Corollary 1. *There is no fixed ordinal γ such that $\aleph_\alpha^{|\alpha|} = \aleph_{\alpha+\gamma}$ for all sufficiently large α.*

Proof. By taking $\beta = 1$ in the previous theorem, we have $\gamma = 0$. If $\gamma = 0$, then for sufficiently large limit ordinal α, $\aleph_\alpha = \aleph_{\alpha+\gamma} = \aleph_\alpha^{|\alpha|} = \prod_{\xi<\alpha} \aleph_\xi > \sum_{\xi<\alpha} \aleph_\xi = \aleph_\alpha$ which is a contradiction.

Corollary 2. *If there is a fixed ordinal γ such that $\aleph_\alpha^{|\alpha|} = \aleph_{\alpha+\gamma}$ for all sufficiently large limit ordinals α, then $\gamma < \omega$.*

Proof. By taking $\beta = \omega$ in the previous theorem, we have $\gamma < \omega$.

Remark. Patai's theorem [2, Theorem XIV] states that if there is a fixed ordinal γ such that $2^{\aleph_\alpha} = \aleph_{\alpha+\gamma}$ for every α, then $\gamma < \omega$.

By the remark preceeding Theorem 1, the hypothesis of Patai's theorem implies the hypothesis of the Corollary 2 in this papaer.

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Clifford theory for p-sections of finite groups *

by

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Abstract. Let K denote an arbitrary field of prime characteristic $p \neq 0$. Let N denote a normal subgroup of the finite group G such that G/N is a p-group. Here, in this situation, we demonstrate some basic results of Clifford theory for irreducible modules and blocks of $K[G]$. These results extend and generalize work of several authors.

1. Introduction and statements. Our notation and terminology are standard and tend to follow the conventions of [5]. In particular, all vector spaces encountered in this article have finite dimension, all modules over an algebra are right and unital, if n is a positive integer and V is a module, then nV denotes the module direct sum of n copies of V and if A is a ring then $U(A)$ denotes the multiplicative group of units of A.

Throughout this paper, G denotes a finite group. N is a normal subgroup of G, K is a field with $\mathrm{char}(K) = p > 0$ but is otherwise arbitrary and $K[G]$ and $K[N]$ are the associated group algebras. Also W denotes an irreducible $K[N]$-module, $I_G(W) = \{g \in G \mid W \otimes g \cong W\}$ denotes the stabilizer of W in G, $P_N(W)$ denotes a projective cover of W and $\mathrm{Irr}(G|W)$ denotes the class of irreducible $K[G]$-modules V such that W is a composition factor of (and hence a direct summand of) V_N. Clearly $I_G(W)$ is a subgroup of G containing N and $\mathrm{Irr}(G|W)$ is non-empty. Also b denotes a block of $K[N]$, $\mathrm{Irr}(b)$ denotes the class of irreducible $K[N]$-modules in b, $I_G(b) = \{g \in G \mid b^g = b\}$ denotes the stabilizer of b and $\mathrm{Bl}(G|b)$ is the set of blocks of $K[G]$ that cover b, cf. [6, Section 6]. Clearly $I_G(b)$ is a subgroup of G containing N and $\mathrm{Bl}(G|b)$ is non-empty.

Our first main result is:

Proposition 1. *Suppose that $I_G(W)/N$ is a p-group. Then all $K[G]$-modules of* $\mathrm{Irr}(G|W)$ *are isomorphic.*

This result seems only to be known in the case that the field K is "sufficiently

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* Part of this research was completed while the author was a visitor at the Mathematics Departments of the Technion, Haifa, Israel; Bar-Ilan University, Ramat Gan, Israel and the University of Kiel, Federal Republic of Germany.

large", cf. [2, III, Corollary 3.16]. Moreover, Proposition 1 seems not to be an immediate consequence of Clifford Theory (cf. [1, Sections 7 and 8]).

Clearly, Proposition 1 and a classical result of Clifford (cf. [4, V. Hauptsatz 17.3]) imply the following generalization of [5, VII, Theorem 16.10b]:

COROLLARY 2. *Suppose that* G/N *is a p-group and let* $\{W_i | 1 \leqslant i \leqslant m\}$ *be a complete set of representatives of the G-conjugacy classes of irreducible* $K[N]$*-modules. For each* W_i *choose a module* V_i *of* $\mathrm{Irr}(G|W_i)$ *for all* $1 \leqslant i \leqslant m$*. Then*

(a) $V_i \cong V_j$ *if and only if* $i = j$ *for all* $1 \leqslant i, j \leqslant m$*; and*

(b) $\{V_i | 1 \leqslant i \leqslant m\}$ *is a complete set of representatives for the isomorphism types of irreducible* $K[G]$*-modules.*

In Section 3 below, we utilize Proposition 1 to present an alternate proof to J. A. Green's elegant proof of [3, Lemma 2.2.3].

