On local 1-connectedness of Whitney continua

by

Hisao Kato (Hiroshima)

Dedicated to Professor Yukihide Kodama on his 60th birthday

Abstract. In [23], S. B. Nadler proved that the property of being locally connected is a Whitney property. In [25], A. Petrus proved that there is a Whitney map $\omega$ on a 2-cell $D$ such that $\omega^{-1}(t)$ is not locally 2-connected and not 2-connected for some $t > 0$, which implies that the property of being locally 2-connected (or 2-connected) is not a Whitney property. Naturally, the following problem will be raised: Is it true that the property of being locally 1-connected is a Whitney property? In this paper, we prove the following. (1) If $X$ is a locally 1-connected continuum contained in a 2-dimensional manifold, then each Whitney continuum of $X$ is also locally 1-connected. Moreover, if $X$ is simply connected, then each Whitney continuum of $X$ is simply connected. (2) There exist a 2-dimensional compact AR $Z$ and a Whitney map $\omega$ for $C(Z)$ such that $\omega^{-1}(t)$ is not locally 1-connected and not simply connected for some $t > 0$.

0. Introduction. By a continuum we mean a nonempty compact connected metric space. Let $X$ be a continuum with metric $d$. By the hyperspace of $X$ we mean $C(X) = \{A | A$ is a subcontinuum of $X\}$ with Hausdorff metric $d_H$. In [29], H. Whitney proved that for any continuum $X$ there exists a map $\omega : C(X) \to [0, \infty)$ satisfying

1. $\omega(x) = 0$ for every $x \in X$, and
2. if $A, B \in C(X)$, $A \subseteq B$ and $A \neq B$, then $\omega(A) < \omega(B)$.

Any such a map $\omega$ is called a Whitney map and $\omega^{-1}(t) (0 \leq t < \omega(X))$ is called a Whitney continuum. A topological property $P$ is called a Whitney property if whenever $X$ has property $P$, so does every Whitney continuum in $C(X)$. Many properties of Whitney continua have been studied by many authors (e.g., see references).

In [23], S. B. Nadler proved that the property of being locally connected is a Whitney property. In [25], A. Petrus proved the following result:

There exists a Whitney map $\omega$ for $C(D)$ such that $\omega^{-1}(t)$ is not locally 2-connected and not 2-connected for some $t > 0$, where $D$ is a 2-cell. But, in [26] J. T. Rogers proved that if $X$ is a continuum with $H^1(X) = 0$, then for any Whitney
map $\omega$ for $C(X)$. Then if $\alpha(A) = \alpha(B)$ and $\varepsilon > 0$, there is neighborhood $U$ of $A$ in $X$ such that $B \in U$ implies $d_B(A, B) < \varepsilon$.

Let $(P, \preceq)$ be a partially ordered space. Then a map $\omega: P \to [0, \infty)$ is said to be a Whitney map if (i) $\omega(p) = 0$ for $p \in \text{Min} P$, (ii) $\alpha(p) < \alpha(q)$ for $p \prec q$, and (iii) $\alpha(p) = \alpha(q)$ for $p, q \in \text{Max} P$. Thus a Whitney map $\omega$ for $C(X)$ is a Whitney map in the above sense for $C(X)$ ordered by inclusion. Then the following fact is very useful.

Let $P$ be a compact metric partially ordered space such that $\text{Min} P$ and $\text{Max} P$ are disjoint closed sets and let $Q$ be a closed subset of $P$ such that $\text{Min} Q \subset \text{Min} P$ and $\text{Max} Q \subset \text{Max} P$. Then a Whitney map for $Q$ can be extended to a Whitney map for $P$.

2. Local 1-connectedness of Whitney continua. In this section, we prove the following theorem.

(2.1) Theorem. Let $X$ be a continuum such that each subcontinuum of $X$ is nearly 1-movable. If $X$ is locally 1-connected, then for any Whitney map $\omega$ for $C(X)$, $\omega^{-1}(1)$ is also locally 1-connected for each $t$. Moreover, if $X$ is simply connected, then $\omega^{-1}(1)$ is also simply connected for each $t$.

