The smallest number of free prime closed filters

by

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Abstract. We show that every non-compact Hausdorff space has at least $\aleph_1$ free prime closed filters, and every non-compact completely regular, Hausdorff space has at least $\aleph_2$ free prime closed filters.

§ 1. Introduction. A filter $\mathscr{F}$ on a topological space X is called a *closed filter* provided that each F in $\mathscr{F}$ is a closed set. A filter $\mathscr{F}$ is called a *prime closed filter* provided $\mathscr{F}$ is a closed filter and for every F in $\mathscr{F}$, if $F = H \cup K$, with both H and K closed sets, then $H \in \mathscr{F}$ or $K \in \mathscr{F}$. A filter $\mathscr{F}$ is called *free* provided it has no adherent points in X (i.e. no point of X is in the closure of every member of $\mathscr{F}$). The natural way to construct a prime closed filter is to start with an ultrafilter u on X and define

$$\mathscr{F}(u) = \{F \subset X\colon F \text{ is a closed set and } F \in u\}.$$

Obviously. $\mathscr{F}(u)$ is a prime closed filter on X. In fact, every prime closed filter on X can be constructed in this way:

1.1. LEMMA (Frolík [2]). *If $\mathscr{F}$ is a prime closed filter on X, then there exists an ultrafilter u on X such that $\mathscr{F} = \mathscr{F}(u)$.*

In this paper we are concerned with the following question:

1.2. Among all non-compact spaces, what is the smallest cardinal number that arises as the number of free prime closed filters on a space?

A countable discrete space obviously has 2^c free prime closed filter (where c denotes the cardinality of the continuum), and as far as we know, the number 2^c is the answer to 1.2 (in the class of T_2-spaces). Our main result shows that under GCH this is indeed the case (in the class of completely regular Hausdorff spaces):

1.3. THEOREM. *Every non-compact completely regular T_2-space has at least $\aleph_2$ free prime closed filters.*

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1.4. Corollary. *Under* GCH, *every non-compact, completely regular T_2-space has at least 2^c free prime closed filters.*

Whether or not GCH can be removed from 1.4 remains unsolved. For the class of Hausdorff spaces, we know even less:

1.5. Every non-compact T_2-space has at least $\aleph_1$ free prime closed filters.

We do not know the full extent of the effect of separation axions on 1.2, but we do have the following example:

1.6. There exists a T_1- (not T_2-) space which has exactly one free prime closed filter.

The proofs of these results are given in Section 2. We conclude this section with a few remarks.

In addition to prime closed filters, there are other kinds of prime filters for which one could ask a question analogous to 1.2. For instance, what is the smallest number of free prime open filters possible on a non-compact space? For T_3-spaces, this is the same question as 1.2 by virtue of the following remark:

1.7. Remark. If X is a T_3 space, then the number of free prime closed filters on X is equal to the number of free prime open filters.

Proof. Let A denote the set of all free prime closed filters on X, and let B denote the set of all free prime open filters on X. We will show that $|A| = |B|$ by the Schroeder–Bernstein theorem. For each free closed prime filter $\mathscr{F}$ on X, pick one ultrafilter u on X such that $\mathscr{F} = \mathscr{F}(u)$, and consider the map defined on A by $\mathscr{F}(u) \mapsto \mathscr{O}(u)$, where we define

$$\mathscr{O}(u) = \{O \subset X\colon\ O \text{ is an open set and } O \in u\}.$$

To show that the map goes into B, it suffices to show that $\mathscr{O}(u)$ is free in X. Let $x \in X$. Since $\mathscr{F}(u)$ is free, there exists a closed set $F \in \mathscr{F}(u)$ such that $x \in (X-F)$. Since X is T_3, there exist disjoint open sets U and V with $x \in U$ and $F \subset V$. Thus $V \in \mathscr{O}(u)$ and $U \cap \mathrm{cl}_X(V) = \emptyset$; so x is not an adherent point of $\mathscr{O}(u)$. To see that the map is one-one, let $\mathscr{F}_1$ and $\mathscr{F}_2$ be distinct members of A, and say that the ultrafilters u and v were the ones chosen so that $\mathscr{F}_1 = \mathscr{F}(u)$ and $\mathscr{F}_2 = \mathscr{F}(v)$. Let F be a closed set in $u-v$. Then $(X-F)$ is an open set in $v-u$; so $\mathscr{O}(u) \neq \mathscr{O}(v)$. Thus $|A| \leqslant |B|$. In a similar manner, we can define a one-one map from B into A (and no separation axioms are needed in this case). Thus $|A| = |B|$.