Combining Proposition 1 with the basic result [3, Lemma 2.2.3], and a well-known result of Clifford (cf. Lemma 2.1), we immediately obtain a generalization of [2, III, Corollary 3.15] and of [6, Theorem 12.17d2] which implies that the integer z in this result always satisfies $z = 1$:

COROLLARY 3. *Suppose that* $I_G(W)/N$ *is a p-group and let* X *be a module of* $\mathrm{Irr}(I_G(W)|W)$*. Then* $X_N \cong W$*, all modules of* $\mathrm{Irr}(I_G(W)|W)$ *are isomorphic to* X*,* X^G *is a module of* $\mathrm{Irr}(G|W)$*, all modules of* $\mathrm{Irr}(G|W)$ *are isomorphic to* X^G *and* W *occurs as a composition factor in* $(X^G)_N$ *with multiplicity* 1.

Our next result generalizes [6, Theorem 12.17d1)]:

PROPOSITION 4. *Suppose that* $I_G(W)/N$ *is a p-group. Let* X *be a module of* $\mathrm{Irr}(I_G(W)|W)$*, so that* $X_N \cong W$*. Set* $V = X^G$*, so that* V *is an element of* $\mathrm{Irr}(G|W)$*, and let* $P_G(V)$ *denote a projective cover of* V*. Then*

(a)
$$\alpha = \frac{\dim_K(\mathrm{End}_{K[N]}(W))}{\dim_K(\mathrm{End}_{K[I_G(W)]}(X))}$$

is a positive integer; and

(b)
$$P_N(W)^G \cong \alpha P_G(V).$$

The next result generalizes [5, VII, Theorem 16.10b), (c) and (d)]:

COROLLARY 5. *Assume that the hypotheses of Proposition 4 hold and that* W *is an absolutely irreducible* $K[N]$*-module. Then*

(a) X *is an absolutely irreducible* $K[I_G(W)]$*-module;*

(b) $V = X^G$ *is an absolutely irreducible* $K[G]$*-module; and*

(c) $P_N(W)^G \cong P_G(V)$.

Remark 6. With regard to Proposition 4, there are examples in which X is *an absolutely irreducible* $K[I_G(W)]$*-module and* W *is not an absolutely irreducible* $K[N]$*-module* (cf. [5, VII, Exercise 70]).

Next we give some consequences of the results above for block theory that genralize [2, V, Lemma 3.5] and [5, VII, Theorem 16.10(a)]:

PROPOSITION 7. *Suppose that* $I_G(b)/N$ *is a p-group. Then there is a unique block* B *of* G *in* $\mathrm{Bl}(G|b)$ *and*

(a) *If* W *is an element of* $\mathrm{Irr}(b)$*, then* $N \leqslant I_G(W) \leqslant I_G(b)$*,* $I_G(W)/N$ *is a p-group, all elements of* $\mathrm{Irr}(G|W)$ *are isomorphic and* $\mathrm{Irr}(G|W) \subseteq \mathrm{Irr}(B)$*;*

(b) *If* V *is an element of* $\mathrm{Irr}(B)$*, then* V *is an element of* $\mathrm{Irr}(G|W)$ *for some* W *in* $\mathrm{Irr}(b)$*; and*

(c) *Assume that* W_i *is an element of* $\mathrm{Irr}(b)$ *and* V_i *is an element of* $\mathrm{Irr}(G|W_i)$ *for* $i = 1$ *and* 2*. Then* $V_1 \cong V_2$ *if and only if* $W_1 \otimes g \cong W_2$ *for some* $g \in I_G(b)$*.*

Remark 8. Assume the situation of Proposition 7. Then the Cartan invariants of B are determined in the following setting. Let $q = |I_G(b)/N|$ and let

$$\{x_1 = 1, x_2, \ldots, x_q\}$$

be a right transversal of N in $I_G(b)$. Also let W_i and V_i for $i = 1$ and 2 be as in Proposition 7(c), so that $P_N(W_i)^G \cong \alpha_i P_G(V_i)$ for a positive integer α_i as in Proposition 4 for $i = 1$ and 2. Then

$$\alpha_1\alpha_2 \dim_K\big(\mathrm{Hom}_{k[G]}(P_G(V_1), P_G(V_2))\big) = \sum_{j=1}^{q} \dim_K\big(\mathrm{Hom}_{K[N]}(P_N(W_1), P_N(W_2 \otimes x_j))\big).$$

