and hence $\sum_i N(h_i, y) \leq 1$ on a positive measure subset of $B \setminus Y_0$. This implies that

there is a $k$ and there is a closed set $C \subset B \setminus Y_0$ such that $\lambda_k(C) > 0$ and for every $y \in C$ we have

$$N(h_i, y) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

We put $D = h_k^{-1}(C)$. Then $D$ is measurable, and $\lambda_k(D) > 0$ since $\lambda_k(D) = 0$ would imply $\lambda_k(C) = \lambda_k(h_k(D)) = 0$.

We prove that $D \subset A \setminus X_0$. Obviously, $D \subset A \setminus X_0$ since $A \setminus X_0$ is the domain of $h_k$. Let $x \in D$ and suppose that $x \notin A \setminus X_0$. Since $h_k(x) \in C \subset B \setminus Y_0 \subset \bigcup h_i(A \setminus X_0)$, we have $h_k(x_i) = h_i(x_i)$ with some $x_i \in A \setminus X_0$. If $i = k$ then $x_i = x_k \in A_k \setminus X_0$, and hence $x \neq x_k$. Thus $N(h_k, h_k(x)) \geq 2$ which is impossible since $h_k(x) \in C$. If $i \neq k$, then we get $N(h_i, h_k(x)) \geq 1$ which also contradicts $h_k(x) \in C$.

Therefore $D \subset A_k \setminus X_0$ and, consequently, $D \cap A_i = \emptyset$ for $i \neq k$. This implies that $\int (A_j \cap D) = f_k(D) = h_k(D) = C$, where $C$ is measurable and

$$0 < \lambda_k(C) \leq M_k \lambda_k(D) \leq M \lambda_k(D).$$

In other words, $D \in \mathcal{K}$. However, $D \cap X_0 = \emptyset$, and hence $D$ is disjoint from the elements of $\mathcal{K}$ which contradicts the maximality of $\mathcal{K}$. This contradiction completes the proof.

References


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An atriotic tree-like continuum with positive span which admits a monotone mapping to a chainable continuum

by

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Abstract. In this paper an example of an atriotic tree-like continuum with positive span is constructed. It is shown that there is a monotone mapping of this continuum onto a chainable continuum such that the only nondegenerate point inverse under the mapping is an arc.

1. Introduction. The following problems appear in the University of Houston Mathematics Problem Book. The first was raised by Howard Cook, the second by Cook and J. B. Fugate.

Problem 92. If $M$ is a continuum with positive span such that each of its proper subcontinua has span zero, does every nondegenerate, monotone, continuous image of $M$ have positive span?

Problem 105. Suppose $M$ is an atriotic 1-dimensional continuum and $G$ is an upper semi-continuous collection of continua filling up $M$ such that $M/G$ and every element of $G$ are chainable. Is $M$ chainable?

These problems also appeared as problems 163 and 15, respectively, in [9]. Several partial positive results concerning these problems have appeared ([2] and [8] for instance).

In this paper we construct an example which answers both questions in the negative. The example is constructed as an inverse limit of simple trees with a single bonding map and has positive span. It is similar in this respect to the examples constructed in [4, 5]. The inspiration for this example was an example of an attractor of a discrete dynamical system presented by Marcy Barge at the 1986 Spring Topology Conference at the University of Southwestern Louisiana [1]. However, this example is not the example he discussed.

1 The first author was partially supported by a grant from the University of Richmond Faculty Research Committee.
All spaces considered in this paper are metric. A continuum is a compact connected space and a mapping is a continuous function.

Suppose $X$ and $Y$ are spaces and $d$ is a metric for $Y$. If $f$ is a mapping of $X$ into $Y$, the span of $f$, denoted by $\sigma f$, is the least upper bound of the set of numbers $\varepsilon$ such that there is a connected subset $Z$ of $X \times X$ with equal first and second projections such that $d(f(x), f(y)) < \varepsilon$ for each $(x, y)$ in $Z$. The span of $X$, denoted by $\sigma X$, is the span of the identity mapping on $X$.