There are four other kinds of prime filters which we could consider: prime z-filters, maximal closed filters, maximal open filters, and maximal z-filters. The analogous version of 1.2 is, however, easily answered in these cases: The ordered space ω_1 has exactly one free maximal closed filter, and exactly one free prime z-filter (hence exactly one free maximal z-filter). To get a space having exactly one maximal open filter, let ω denote the discrete space of natural numbers, and $\beta(\omega)$ its Stone–Čech compactification. For the space, take $\beta(\omega)-\{u\}$, where

$$u \in \omega^* = \beta(\omega)-\omega.$$

If we consider all prime closed filters (both free and non-free) or if we consider only non-free prime closed filters, then again the analogous version of 1.2 is easily answered. Every infinite T_2-space has at least 2^c prime closed filters (of both kinds) by Lemma 2.1, and the countable discrete space has exactly $\aleph_0$ non-free prime closed filters (the minimum number possible in an infinite T_2-space).

Notation. We let α, β, γ, δ, η, and τ denote ordinal numbers, $\varkappa$ an infinite cardinal number, and $\beta(\varkappa)$ the Stone–Čech compactification of $\varkappa$, with the discrete topology on $\varkappa$. Recall that there are $2^{2^\varkappa}$ uniform ultrafilters on $\varkappa$. Some elementary facts about prime filters can be found in [4, 12E].

§ 2. Proofs.

2.1. Lemma. *If X has a discrete subspace Z such that Z has no complete accumulation point in X and $|Z| = \varkappa$, then X has at least $2^{2^\varkappa}$ free prime closed filters.*

Proof. It suffices to show that if u and v are distinct ultrafilters on Z, then $\mathscr{F}(u) \neq \mathscr{F}(v)$. Let $A \in u-v$. Then $\mathrm{cl}_X(A) \in \mathscr{F}(u)$, but $\mathrm{cl}_X(A) \notin \mathscr{F}(v)$ since $\mathrm{cl}_X(A) \cap Z = A \notin v$.

An infinite open cover $\mathscr{U}$ of a space X is called *inflexible* provided for every $\mathscr{V} \subset \mathscr{U}$ if $\mathscr{V}$ covers X, then $|\mathscr{V}| = |\mathscr{U}|$.

2.2. Lemma. *Let X be a T_3-space and $\mathscr{U}$ an inflexible open cover of X. If $|\mathscr{U}| = \varkappa$ is regular, then X has at least $\varkappa$ free prime closed filters.*

Proof. By 2.1, we may assume that X does not have a discrete subset of cardinality $\varkappa$. Let $\{U_\alpha\colon\ \alpha < \varkappa\}$ be an inflexible open cover of X. We may assume that there exist points