COROLLARY 9. *Suppose that* G/N *is a p-group. Then each block of* N *is covered by a unique block of* G *and this correspondence induces a one-to-one correspondence between* $\mathrm{Bl}(G)$ *and the orbits of* G *on* $\mathrm{Bl}(N)$*.*

In Section 2, we present two general results that hold for arbitrary fields that we require for the proofs of our main results given in Section 3.

2. Preliminary results. For the convenience of the reader, we present two general results that we require in Section 3.

Since the results of this section hold for all fields, we shall, in this section, let k denote an arbitrary field. As above, N is a normal subgroup of the finite group G and $k[N]$ and $k[G]$ denote the associated group algebras, etc. Also W denotes an arbitrary $k[N]$-module, V denotes an arbitrary $k[G]$-module, b is a block of $k[N]$ and B is a block of $k[G]$. Moreover, $I_G(W) = \{g \in G | W \otimes g \cong W\}$ and if U is an irreducible $k[N]$-module then $\mathrm{mult}(U \text{ in } W)$ denotes the multiplicity of U as a composition factor of W.

THEOREM 2.1 (Clifford). *Assume that* W *is an irreducible* $k[N]$*-module and set* $T = I_G(W)$*, so that* $N \leqslant T \leqslant G$*. Then the following five conditions hold:*

(a) *If* $X \in \mathrm{Irr}(T|W)$*, then* $X^G \in \mathrm{Irr}(G|W)$ *and* $X_N \cong \big(\mathrm{mult}(W \text{ in } (X^G)_N\big) W$*;*

(b) *If* $X \in \mathrm{Irr}(T|W)$*, then* X *is isomorphic to a submodule of* $(X^G)_T$ *and* $(X^G)_T$ *has precisely one composition factor in* $\mathrm{Irr}(T|W)$ *(and that composition factor is isomorphic to* X*);*

(c) *Suppose that* $X, Y \in \mathrm{Irr}(T|W)$*. Then* $X^G \cong Y^G$ *if and only if* $X \cong Y$*;*

(d) *Suppose that* $V \in \mathrm{Irr}(G|W)$*. Then* $V \cong X^G$ *for some* $X \in \mathrm{Irr}(T|W)$*; and*

(e) *If* $X \in \mathrm{Irr}(T|W)$, *then* $\mathrm{End}_{k[T]}(X) \cong \mathrm{End}_{k[G]}(W^G)$ (*as algebras*) *and* $P_T(X)^G \cong P_G(X^G)$.

Proof. Let $X \in \mathrm{Irr}(T|W)$ and let V be an irreducible constituent of the head $H(X^G)$ of X^G. Since $\mathrm{Hom}_{k[G]}(X^G, V) \cong \mathrm{Hom}_{k[T]}(X, V_T)$ by [5, VII, Theorem 4.5], we conclude that X is isomorphic to a submodule of V_T and hence $V \in \mathrm{Irr}(G|W)$. Set $t = |G:T|$ and let $g_1 = 1, g_2, \ldots, g_t$ be a right transversal of T in G. Then, by a Theorem of Clifford ([4, V, Hauptsatz 17.3]), there are positive integers e, f such that

$$V_N \cong e(\bigoplus_{j=1}^{t} W \otimes g_j) \quad \text{and} \quad X_N \cong fW.$$

Consequently $f \leqslant e$. On the other hand,

$$\dim_k(V) = et\dim_k(W) \leqslant \dim_k(X^G) = tf\dim_k(W),$$

so that $e \leqslant f$. Thus $e = f$, $V = X^G \in \mathrm{Irr}(G|W)$ and (a) holds.

We have seen that X is isomorphic to a submodule of $(X^G)_T$. Since

$$f = \mathrm{mult}(W \text{ in } X_N)$$

and $e = \mathrm{mult}(W \text{ in } (X^G)_N)$, (b) also holds. Moreover, it is clear that (c) is demonstrated.