An inverse sequence is a pair $\{X_i, f_i\}$ whose first term is a sequence $X_1, X_2, X_3, ...$ of spaces and whose second term is a sequence of mappings (called the bonding maps of the system) $f_i : X_{i+1} \rightarrow X_i$. The inverse limit of the inverse sequence $\{X_i, f_i\}$ is the subspace $X$ of $\prod X_i$ consisting of all points $(x_1, x_2, x_3, ...)$ in $\prod X_i$ such that $f_i(x_{i+1}) = x_i$ for each $i$. The projection of $X$ onto the $i$th factor space will be denoted by $\pi_i$.

2. The continua $X$ and $Y$ and the mapping $\mu$. Let $A, B, C$, and $O$ denote the complex numbers $-1, 1$, and $0$ respectively. Let $T$ be the simple triod $[O, A] \cup [O, B] \cup [O, C]$ lying in the complex plane. Let $f : T \rightarrow T$ be the unique piecewise linear mapping such that $f(O) = \frac{B}{2}$, $f(A) = B$, $f(B) = C$, $f(\frac{B}{2}) = \frac{C}{2}$, $f(\frac{C}{2}) = 0$, $f(C) = C$, $f(\frac{3C}{8}) = O$, $f(\frac{C}{4}) = A$, and $f(\frac{C}{8}) = O$, and such that $f$ is linear on $[O, A]$, $[O, B/4]$, $[B/4, B/2]$, $[B/2, B]$, $[O, C/8]$, $[C/8, C/4]$, $[C/4, 3C/8]$, $[3C/8, C/2]$, and $[C/2, C]$. Figure 1 depicts this mapping by showing the domain simple triod embedded in a "thickened" simple triod, following the pattern given by $f$. The diagram might be thought of as a "graph" of the function $f$.

For each positive integer $n$, let $T_n = T$ and $f_n = f$. Let $X$ be the inverse limit of the inverse sequence $\{T_n, f_n\}$. An embedding of $X$ in the plane is indicated in the left half of Fig. 2.

Denote the unit interval $[0, 1]$ by $I$. Define $g : I \rightarrow I$ to be the mapping such that $g(0) = 0$, $g(1/8) = 1/2$, $g(3/8) = 1/2$, $g(1/2) = 1$, $g(1) = 0$, and such that $g$ is linear on each of $[0, 1/8]$, $[1/8, 3/8]$, $[3/8, 1/2]$, $[1/2, 1]$. The graph of $g$ is shown in Fig. 3. For each positive integer $n$, let $Y_n = I$ and $g_n = g$. Let $Y$ be the inverse limit of the inverse sequence $\{Y_n, g_n\}$. The continuum $Y$ is homeomorphic to the Brouwer-Janiszewski–Knaster Continuum [6, p. 204], shown in the right half of Fig. 2.

Define $h : T \rightarrow I$ as the mapping such that $h((C, C/2)) = 0$, $h((O, A)) = 1/2$, $h(B/2, B) = 1$, and such that $h$ is linear on $[C/2, O]$, and $[O, B/2]$. The graph of $h$ restricted to $[C, O] \cup [O, B]$ is also shown in Fig. 3.

**Lemma 1.** The mapping $h$ maps $T$ onto $I$, and $hf = gh$. Thus, $\mu : X \rightarrow Y$ defined by $\mu(x_1, x_2, ...) = (h(x_1), h(x_2), ...)$ is a mapping of $X$ onto $Y$.

**Proof:** Since each of $f$, $g$, and $h$ is piecewise linear, it suffices to check that $hf(x) = gh(x)$ at each of the points $x$ in $\{A, C, C/2, 3C/8, C/4, C/8, B, B/2, B/4, O\}$. We leave this to the reader.
THEOREM 1. The mapping $\mu$ is monotone, and the only nondegenerate point inverse under $\mu$ is the arc

$$\mu^{-1}(0, 0, 0, \ldots) = \{(t, t, t, \ldots) : t \in [C/2, C]\}.$$ 

Proof. That $\mu^{-1}(0, 0, 0, \ldots)$ is the set indicated above follows from the facts that $h^{-1}(0) = [C/2, C]$ and $f$ restricted to $[C/2, C]$ is the identity. This set is clearly homeomorphic to an arc and is thus connected.