Proof. Let $\omega$ be a Whitney map for $C(X)$ and let $0 < t < \infty$. Consider the following subset $Y$ in $X \times \omega^{-1}(t)$:

$Y = \{ (x, A) \in X \times \omega^{-1}(t) \mid x \in A \}$.

Then $Y$ is a continuum. In fact, the map $p = p_1: Y \to \omega^{-1}(t)$ is an open and monotone map, where $p_1: X \times \omega^{-1}(t) \to \omega^{-1}(t)$ is the projection map. Note that $\omega^{-1}(A) \subset A$ for each $A \in \omega^{-1}(t)$ and $A$ is nearly 1-movable. Hence, if $Y$ is locally 1-connected, by (1.3) we see that $\omega^{-1}(t)$ is locally 1-connected. Now, we shall show that $Y$ is locally 1-connected. First, we shall show that $Y$ is locally connected. Let $(x, A) \in Y$ and $\mathcal{W}$ be any neighborhood of $(x, A)$ in $Y$. Take $\varepsilon > 0$ and a neighborhood $U$ of $x$ in $X$ such that if $y \in U$ and $d_B(A, B) < \varepsilon$, then $(y, B) \in \mathcal{W}$. By (1.4), there is a closed neighborhood $A^* \subset A$ in $X$ such that if $B \in \omega^{-1}(t) \cap C(A^*)$, then $d_B(A, B) < \varepsilon$. We may assume that $A^*$ is a locally connected continuum. Take a path connected neighborhood $V$ of $x$ in $X$ such that $V \subset U \cap A^*$. Set $e^{-1}(x, A) \subset V$ and $e \subset \mathcal{W}$. We show that $e$ is path-connected. Let $(y, B) \in e$. Since $y \in V$, there is an arc $\gamma: 0 \to 1$ such that $\gamma(0) = x$, $\gamma(1) = y$. By [14], there are two segments $\beta_1, \beta_2: I \to \omega^{-1}(0, t)$ such that $\beta_1(0) = x$, $\beta_2(0) = y$ and $\beta_2(t) = B$. Define an arc $\gamma$ from $A$ to $B$ in $\omega^{-1}(0, t)$ by

$\gamma(s) = \begin{cases} \beta_1(1 - 3s), & 0 < s \leq 1/3, \\ \gamma(3s - 1), & 1/3 < s \leq 2/3, \\ \beta_2(3s - 2), & 2/3 < s \leq 1. \end{cases}$

$e$ is path-connected.
Since $A^*$ is locally connected, there is a retraction

$$R: \omega^{-1}(0, t) \cap C(A^*) \to \omega^{-1}(0, t) \cap C(A^*)$$

such that $R(C) \supset C$ for each $C \in \omega^{-1}(0, t) \cap C(A^*)$ (see [4, (1.2)]). Then $\sigma = R^{-1}: I \to \omega^{-1}(t) \cap C(A^*)$ is an arc from $A$ to $B$ such that $x < \sigma(s)$ for $0 < s < 1/3$, $y < \sigma(s)$ for $1/3 < s < 2/3$, and $y < \sigma(s)$ for $2/3 < s < 1$. Define an arc $\gamma$ from $x$ to $y$ in $V$ by

$$\gamma(s) = \begin{cases} x, & \text{for } 0 < s < 1/3, \\ g(3s - 1), & \text{for } 1/3 < s < 2/3, \\ y, & \text{for } 2/3 < s < 1. \end{cases}$$

