# Infinite products of alephs

by

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**Abstract.** F. Bagemihl [1, Theorem 2] proved that if for some fixed ordinal $\gamma$, $\aleph_\alpha^{|\alpha|} = \aleph_{\alpha+\gamma}$ for every $\alpha \geqslant \omega$, then $\gamma < \omega$. The purpose of this paper is to show that there is no fixed ordinal $\gamma$ such that $\aleph_\alpha^{|\alpha|} = \aleph_{\alpha+\gamma}$ for all sufficiently large $\alpha$ and if there is a fixed ordinal $\gamma$ such that $\aleph_\alpha^{|\alpha|} = \aleph_{\alpha+\gamma}$ for all sufficiently large limit ordinals $\alpha$, then $\gamma < \omega$.

The following theorem leads to show that there is no fixed ordinal $\gamma$ such that $\aleph_\alpha^{|\alpha|} - \aleph_{\alpha+\gamma}$ for all sufficiently large $\alpha$ and if there is a fixed ordinal $\gamma$ such that $\aleph_\alpha^{|\alpha|} - \aleph_{\alpha+\gamma}$ for all sufficiently large limit ordinals $\alpha$, then $\gamma < \omega$, which is much stronger than theorem 2 in [1], which says that if there is a fixed ordinals $\gamma$ such that $\aleph_\alpha^{|\alpha|} = \aleph_{\alpha+\gamma}$ for every $\alpha \geqslant \omega$, then $\gamma < \omega$. It is interesting to note that if there is a fixed finite ordinal $m$ such that $2^{\aleph_\xi} = \aleph_{\xi+m}$ for every ordinal $\xi$, then for every limit ordinal $\alpha$, $\aleph_\alpha$ is a strong limit cardinal and hence by a well-known result in [3, P.50, (6.21)], $\aleph_\alpha^{|\mathrm{cf}\,\alpha|} = 2^{\aleph_\alpha}$, (where cf$\alpha$ is the least ordinal cofinal with $\alpha$) which implies that $\aleph_\alpha^{|\alpha|} = 2^{\aleph_\alpha} = \aleph_{\alpha+m}$ for every limit ordinal $\alpha$.

THEOREM 1. *If there are fixed ordinals $\gamma$ and $\beta \neq 0$ such that $\aleph_\alpha^{|\alpha|} = \aleph_{\alpha+\gamma}$ for all sufficiently large $\alpha$ of the form $\beta\cdot\xi$, then $\gamma < \beta$.*

Proof. Assume the conclusion to be false. Then there are ordinals $\varrho$ and $\eta \neq 0$ such that $\gamma = \beta\cdot\eta+\varrho$, $\varrho < \beta$, and $\aleph_\alpha^{|\alpha|} = \aleph_{\alpha+\gamma}$ for all sufficiently large $\alpha$ of the form $\beta\cdot\xi$. Then

$$\aleph_{\alpha+\beta\cdot\eta}^{|\alpha+\beta\cdot\eta|} = \aleph_{\alpha+\beta\cdot\eta+\gamma} \quad \text{and} \quad \aleph_{\alpha+\beta\cdot\eta}^{|\alpha+\beta\cdot\eta|} = \aleph_{\alpha+\beta\cdot\eta}^{|\alpha|} \leqslant \aleph_{\alpha+\gamma}^{|\alpha|} = (\aleph_\alpha^{|\alpha|})^{|\alpha|} = \aleph_\alpha^{|\alpha|} = \aleph_{\alpha+\gamma}.$$

Hence $\aleph_{\alpha+\beta\cdot\eta+\gamma} \leqslant \aleph_{\alpha+\gamma}$, and consequently

$$\alpha+\beta\cdot\eta+\gamma \leqslant \alpha+\gamma. \tag{1}$$

Since $\varrho < \beta \leqslant \beta\cdot\eta$, we have

$$\alpha+\beta\cdot\eta+\gamma = \alpha+\beta\cdot\eta+\beta\cdot\eta+\varrho \geqslant \alpha+\beta\cdot\eta+\beta\cdot\eta \geqslant \alpha+\beta\cdot\eta+\beta > \alpha+\beta\cdot\eta+\varrho = \alpha+\gamma,$$

which contradicts (1).