Suppose that $y = (y_1, y_2, y_3, \ldots)$ is a point of $Y$ and that $\mu^{-1}(y)$ is nondegenerate. Let $w = (w_1, w_2, w_3, \ldots)$ and $x = (x_1, x_2, x_3, \ldots)$ be points of $\mu^{-1}(y)$ such that $w \neq x$. Then there is a positive integer $n$ such that $w_n \neq x_n$.

Suppose that $l > n$. Then $w_{l+n} = x_{l+n}$. Recall, from the definition of $\mu$, that $h(w_{l+n}) = h(x_{l+n}) = y_{l+n}$. Since $h$ is a homeomorphism on $[B, 2] \cup (O, C]$, both $w_{l+n}$ and $x_{l+n}$ belong to $B \cup [B, 2] \cup (C/2, C] \cup [O, A]$. Now $f^l(O, A) = f^l([O, A]) = f^l([B, 2], B) = f^l([C/2, C]) = [C/2, C]$, hence $w_l$ and $x_l$ are both in $[C/2, C]$, regardless of which of the intervals $w_{l+n}$ and $x_{l+n}$ are in. If $l < n$, $w_l = x_l$ and $w_l = x_l$ since $f$ restricted to $[C/2, C]$ is the identity. Thus $w_0$ and $x_0$ are in $[C/2, C]$, and $w_l = w_0$ and $x_l = x_0$ for all $l$. Therefore $w$ and $x$ both belong to $\mu^{-1}(0, 0, 0, \ldots)$.

3. Atridiocy of $X$. A continuum $M$ is a triod provided there is a subcontinuum $K$ of $M$ such that $M \setminus K$ has at least three components. A continuum is atriodic if it contains no triod.

THEOREM 2. Every proper subcontinuum of $X$ is an arc, and thus $X$ is atriodic.

Proof. Suppose that $H$ is a proper subcontinuum of $X$. There exists a positive integer $n$ such that if $i > n$ then $O$ is not in $\pi_i(H)$. If not, for infinitely many integers $i$, $O$ belongs to $\pi_i(H)$. Then every projection of $H$ contains $C/2$ since $f(O) = B/2$ and $f(B/2) = C/2$ which is a fixed point for $f$. Therefore, for infinitely many integers $i$, $(O, C/2) \subseteq \pi_i(H)$. However, $f^i((O, C/2)) = T$, which implies that every projection of $H$ is all of $T$. This contradicts the assumption that $H$ is a proper subcontinuum.

Note that there is a positive integer $k$ such that $A$ is not in $\pi_k(H)$ if $i > k$. If not there are infinitely many projections of $H$ containing $A$, but $f(\pi_k(A)) = C$ and $C$ is fixed by $f$ so all projections of $H$ contain $C$. Thus infinitely many of the projections of $H$ contain both $A$ and $C$ infinitely many projections contain $O$, which we have just shown to be impossible.

Since $f^{-1}(B/2) = \{O\}$, for $i > n - 1$, $B/2$ is not in $\pi_i(H)$. Thus for $i > n + k$, $f(\pi_i(H))$ is a homeomorphism of an arc, so $H$ is an arc.

Thus every proper subcontinuum of $X$ is an arc, and by [3, Theorem 3], $X$ is atriodic.

4. The continuum $X$ has positive span. In the definition and theorems that follow we employ the following notation, similar to that used in [4] and [5]. The symbols $\langle t, u \rangle$ and $\langle t, u \rangle'$ will be used only to denote subcontinua of $X \times T$ such that the first projection is the subarc of $T$ and the second projection is the subarc of $T$. If $p$ and $q$ are points of $T$ the unique arc in $T$ which is irreducible from $p$ to $q$ will be denoted by $pq$ unless $O$ is a point of the arc and is not an endpoint of it, in which case it will be denoted by $pq$. 