Let $U \in \mathrm{Irr}(G|W)$ and choose a submodule Z of U_N such that $Z \cong W$. Set $P = \sum_{t \in T} Zt = Zk(T)$, so that P is a submodule of U_T. As $k[N]$-modules we have $Zt \cong Z \otimes t \cong Z$ for all $t \in T$. Thus $P_N \cong \alpha W$ for some positive integer α. Let X be a $k[T]$-irreducible submodule of P, so that $X_N \cong \beta W$ for some positive integer $\beta \leqslant \alpha$ and $X \in \mathrm{Irr}(T|W)$. As $(0) \neq \mathrm{Hom}_{k[T]}(X, U_T) \cong \mathrm{Hom}_{k[G]}(X^G, U)$ by [5, VII, Theorem 4.5] and X^G is an irreducible $k[G]$-module by (a), we have $X^G \cong U$ and (d) holds.

For (e), let $X \in \mathrm{Irr}(T|W)$. Then, by [5, VII, Theorem 4.5],

$$\mathrm{Hom}_{k[G]}(X^G, X^G) \cong \mathrm{Hom}_{k[T]}(X, (X^G)_T).$$

Since $\mathrm{Hom}_{k[T]}(X, (X^G)_T) \cong \mathrm{Hom}_{k[T]}(X, X)$ by (b), we have

$$\mathrm{End}_{k[T]}(X) \cong \mathrm{End}_{k[G]}(X^G)$$

as algebras. Clearly, $P_T(X)^G$ is a projective $k[G]$-module, $P_N(W)^T$ is a projective $k[T]$-module and $I_G(P_N(W)) = T$. Since

$$\mathrm{Hom}_{k[T]}(P_N(W)^T, X) \cong \mathrm{Hom}_{k[N]}(P_N(W), X_N) \neq (0),$$

we conclude that $P_T(X)|P_N(W)^T$ by [5, VII, Theorem 10.9a)]. Then [5, VII, Theorem 9.6(a)] implies that $P_T(X)^G$ is indecomposable. As

$$\mathrm{Hom}_{k[G]}(P_T(X)^G, X^G) \cong \mathrm{Hom}_{k[T]}(P_T(X), (X^G)_T) \neq (0)$$

by we have $P_T(X)^G \cong P_T(X^G)$ by [5, VII, Theorem 10.9(a)] which completes our proof of this Theorem.

For the next result, let f be the primitive central idempotent of $k[N]$ that determines b, let $t = |G:I_G(b)|$, let $\{x_1 = 1, x_2, \ldots, x_t\}$ be a right transversal of $I_G(b)$ in G and set $x = \sum_{j=1}^{t} f^{x_j}$. Also let e denote the primitive central idempotent of $k[G]$ that determines B.

Lemma 2.2. *Assume that W is a non-zero module in the block b of $k[N]$ and that V is a non-zero module in the block B of $k[G]$. Then*

(a) *B covers b if and only if V_N has a composition factor in b; and*

(b) *If a composition factor of W^G lies in B, then B covers b.*

Proof. Either (i) $xe = e$ and B covers b or (ii) $xe = 0$ and B does not cover b. Assume that V_N has a composition factor in b. Then $(0) \neq Vf \subseteq Vx = Vex$, so that $xe \neq 0$ and B covers b. Assume that B covers b and choose an irreducible $k[G]$-submodule X of V. Then $X \in B$ and $(0) \neq X = Xe = Xx$. Restricting to N and applying [4, V, Hauptsatz 17.3], we obtain an irreducible $k[N]$-submodule Y of X_N and an integer j with $1 \leqslant j \leqslant t$ such that $Yf^{x_j} \neq (0)$. Then $Yx_j^{-1}f \neq (0)$, X_N has an irreducible k[N]-submodule in (b) and (a) follows. Assume that W^G has a composition factor in B. Since $W^Gx = W^G$, we have $(0) \neq W^Ge = W^Gxe$, whence $xe \neq 0$ and are done.

3. Proofs of our main results.

A proof of Proposition 1. Assume the hypotheses of this Proposition and let $q = |I_G(W)/N|$ so that q is a p-power. Clearly, Theorem 2.1 implies that it suffices to assume that $G = I_G(W)$. Let X and Y be $K[G]$-modules of $\mathrm{Irr}(G|W)$. Then there are positive integers r and s such that $X_N \cong rW$ and $Y_N \cong sW$. Setting $V = (rsW)^G$ and applying [5, VII, Lemma 4.15b], we have $V \cong ((sX)_N)^G \cong s((X_N)^G) \cong s(X \otimes_K K[G/N])$ where the $K[G]$-module structure of $K[G/N]$ is given by $(Ng_1)g_2 = Ng_1g_2$. Since the $K[G]$-module $K[G/N]$ has precisely q composition factors each of which ist he trivial $K[G]$-module by [5, VII, Theorem 5.2a], it follows that all composition factors of V are isomorphic to X. Similarly all composition factors of V are isomorphic to Y and the Jordan–Holder Theorem completes the proof.