Our assumption is therefore untenable, and the theorem is true.

COROLLARY 1. *There is no fixed ordinal $\gamma$ such that $\aleph_\alpha^{|\alpha|} = \aleph_{\alpha+\gamma}$ for all sufficiently large $\alpha$.*

Proof. By taking $\beta = 1$ in the previous theorem, we have $\gamma = 0$. If $\gamma = 0$, then for sufficiently large limit ordinal $\alpha$, $\aleph_\alpha = \aleph_{\alpha+\gamma} = \aleph_\alpha^{|\alpha|} = \prod_{\xi<\alpha} \aleph_\xi > \sum_{\xi<\alpha} \aleph_\xi = \aleph_\alpha$ which is a contradiction.

COROLLARY 2. *If there is a fixed ordinal $\gamma$ such that $\aleph_\alpha^{|\alpha|} = \aleph_{\alpha+\gamma}$ for all sufficiently large limit ordinals $\alpha$, then $\gamma < \omega$.*

Proof. By taking $\beta = \omega$ in the previous theorem, we have $\gamma < \omega$.

Remark. Patai's theorem [2, Theorem XIV] states that if there is a fixed ordinal $\gamma$ such that $2^{\aleph_\alpha} = \aleph_{\alpha+\gamma}$ for every $\alpha$, then $\gamma < \omega$.

By the remark preceeding Theorem 1, the hypothesis of Patai's theorem implies the hypothesis of the Corollary 2 in this papaer.

### References


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# Clifford theory for *p*-sections of finite groups *

by

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**Abstract.** Let $K$ denote an arbitrary field of prime characteristic $p \neq 0$. Let $N$ denote a normal subgroup of the finite group $G$ such that $G/N$ is a $p$-group. Here, in this situation, we demonstrate some basic results of Clifford theory for irreducible modules and blocks of $K[G]$. These results extend and generalize work of several authors.

**1. Introduction and statements.** Our notation and terminology are standard and tend to follow the conventions of [5]. In particular, all vector spaces encountered in this article have finite dimension, all modules over an algebra are right and unital, if $n$ is a positive integer and $V$ is a module, then $nV$ denotes the module direct sum of $n$ copies of $V$ and if $A$ is a ring then $U(A)$ denotes the multiplicative group of units of $A$.

Throughout this paper, $G$ denotes a finite group. $N$ is a normal subgroup of $G$, $K$ is a field with $\mathrm{char}(K) = p > 0$ but is otherwise arbitrary and $K[G]$ and $K[N]$ are the associated group algebras. Also $W$ denotes an irreducible $K[N]$-module, $I_G(W) = \{g \in G \mid W \otimes g \cong W\}$ denotes the stabilizer of $W$ in $G$, $P_N(W)$ denotes a projective cover of $W$ and $\mathrm{Irr}(G|W)$ denotes the class of irreducible $K[G]$-modules $V$ such that $W$ is a composition factor of (and hence a direct summand of) $V_N$. Clearly $I_G(W)$ is a subgroup of $G$ containing $N$ and $\mathrm{Irr}(G|W)$ is non-empty. Also $b$ denotes a block of $K[N]$, $\mathrm{Irr}(b)$ denotes the class of irreducible $K[N]$-modules in $b$, $I_G(b) = \{g \in G \mid b^g = b\}$ denotes the stabilizer of $b$ and $\mathrm{Bl}(G|b)$ is the set of blocks of $K[G]$ that cover $b$, cf. [6, Section 6]. Clearly $I_G(b)$ is a subgroup of $G$ containing $N$ and $\mathrm{Bl}(G|b)$ is non-empty.

Our first main result is:

PROPOSITION 1. *Suppose that $I_G(W)/N$ is a $p$-group. Then all $K[G]$-modules of* $\mathrm{Irr}(G|W)$ *are isomorphic.*

This result seems only to be known in the case that the field $K$ is "sufficiently

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