DEFINITION. A subcontinuum of $X \times T$ is said to have property $L$ provided that

(a) it is the union of twenty subcontinua

$$\langle O^B, O^C, O^B \rangle, \langle O^C, O^B \rangle, \langle O^C, AO^C \rangle, \langle AO^C, O^B \rangle, \langle AO^B, O^C \rangle, \langle AO^C, AO^B \rangle,$$

$$\langle AO^B, BO^C \rangle, \langle OB, O^C \rangle, \langle OC, OB \rangle, \langle OB, BO^C \rangle, \langle OB, \frac{3B}{8} \rangle, \langle \frac{3B}{8}, OB \rangle,$$

$$\langle OB, C^C \rangle, \langle C^C, OB \rangle, \langle OC, C^C \rangle, \langle C^C, C^C \rangle, \langle AO^C, AO^C \rangle, \langle AO^C, \frac{3B}{8} \rangle,$$

$$\langle AO^C, AO^C \rangle, \langle AO^C, \frac{3B}{8} \rangle, \langle AO^C, AO^C \rangle, \langle AO^C, AO^C \rangle,$$

$$\langle AO^B, OB \rangle, \langle OB, AO^B \rangle;$$

(b) $\langle t, u \rangle^{-1} = \langle u, t \rangle$ for each $\langle t, u \rangle$ in the list above;

(c) there exist eight points $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ and $x_9$ such that

(c.1) $x_1$ is in $\frac{B}{2}$ and $(x_1, O)$ is in $\langle O^B, O^C \rangle$ and $\langle O^B, AO^C \rangle$,

(c.2) $x_2$ is in $AO^C$, $\frac{3B}{8}$ and $(x_2, O)$ is in $\langle AO^C, O^B \rangle$ and $\langle AO^C, \frac{3B}{8} \rangle$,

(c.3) $x_3$ is in $\frac{3B}{8}$ and $(x_3, O)$ is in $\langle OB, O^B \rangle$ and $\langle \frac{3B}{8}, OB \rangle$,

(c.4) $x_4$ is in $\frac{C}{4}$ and $(x_4, O)$ is in $\langle C^C, O^B \rangle$ and $\langle C^C, C^C \rangle$ and $\langle C^C, AO^C \rangle$,

(c.5) $x_5$ is in $\frac{B}{2}$ and $(x_5, O)$ is in $\langle O^B, O^C \rangle$ and $\langle O^B, C^C \rangle$,

(c.6) $x_6$ is in $\frac{B}{2}$ and $(x_6, O)$ is in $\langle AO^C, O^B \rangle$ and $\langle AO^C, C^C \rangle$, and

(c.7) $x_7$ is in $\frac{3B}{8}$ and $(x_7, O)$ is in $\langle AO^C, AO^C \rangle$ and $\langle AO^C, AO^C \rangle$,

(c.8) $x_8$ is in $\frac{B}{2}$ and $(x_8, O)$ is in $\langle AO^C, AO^C \rangle$ and $\langle OB, AO^B \rangle$;

(c.9) $x_9$ is in $\frac{B}{2}$ and $(x_9, O)$ is in $\langle AO^C, AO^C \rangle$ and $\langle OB, AO^B \rangle$.

1 Fundamenta Mathematicae 31. 1
Lemma 2. If $Z$ is a subcontinuum of $T \times T$ with property $L$ then there is a subcontinuum $Z'$ of $T \times T$ with property $L$ such that $f \times f(Z') = Z$.

Proof. If $(t, u)$ is a subcontinuum of $Z$ and $v$ and $w$ are arcs in $T$ such that $f|v$ and $f|w$ is a homeomorphism throwing $v$ onto $t$ and $f|w$ is a homeomorphism throwing $w$ onto $t$ then $L((t, u), v, w)$ denotes the continuum $(f|v)^{-1} \times (f|w)^{-1}((t, u))$, called the lift of $(t, u)$ with respect to $v$ and $w$, and having the property that its first and second projections are $v$ and $w$ respectively.