An alternate proof of [3, Lemma 2.2.3]: Clearly, by Theorem 2.1, the following lemma is equivalent to [3, Lemma 2.2.3] and combined with Proposition 1 generalizes [2, III, Corollary 3.15]. Our proof of the following result differs significantly from J. A. Green's elegant proof of [3, Lemma 2.2.3] and uses Proposition 1.

Lemma 3.1. *Assume that K is an arbitrary field of characteristic $p > 0$, that N is a normal subgroup of G such that G/N is a p-group, that W is an irreducible*

K[N]-module and that V is an irreducible K[G]-module. Then mult(W in V_N) = 0 *or* 1

Proof By Theorem 2.1, we may assume that $I_G(W) = G$ and that $V_N \cong eW$ for a positive integer e. It suffices to demonstrate that $e = 1$. Let L denote an algebraic closure of K. Then $V_L \cong \bigoplus_{i=1}^{r} U_i$ where $U_1, \ldots U_r$ are pairwise non-isomorphic irreducible $L[G]$-modules for some positive integer r by [5, VII, Lemma 1.15]. Similarly $W_L \cong \bigoplus_{j=1}^{s} X_j$ where $X_1, \ldots, X_s$ are pairwise non-isomorphic irreducible $L[N]$-modules for some positive integer s. Hence

$$(V_L)_N \cong (V_N)_L \cong e(\bigoplus_{j=1}^{s} X_j) \cong \bigoplus_{i=1}^{r} ((U_i)_N).$$

Since $U_1, \ldots, U_r$ are pairwise non-isomorphic, we may assume by Proposition 1 that U_i is not in $\mathrm{Irr}(G|X_1)$ if $i > 1$ and U_1 is in $\mathrm{Irr}(G|X_1)$. Hence

$$\mathrm{mult}(X_1 \text{ in } (U_1)_N) = e.$$

It is now apparent that we may assume that $K = L$ is algebraically closed from the beginning. We now proceed by induction on $|G/N|$. Clearly we may assume that $N \neq G$. Let M be a maximal subgroup of G such that $N \leqslant M$. Then $M \trianglelefteq G$ and $|G/M| = p$. Choose X in $\mathrm{Irr}(M|W)$. Then $X_N \cong W$ by the inductive hypothesis and $I_G(X) = G$ by Proposition 1 since $I_G(W) = G$. Moreover, V is a module in $\mathrm{Irr}(G|X)$ by Proposition 1. Now [5, VII, Theorem 9.9(a)] and Proposition 1 imply that $V_M \cong X$ and hence $V_N \cong W$ which concludes our proof.

A proof of Proposition 4 and Corollary 5. Assume the hypotheses of Proposition 4. Clearly, Theorem 2.1 implies that we may assume that $G = I_G(W)$ and $V = X$. Since $P_N(W)^G$ is a projective $K[G]$-module, it suffices by [5, VII, Theorem 10.9(a)] to show that the head of $P_N(W)^G$ satisfies: $H(P_N(W)^G) \cong \alpha V$. Now $P_N(W)^G J(K[N])$ is a $K[G]$-submodule of $P_N(W)^G$ and

$$P_N(W)^G J(K[N]) \subseteq P_N(W)^G J(K[G]).$$

Note that $P_N(W)/(P_N(W)J(K[N])) \cong W$ and hence

$$P_N(W)^G/(P_N(W)^G J(K[N])) \cong (P_N(W)/P_N(W)J(K[N]))^G \cong W^G.$$

Consequently,

$$H(P_N(W)^G \cong H(W^G).$$

Since $(W^G)_N \cong |G/N|W$, we conclude that $H(W^G) \cong \beta V$ for some positive integer β by Proposition 1. Then Proposition 1 and [5, VII, Theorem 4.13a)] complete the proof of Proposition 4

Next, as in Corollary 5, we have $\dim_K(\mathrm{End}_{K[N]}(W)) = 1$. Then Proposition 4 forces

$$\dim_K(\mathrm{End}_{K[I_G(W)]}(X)) = 1 \quad \text{and} \quad P_N(W)^G \cong P_G(V).$$

Since $\mathrm{End}_{K[G]}(X^G) \cong \mathrm{End}_{K[I_G(W)]}(X)$ by Theorem 2.1(e), we are done.