Define an arc $\theta: I \to X \times \omega^{-1}(t) \cap C(A^*)$ by $\theta(s) = (\gamma(s), \sigma(s))$ for $s \in I$. Then $\theta(s) \in Y$ and $\theta(s) \in \mathcal{W}$ for each $s \in I$. This implies that $Y$ is locally connected. Next, we shall show that $Y$ is approximately locally 1-connected. Let $(x, A) \in Y$ and $U$ be any neighborhood of $(x, A)$ in $Y$. Take a neighborhood $A^*$ of $A$ in $X$ and a neighborhood $U$ of $x$ in $X$ such that $(U \times C(A^*)) \cap Y \subset U$. We may assume that $A^*$ is a locally connected continuum and $U \subset A^*$. Since $X$ is locally 1-connected, there is a neighborhood $V$ of $x$ in $U$ such that any loop $f: S^1 \to V$ is null homotopic in $U$. Set $\gamma = (V \times C(A^*)) \cap Y$. Let $\epsilon > 0$ and let $f: S^1 \to Y$ be any map with $(x, A) \in f(S^1)$. We assume that $f(0) = (x, A)$. Consider the following maps $f_1 = p_1 f: S^1 \to V$ and $f_2 = p_2 f: S^1 \to \omega^{-1}(t) \cap C(A^*)$, where $p_1: X \times \omega^{-1}(t) \to X$ and $p_2: X \times \omega^{-1}(t) \to \omega^{-1}(t)$ are the projection maps. Since $f_1$ is null homotopic in $U$, there is an extension $f_1: D \to U$ of $f_1$. Choose points $a_i, a_{i+1}, a_0, a_n = 0$ of $S^1$ such that diam$(f_1([a_i, a_{i+1}])) < \epsilon/2$, diam$([C(\bigcup \{ B \in f_2([a_i, a_{i+1}])\}]) \cap \omega^{-1}(t)) < \epsilon/2$ (see (1.5)), and diam$(f_1([a_i, a_{i+1}])) < \epsilon/2$ for each $i = 0, 1, \ldots, n$ (see Figure 1).

![Figure 1](image)

For each $i$, take a neighborhood $A_i$ of $\bigcup \{ B \in f_2([a_i, a_{i+1}])\}$ in $A^*$ such that diam$(C(\omega^{-1}(t)) \cap A_i) < \epsilon/2$. We may assume that each $A_i$ is a locally connected continuum. As before, take arcs $\alpha_i: I \to \omega^{-1}(0, t) \cap C(A^*)$ such that $\alpha_i(0) = f_1(a_i)$, $\alpha_i(1) = f_1(a_{i+1})$ and $\alpha_i(0) \subset f_1(a_i)$ if $s \leq s'$. Since each $A_i$ is locally connected, there is a strong deformation retract $R_i: C(A^*) \cap \omega^{-1}(0, t) \cap C(A_i) \to C(A_i) \cap \omega^{-1}(t)$ such that $R_i(B, s) \subset R_i(B, s')$ if $s \leq s'$. We identify $[a_i, a_{i+1}]$ with $I$, where $[a_i, a_{i+1}]$ is the arc from $A_i$ to $A_{i+1}$ in $S^1$. Define maps $g_i: \times I \to C(A_i) \cap \omega^{-1}(0, t)$ by

$$g_i((1-z)x + zh(s)) = \begin{cases} R_i(zx, x), & \text{for } z \in [0, 1), \\ f_1(a_i), & \text{for } z \in [1, \infty), \end{cases}$$

where $0 < s < 1$, and for each $z \in [0, 1) \times I \times \omega^{-1}(0, t)$, $h(z)$ is the unique point of $I \times \omega^{-1}(t)$ such that the segment in $I \times [0, 2]$ from the point $(1/2, 3/2)$ to $z$ contains $h(z)$ (see Figure 2). Define a map $g: D \times S^1 \times I \to C(A^*) \cap \omega^{-1}(0, t)$ by

$$g(D \times [0, 1]) = f_1 \times [0, 1],$$

Since $A^*$ is locally connected, there is a strong deformation retract $R: (C(A^*) \cap \omega^{-1}(0, t)) \times I \to C(A^*) \cap \omega^{-1}(t)$ such that $R(B, s) \subset R(B, s')$ if $s \leq s'$. Define a map $F_2: D \times S^1 \times I \to C(A^*) \cap \omega^{-1}(t)$ by $F_2(y) = R_2(y, 1)$. Then $d_0(F_2) S^1 \times \{0\} < \epsilon$. By identifying $[a_i, a_{i+1}]$ with $I$, define maps $k_i: \times I \to A_i$ by

$$k_i((1-z)x + zh(s)) = \begin{cases} f_1(a_i), & \text{for } z \in [0, 1), \\ f_1(a_0), & \text{for } z \in [1, \infty). \end{cases}$$