The construction of the twenty continua whose union is $Z'$ may be read from the following table. The first column contains the name, $(t, u)',$ of the subcontinuum of $Z'$ under construction: $(t, u)'$ is the union of the subcontinua $L_i$ listed as corresponding to it in the second column. Each $L_i$ is the lift of $(t, u)$ with respect to $v$ and $w$. The subcontinuum of $Z$ shown in the third column, $r$ and $s$ shown in the fourth and fifth columns. The construction is identical in nature to that of [4].

If $(t, u)'$ is one of the ten liftings constructed below let $(t', u')'$ denote $(t, u)'^{-1}$ and let $Z'$ be the union of these twenty sets.

To see that $Z'$ is connected, note that $\left((f|O_2)^{-1}(x_2), \frac{3C}{8}\right)$ is common to $L_1$ and $L_2$, $\left(f|O_2, \frac{B}{2}\right)$ is in $L_5$ and $L_6$, and $\left(f|O_2, \frac{B}{2}\right)$ is in $L_3$ and $L_4$. Then note that $\left(f|O_2, \frac{B}{2}\right)$ is in $L_6$ and $L_7$ (since $x_2$ is in $\left(f|O_2, \frac{B}{2}\right)$). The only observation necessary to see that the third of the ten sets constructed below is connected is to note that $L_1$ is a subset of $L_2$ and this also shows that $\left(f|O_2, \frac{B}{2}\right)$ is a subset of $\left(f|O_2, \frac{B}{2}\right)$. Further note that $L_4$ and $L_5$ both contain the point $\left(f|O_2, \frac{B}{2}\right)$, the point $\left(f|O_2, \frac{B}{2}\right)$ belongs to $L_5$ and $L_10$. The point $\left(f|O_2, \frac{B}{2}\right)$ belongs to $L_1$ and $L_11$ while $L_11$ and $L_12$ both contain $\left(f|O_2, \frac{B}{2}\right)$. $L_1$ and $L_14$ contain the point $\left(f|O_2, \frac{B}{2}\right)$ while $\left(f|O_2, \frac{B}{2}\right)$.

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<td>L_1</td>
<td>$\left(O_2, \frac{B}{2}\right)$</td>
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<td>$\frac{3C}{8}$</td>
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<td>$\frac{C}{4}$</td>
</tr>
<tr>
<td>L_5</td>
<td>$\left(O_2, \frac{B}{2}\right)$</td>
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<td>$\frac{C}{8}$</td>
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<tr>
<td>L_6</td>
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<tr>
<td>L_7</td>
<td>$\left(O_2, \frac{B}{2}\right)$</td>
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<td>$\frac{C}{8}$</td>
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<tr>
<td>L_8</td>
<td>$\left(O_2, \frac{B}{2}\right)$</td>
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<td>$\frac{C}{8}$</td>
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<tr>
<td>L_9</td>
<td>$\left(O_2, \frac{B}{2}\right)$</td>
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<td>$\frac{C}{8}$</td>
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<td>L_10</td>
<td>$\left(O_2, \frac{B}{2}\right)$</td>
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<td>$\frac{C}{8}$</td>
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The construction is identical in nature to that of [4].
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<td>$\langle \frac{3B}{8}, o^B, o^B \rangle$</td>
<td>$L_9$</td>
<td>$\left\langle \frac{B}{4}, C, O^C \right\rangle_2$</td>
<td>$\frac{3B}{8}, B \overline{C} \frac{B}{4}$</td>
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<tr>
<td>$L_{10}$</td>
<td>$\left\langle \frac{C}{4}, O^C \right\rangle_2$</td>
<td>$\frac{3B}{8}, B \overline{C} \frac{B}{4}$</td>
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| $\langle o^B, o^B \rangle_2$ | $L_2$ | $\left\langle \frac{B}{4}, O^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |
| $L_4$ | $\left\langle \frac{C}{4}, o^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |

| $\langle o^C, o^C \rangle$ | $L_11$ | $\left\langle \frac{B}{4}, C, O^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |
| $L_{12}$ | $\left\langle \frac{C}{4}, O^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |

