant n$, and our Theorem 3.2 is now completely proved.

3.3. THEOREM. *A non-compact space X has a compactification αX with $\dim(\alpha X - X) = n$ if and only if* $\mathrm{Skl}\,X \leqslant n$.

Proof. It suffices to prove the "if" part. Let δX be a compactification of X with $\dim(\delta X - X) = \mathrm{def}\,X = \mathrm{Skl}\,X$. Take a point $y \in \delta X - X$ and set $Y = \delta X - \{y\}$. Then Y is a locally compact, non-compact space. By [1], Y has a compactification Z such that $Z - Y$ is homeomorphic to I^n, where $I = [0, 1]$. Let $\alpha X = Z$. Then αX is a compactification of X. Since I^n is embedded in $\alpha X - X$, we have $\dim(\alpha X - X) \geqslant n$. On the other hand, for every closed subset F of $\alpha X - X$ with $F \cap I^n = \varnothing$ we have $\dim F \leqslant n$, because $F \subset \delta X - X$. Hence, by [8, Theorem 9.11], $\dim(\alpha X - X) = n$. Theorem 3.3 is proved.

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