By using $k_i$, we have a map $F_1: D \times [0, 1] \cup S^1 \times I \to U$ such that $F_1(D \times [0, 1]) = f_1$ and $F_1([a_i, a_{i+1}] \times I) = k_i$. Then we have $d(F_1 S^1 \times \{1\}, f_1) < \epsilon$. By the construction, $F_1(y) \in F_2(y)$ for each $y \in D \times S^1 \times I$. Define a map

$$F: D \times [0, 1] \cup S^1 \times I \to U \times C(A^*) \cap Y,$$
for \( y \in D \setminus \{0\} \cup S^1 \times I \). Then \( F[S^2 \times \{1\}] \) is sufficiently near to \( f \). Hence \( Y \) is approximately locally 1-connected. By (1.1), we see that \( Y \) is locally 1-connected. Hence we conclude that \( \omega^{-1}(t) \) is locally 1-connected. Similarly, moreover, if \( X \) is simply connected, we see that \( \omega^{-1}(t) \) is also simply connected. This completes the proof.

Let \( X \) be a continuum in a 2-dimensional manifold. Then \( X \) is locally 1-connected if and only if \( X \) is an ANR. Hence we have

(2.2) Corollary. Let \( X \) be a continuum in a 2-dimensional manifold. If \( X \) is an ANR, then for each Whitney map \( \omega \) for \( C(X) \), \( \omega^{-1}(t) \) is locally 1-connected for each \( t \). In particular, every Whitney continuum of a 2-cell \( D \) is locally 1-connected and simply connected.

Proof. (1.2) and (2.1) imply (2.2).

3. The property of being locally 1-connected is not a Whitney property. In this section, we prove, by showing a counterexample, that the property of being locally 1-connected (or simply connected) is not a Whitney property.

In fact, we show that there exist a 2-dimensional AR \( Z \) and a Whitney map \( \omega \) for \( C(Z) \) natural that \( \omega^{-1}(t) \) is not locally 1-connected and not simply connected for some \( t > 0 \).

It is known that if \( Z \) is a 1-dimensional ANR (resp. AR), then for any Whitney map \( \omega \) for \( C(Z) \), \( \omega^{-1}(t) \) is an ANR (resp. AR) for each \( t \) (see [20]).

(3.1) Example. Let \( X = \{x_n, p_{n+1}, N\} \) be the inverse sequence as follows:

1. \( X_1 = \{y\} \) and \( X_n(n \geq 2) \) is the unit circle \( S^1 \), and
2. the bonding maps \( p_{n+1}: X_{n+1} \rightarrow X_n \) are covering projections with degree \( 2 \) \((n \geq 2)\), and \( p_{1,2}: X_2 \rightarrow X_1 \) is the constant map.

Set \( X = \text{invlim} X \). Then \( X \) is the dyadic solenoid, and hence it is not nearly 1-movable. Let \( p_2: X \rightarrow X_2 \) be the projection. Now, consider the infinite telescope \( T(X) = \bigcup_{n=1}^\infty M(p_{n+1}) \), where \( M(p_{n+1}) \) denotes the mapping cylinder obtained by \( p_{n+1}: X_{n+1} \rightarrow X_n \), i.e., \( M(p_{n+1}) \) is obtained by identifying points \((x, 1/n) \in X_{n+1} \times \{1/n\}\)

and \( p_{n+1}(x) \in X_n \) for \( x \in X_{n+1} \) in a topological sum \( X_n \cup (X_{n+1} \times \{1/\{n+1\}, 1/n\}) \), and \( T(X) \) is obtained by identifying each point of \( X_n \times \{1/n\} \) in \( M(p_{n+1}) \) and the corresponding point of \( X_n \) in \( M(p_{n+1}) \).