| $\langle o^B, o^B \rangle^5$ | $L_{13}$ | $\left\langle \frac{B}{4}, O^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |
| $L_{14}$ | $\left\langle \frac{C}{4}, o^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |

| $\langle o^C, o^C \rangle$ | $L_{15}$ | $\left\langle \frac{B}{4}, C, O^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |
| $L_{16}$ | $\left\langle \frac{C}{4}, o^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |

| $\langle o^B, o^B \rangle^5$ | $L_{17}$ | $\left\langle \frac{B}{4}, O^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |
| $L_{18}$ | $\left\langle \frac{C}{4}, o^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |

| $\langle o^C, o^C \rangle$ | $L_{19}$ | $\left\langle \frac{B}{4}, O^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |
| $L_{20}$ | $\left\langle \frac{C}{4}, o^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |

| $\langle o^B, o^B \rangle^5$ | $L_{21}$ | $\left\langle \frac{B}{4}, O^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |
| $L_{22}$ | $\left\langle \frac{C}{4}, o^C \right\rangle_2$ | $\frac{B}{4}, C \overline{C} \frac{B}{4}$ |

is in $L_{14}$ and $L_{15}$, $\left(\frac{f}{C} \frac{13C}{4}, \frac{C}{2}\right)$ is in $L_{15}$ and $L_{16}$, and $\left(\frac{3C}{8}, \frac{f}{C} \frac{7C}{4}\right)$ is in $L_{14}$ and $L_{15}$. Not that $L_{14}$ and $L_{15}$ both contain the point $\left(\frac{f}{C} \frac{13C}{4}, \frac{C}{2}\right)$ while $\left(\frac{f}{C} \frac{13C}{4}, \frac{C}{2}\right)$ is in $L_{15}$ and $L_{16}$. Note that $L_{13}$ and $L_{20}$ both contain the point $\left(\frac{f}{C} \frac{13C}{4}, \frac{C}{2}\right)$. Finally, $L_{14}$ and $L_{21}$ both contain the point $\left(\frac{f}{C} \frac{13C}{4}, \frac{C}{2}\right)$ while $\left(\frac{f}{C} \frac{13C}{4}, \frac{C}{2}\right)$ is in $L_{21}$ and $L_{22}$.

Because each of these liftings constructed projects onto its corresponding arcs there exist points $X_1, X_2, X_3, X_4, X_5, X_6$ and $X_7$ such that $\langle X_1, B \rangle$ is in $\langle X_2, C \rangle$, $\langle X_2, B \rangle$ is in $\langle X_3, B \rangle$, $\langle X_3, B \rangle$ is in $\langle X_4, B \rangle$, $\langle X_4, B \rangle$ is in $\langle X_5, B \rangle$, $\langle X_5, B \rangle$ is in $\langle X_6, B \rangle$, $\langle X_6, B \rangle$ is in $\langle X_7, B \rangle$, and $\langle X_7, B \rangle$ is in $\langle X_8, B \rangle$. 

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</tr>
<tr>
<td>$L_{13}$</td>
<td>$\left\langle \frac{C}{4}, O^C \right\rangle_2$</td>
<td>$\frac{3B}{8}, B \overline{C} \frac{B}{4}$</td>
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</tbody>
</table>

| $\langle o^B, o^B \rangle_2$ | $L_{10}$ | $\left\langle \frac{B}{4}, C, O^C \right\rangle_2$ | $\frac{3B}{8}, B \overline{C} \frac{B}{4}$ |
| $L_{12}$ | $\left\langle \frac{C}{4}, O^C \right\rangle_2$ | $\frac{3B}{8}, B \overline{C} \frac{B}{4}$ |

| $\langle o^C, o^C \rangle$ | $L_{11}$ | $\left\langle \frac{B}{4}, C, O^C \right\rangle_2$ | $\frac{3B}{8}, B \overline{C} \frac{B}{4}$ |
| $L_{14}$ | $\left\langle \frac{C}{4}, O^C \right\rangle_2$ | $\frac{3B}{8}, B \overline{C} \frac{B}{4}$ |