A proof of Proposition 7 and Corollary 9. Assume the hypotheses of Proposition 7 and let $B \in \mathrm{Bl}(G|b)$. Then, as is well known, if $Y \in \mathrm{Irr}(b)$, then $N \leqslant I_G(Y) \leqslant I_G(b)$, and hence all elements of $\mathrm{Irr}(G|Y)$ are isomorphic by Proposition 1. Also if V is an element of $\mathrm{Irr}(B)$, then V_N possesses a composition factor W in $\mathrm{Irr}(b)$ by Lemma 2.2(a). We assert:

(3.1) If W is an element of $\mathrm{Irr}(b)$, U is a composition factor of $P_N(W)$, V is an element of $\mathrm{Irr}(G|W)$ and X is an element of $\mathrm{Irr}(G|U)$, then V and X belong to the same block of G.

To see this, observe that $\mathrm{Hom}_{K[N]}(P_N(U), P_N(W)) \neq (0)$ since U is a composition factor of $P_N(W)$. Since $P_N(W)|(P_N(W)^G)_N$, we conclude from [5, VII, Theorem 4.5] that $\mathrm{Hom}_{K[G]}(P_N(U)^G, P_N(W)^G) \cong \mathrm{Hom}_{K[N]}(P_N(U), (P_N(W)^G)_N) \neq (0)$. Thus $\mathrm{Hom}_{K[G]}(P_G(X), P_G(V)) \neq (0)$ by Proposition 4(b) and (3.1) holds.

Now Lemma 2.2(a) and [5, VII, Theorem 12.4] imply that B is unique and that Proposition 7(a) and (b) and Corollary 9 hold. Next let W_i and V_i be as in Proposition 7(c) for $i = 1$ and 2. Suppose that $W_1 \otimes g \cong W_2$ for some $g \in G$. Thus $g \in I_G(b)$ and $P_N(W_2)^G \cong (P_N(W_1 \otimes g))^G \cong (P_N(W_1) \otimes g)^G \cong P_N(W_1)^G$. Then Proposition 4(b) implies that $P_G(V_1) \cong P_G(V_2)$ and hence $V_1 \cong V_2$. Conversely assume that $V_1 \cong V_2$ and let X_i be an element of $\mathrm{Irr}(I_G(W_i)|W_i)$ so that $(X_i)_N \cong W_i$ and $V_i \cong (X_i)^G$ for $i = 1$ and 2 by Corollary 3. Set $s = |G:I_G(W_1)|$ and choose a right transversal $\{x_1 = 1, x_2, \ldots, x_s\}$ of $I_G(W_1)$ in G. Then

$$(V_1)_N \cong ((X_1)^G)_N \cong \bigoplus_{j=1}^{s} (W_1 \otimes x_j).$$

Note that $W_1 \otimes x_j$ is an element of $\mathrm{Irr}(b^{x_j})$ for all $1 \leqslant j \leqslant s$. A similar result holds for $(V_2)_N$. Since $(V_1)_N \cong (V_2)_N$, the Jordan-Holder Theorem implies the desired conclusion.

A proof of Remark 8. Assume the hypotheses of this remark. Let $r = |G:I_G(b)|$ and choose a right transversal $\{y_1 = 1, y_2, \ldots, y_r\}$ of $I_G(b)$ in G. Thus $\{x_i y_j | 1 \leqslant i \leqslant q, 1 \leqslant j \leqslant r\}$ is a right transversal of N in G. By [5, XII, Theorem 4.5], we have $\mathrm{Hom}_{K[G]}(P_N(W_1)^G, P_N(W_2)^G) \cong \mathrm{Hom}_{K[N]}(P_N(W_1),$

$$\bigoplus_{j=1}^{r} \bigoplus_{i=1}^{q} (P_N(W_2) \otimes x_i y_j)) \cong \mathrm{Hom}_{K[N]}(P_N(W_1), \bigoplus_{j=1}^{r} \bigoplus_{i=1}^{q} (P_N(W_2 \otimes x_i y_j)).$$

Since $W_2 \otimes x_i y_j$ is an element of $\mathrm{Irr}(b^{y_j})$ for all $1 \leqslant i \leqslant q$ and $1 \leqslant j \leqslant r$, the desired conclusion follows directly.

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