Let \( Z = S(X) = T(X) \cup X \) be an AR having the same topology as \([17, (4.1)] \) (see Figure 3). Define a map \( \mu: Z \rightarrow I \) by \( \mu(x, t) = t \) if \( (x, t) \in T(X) \) and \( \mu(x) = 0 \) if \( x \in X \). Also, define a natural retraction \( \psi_t: Z \rightarrow \mu^{-1}(t), (t \in I) \) by \( \psi_t(x) = \rho_0(t) \) for \( x \in X \), \( \psi_t(x) = (x, t) \) if \( x \in X \) for \( x \in X \), \( \psi_t(x) = (x, t) \) if \( x \in X \) and \( \mu(x) = t \) for \( x \in \mu^{-1}(t, 1) \), where \( q(t) \) is the natural number such that \( 1/q(t) \leq t < 1/q(t) - 1 \) (for more detail construction of \( S(X) \), see [17]). Note that \( \mu^{-1}(t) \) is homeomorphic to \( S^1 \) for \( 0 < t < 1 \). Then \( Z \) is a 2-dimensional AR. Next, we construct a Whitney map \( \omega \) for \( C(Z) \) as follows: Consider the arc \( A = \{(q^{-1}(t) = \psi_1(X), 0 \leq t \leq 1) \in C(Z) \). Define a map \( \omega_1: A \rightarrow [0, 1] \) by \( \omega_1(q^{-1}(t)) = t + 1 \) for \( 0 \leq t \leq 1/2 \), and \( \omega_1(q^{-1}(t)) = -3t + 3 \) for \( 1/2 \leq t \leq 1 \). By (1.6), there exists a Whitney map \( \omega_2 \) for \( C(Z) \) which is an extension of \( \omega_1 \). Define a map \( \omega: C(Z) \rightarrow [0, \infty) \) by \( \omega(x) = \sup \{\omega_2(\psi_1(x)) | t \in M(A)\} + (\text{diam } M(A)) \omega_2(A) \).

Clearly, \( \omega(x) = 0 \) for \( x \in Z \). Suppose that \( A, B \in C(Z) \), \( A \subset B \) and \( A \neq B \). If \( B \subset \mu^{-1}(t) \) for some \( t \), then \( \omega(A) = \omega_2(A) + \omega_2(B) = \omega(B) \). If \( \mu(B) > 0 \), then \( \sup \{\omega_2(\psi_1(x)) | t \in M(A)\} \leq \sup \{\omega_2(\psi_1(x)) | t \in M(B)\} \),

\[ \text{diam } M(A) \omega_2(A) < \text{diam } M(B) \omega_2(B), \]

hence \( \omega(A) < \omega(B) \). This implies that \( \omega \) is a Whitney map for \( C(Z) \). Note that \( X \in \omega^{-1}(1) \). Now, we shall show that \( \omega^{-1}(1) \) is not locally 1-connected. Consider the following decomposition space \((Z', *) = (\mu^{-1}([0, 1/2]), X, *)\).
Let \( z \in Z' \) and \( z \neq * \). Then \( q^{-1}(z) \) is one point set. Choose unique two arcs \( A_1 \) and \( A_2 \) in \( \mu^{-1}(\{q^{-1}(z)\}) \) such that \( A_1 \cap A_2 = \{q^{-1}(z)\} \), \( \alpha(A_1) = \alpha(A_2) \) and \( \omega(A_1 \cup A_2) = 1 \). Set \( \beta(z) = A_1 \cup A_2 \). Define a map \( G : Z' \to \Omega \) by

\[
G(z) = \begin{cases} 
\beta(z) & \text{for } z \in Z' \text{ and } z \neq *, \\
X & \text{for } z = * \in Z'.
\end{cases}
\]

Then \( FG = I_Z \). This implies that \( \omega^{-1}(1) \) is not locally 1-connected at \( X \). Similarly, we can conclude that \( Z/X \) is a retract of \( \omega^{-1}(1) \). Since \( Z/X \) is not simply connected (see [22, Theorem 1]), \( \omega^{-1}(1) \) is not simply connected.

References

[9] — Various types of Whitney maps on \( n \)-dimensional compact connected polyhedra \( (n \geq 2) \), Topology Appl. to appear.

[27] — Whitney continua in the hyperspace \( C(X) \), Pacific J. Math. 58 (1975), 569-584.
[29] H. Whitney, Regular families of curves, Ann. of Math. 34 (1933), 244-270.

Received 9 February 1987;
In revised form 25 May 1987