| $\langle o^B, o^B \rangle^5$ | $L_{15}$ | $\left\langle \frac{B}{4}, C, O^C \right\rangle_2$ | $\frac{3B}{8}, B \overline{C} \frac{B}{4}$ |
| $L_{16}$ | $\left\langle \frac{C}{4}, O^C \right\rangle_2$ | $\frac{3B}{8}, B \overline{C} \frac{B}{4}$ |

| $\langle o^C, o^C \rangle$ | $L_{17}$ | $\left\langle \frac{B}{4}, C, O^C \right\rangle_2$ | $\frac{3B}{8}, B \overline{C} \frac{B}{4}$ |
| $L_{18}$ | $\left\langle \frac{C}{4}, O^C \right\rangle_2$ | $\frac{3B}{8}, B \overline{C} \frac{B}{4}$ |

| $\langle o^B, o^B \rangle^5$ | $L_{19}$ | $\left\langle \frac{B}{4}, C, O^C \right\rangle_2$ | $\frac{3B}{8}, B \overline{C} \frac{B}{4}$ |
| $L_{20}$ | $\left\langle \frac{C}{4}, O^C \right\rangle_2$ | $\frac{3B}{8}, B \overline{C} \frac{B}{4}$ |

| $\langle o^C, o^C \rangle$ | $L_{21}$ | $\left\langle \frac{B}{4}, C, O^C \right\rangle_2$ | $\frac{3B}{8}, B \overline{C} \frac{B}{4}$ |
| $L_{22}$ | $\left\langle \frac{C}{4}, O^C \right\rangle_2$ | $\frac{3B}{8}, B \overline{C} \frac{B}{4}$ |
Let $x_1 = \left( f \left( \frac{3B}{2} \right) \right)^{-1} (y_1)$, $x_2 = \left( f \left( \frac{AO C}{2} \right) \right)^{-1} (y_2)$, $x_3 = \left( f \left( \frac{3B}{8} \right) \right)^{-1} (y_3)$, $x_4 = \left( f \left( \frac{C}{4} \right) \right)^{-1} (y_4)$, $z_1 = \left( f \left( \frac{BO}{2} \right) \right)^{-1} (y_5)$, $z_2 = \left( f \left( AO C \right) \right)^{-1} (y_6)$, $z_3 = \left( f \left( \frac{3C}{8} \right) \right)^{-1} (y_7)$, and $z_4 = \left( f \left( \frac{BO}{2} \right) \right)^{-1} (y_8)$. So far, we have shown that each of the ten liftings listed above is connected. Observe that for the collection $C_1 = \left\{ \left( \frac{B}{2}, \frac{C}{2} \right), \left( AO \frac{C}{2} \right), \left( OB, AO \frac{B}{2} \right), \left( \frac{B}{2}, \frac{C}{4} \right) \right\}$, the first member listed is a subset of the second and third members and intersects the fourth. Thus $C_1$ is connected. Let $C_2 = \left\{ \left( \frac{3B}{8}, \frac{B}{2} \right), \left( OB, AO \frac{B}{2} \right) \right\}$ and note that $C_2$ is connected. Let $C_3 = \left\{ \frac{2C}{4}, C, \frac{C}{8}, C \right\}$ and since the first member listed intersects each of the other two, $C_3$ is a continuum. Observe that $(x_1, O)$ is in $\left( \frac{B}{2}, \frac{C}{2} \right)$ and $\left( \frac{B}{2}, AO \frac{C}{2} \right)$, while $(x_4, A)$ is in $\left( \frac{B}{2}, AO \frac{C}{2} \right)$ and $\left( OB, AO \frac{B}{2} \right)$ and $(O, x_4)$ is in $\left( \frac{B}{2}, AO \frac{C}{2} \right)$ and $\left\{ \frac{C}{2}, C \right\}$. These observations imply $C_3 \cup C_2 \cup C_1 \cup \left( \frac{B}{2}, AO \frac{C}{2} \right)$ is a continuum, $H$. Since the point $(x_2, O)$ is in $\left( AO \frac{C}{2}, \frac{B}{2} \right)$ and $\left( AO \frac{C}{2}, \frac{C}{8} \right)$, $Z' = H \cup H^{-1}$ is a continuum.

**Lemma 3.** Suppose that for each integer $n > 1$, $f^n$ is the $(n-1)$-fold composite of $f$ with itself. Then if $n$ is an integer, $o(f^n) > 1/4$.

**Proof.** Let

\[
\begin{align*}
\left( \frac{B}{2}, \frac{C}{2} \right) &= \left( \frac{B}{2}, \left\{ \frac{C}{2} \right\} \right) \cup \left( \frac{B}{2}, \frac{C}{8} \right), \\
\left( \frac{B}{2}, AO \frac{C}{2} \right) &= \left( \frac{B}{2}, \left\{ \frac{A}{2} \right\} \right) \cup \left( \frac{B}{2}, AO \frac{C}{8} \right), \\
\left( \frac{C}{2}, AO \frac{B}{2} \right) &= \left( \left\{ \frac{C}{2} \right\}, AO \frac{B}{2} \right) \cup \left( \frac{C}{2}, \left\{ \frac{B}{2} \right\} \right), \\
\left( \frac{C}{2}, OB \right) &= \left( \frac{C}{2}, \left\{ \frac{B}{2} \right\} \right) \cup \left( \frac{C}{2}, OB \right), \\
\left( \frac{B}{2}, \frac{3B}{8} \right) &= \left( \frac{B}{2}, \left\{ \frac{B}{2} \right\} \right) \cup \left( \frac{3B}{8} \right).
\end{align*}
\]
Solution to a compactification problem of Skyarenko

by
Takashi Kimura (Ibaraki)

Dedicated to Professor Yukihiro Kodama
On his 60th birthday

Abstract. Concerning a function SkI originally introduced by Skyarenko to study compactness deficiency defX, we establish a theorem that SkI X = defX for every separable metrizable space X. This answers a problem of Skyarenko affirmatively.

1. Introduction. All spaces considered in this paper are assumed to be separable and metrizable. By a compactification of a space X, we mean a compact metrizable space containing X as a dense subspace. For undefined notion see [3] and [5].

The compactness deficiency defX of a space X is the least integer n for which X has a compactification aX with dim(aX\–X) = n.

J. de Groot [4] proved that a space X has a compactification aX with dim(aX\–X) ≤ 0 if and only if X is rim-compact. Motivated by this result, to study further defX he introduced the small (resp. large) inductive compactness degree cmpX (resp. CmpX) of a space X. In general, the inequality cmpX ≤ defX is known to hold [5]. The well-known conjecture of de Groot that cmpX = defX has been negatively solved by R. Pol [9]; the space X of Pol’s example has cmpX = 1 and CmpX = defX = 2. It is unknown whether there is a space X with cmpX < defX. 1

Another condition to study defX is due to E. Skyarenko [10], [11], which is denoted by SkI X ≤ n in Isbell’s book [6]; a space X has SkI X ≤ n if X has a base Σ such that Bd Bn \ Bd B_{n+1} \ ... \ Bd B_1 is compact for any n + 1 distinct members of Σ. Skyarenko proved that SkI X ≤ defX [10] and asked whether SkI X = defX for every space X [11]. Recently, J. M. Aarts, J. Bruijning and J. van Mill [2] proved that CmpX ≤ SkI X. In this paper we give an affirmative answer to Skyarenko’s problem above. Namely, we shall establish a theorem that SkI X = defX for every space X. As an application it will be shown that a non-compact space X has a compactification aX with dim(aX\–X) = n if and only if SkI X ≤ n.

2. Preliminaries and lemmas. Let Σ be a collection of subsets of a space X. We shall write [Σ]n for {Σ′: Σ′ is a subcollection of Σ with |Σ′| = n}, ∩ Σ

1 Added in proof. Recently the author has constructed such a space.