$$y_\alpha \in U_\alpha - \bigcup\{U_\beta\colon\ \beta < \alpha\}$$

for $\alpha < \varkappa$. Clearly, $Y = \{y_\alpha\colon\ \alpha < \varkappa\}$ is a right separated subset of X and has no complete accumulation point. By transfinite induction. we construct discrete sets $Z_\alpha \subset Y$ and an increasing sequence η_α of ordinals less than $\varkappa$ such that $\eta_0 = 0$ and for all $\alpha < \gamma$, Z_α is a dense subset of $\{y_\tau\colon\ \eta_\alpha \leqslant \tau < \varkappa\}$. Construct Z_γ and η_γ as follows: Put $\eta_\gamma = \sup\{\bigcup\{\eta_\alpha\colon \alpha < \gamma\}\}$. Since the Z_α are discrete, $|Z_\alpha| < \varkappa$, and since $\varkappa$ is regular, $\eta_\gamma < \varkappa$. Let Z_γ be a discrete, dense subset of the set $Y_\gamma = \{y_\tau\colon\ \eta_\gamma \leqslant \tau < \varkappa\}$ (this is possible since Y_γ is right separated). This completes the construction of the sets Z_α for $\alpha < \varkappa$. Since the closure in X of Z_α is not compact, and X is T_3, there exists an ultrafilter u_α on Z_α such that u_α is free on X. We claim that if $\alpha < \beta$, then $\mathscr{F}(u_\alpha) \neq \mathscr{F}(u_\beta)$. By construction of Y, there exists an open set $U \in \mathscr{U}$ such that $Z_\alpha \subset U$ and $Z_\beta \subset (X-U)$. Thus the closed set $X-U \in u_\beta - u_\alpha$; so $\mathscr{F}(u_\alpha) \neq \mathscr{F}(u_\beta)$. Thus there are at least $\varkappa$ free closed prime filters on X.

A point x in a space X is called a *$\varkappa$-point* provided there exists a family $\{U_\alpha\colon\ \alpha < \varkappa\}$ of pairwise disjoint non-empty open subsets of X such that

$$x \in \mathrm{cl}_X(U_\alpha) - U_\alpha \quad \text{for all } \alpha < \varkappa.$$

2.3. LEMMA. *If x is a $\varkappa$-point in X, then there exist $2^{2^\varkappa}$ prime closed filters converging to x in X.*

Proof. Let $\{U_\alpha: \alpha<\varkappa\}$ be a family of pairwise disjoint non-empty open sets such that x is on the boundary of each U_α and let u_α be an ultrafilter converging to x with $U_\alpha \in u_\alpha$ for all $\alpha<\varkappa$. For each uniform ultrafilter v on $\varkappa$, define

$$\mathscr{P}(v) = \{F\subset X: F \text{ is a closed set, and } \{\alpha<\varkappa: F\in u_\alpha\}\in v\}.$$

If $v_1 \neq v_2$, let $A\in v_1-v_2$, and put $F = \mathrm{cl}_X(\bigcup\{U_\alpha: \alpha\in A\})$. Clearly, we have $F\in\mathscr{P}(v_1)-\mathscr{P}(v_2)$.

2.4. LEMMA. *If is a non-compact, completely regular T_2-space, and for every x in X there are fewer than 2^c prime closed filters converging to x in X, then X has at least 2^c free prime closed filters.*

Proof. Let $\mathscr{U}$ be an open cover of X with no finite subcover, and having smallest possible cardinality. Then $\mathscr{U}$ is an inflexible open cover of regular cardinality. As in the proof of 2.1, by using $\mathscr{U}$ we may pick a right separated set Y which has no complete accumulation point in X. Let Z be a dense, discrete subset of Y. Since we are concerned with free filters, we may assume that $X = \mathrm{cl}_X(Z)$. Let $\varkappa = |Z|$, and let $f\colon \beta(\varkappa)\to\beta(X)$ be a continuous map such that $f|\varkappa$ is one-one onto Z. We have two cases.

Case 1. f is not finite-to-one. Then there exists $y\in\beta(X)$ such that $f^{-1}(y)$ is infinite; so $|f^{-1}(y)|\geqslant 2^c$. Every $u\in f^{-1}(y)$ can be considered as an ultrafilter on $\varkappa$; so every $f(u)$ can be considered as an ultrafilter on the set Z. If $u\in f^{-1}(y)$, by continuity $f(u)$ converges to y. Since Z is discrete, these distinct ultrafilters contain distinct prime closed filters (converging to y). Hence, by hypothesis, $y\in\beta(X)-X$. Thus there are at last 2^c free prime closed filters on X.

Case 2. f is finite-to-one. Thus $f|\omega^*$ is finite to one. Hence, $\beta(X)$ contains a homeomorfiphic copy T of ω^* [3]. We make use of the recent result of Balcar and Vojtáš [1] which says that every point in ω^* is a c-point of ω^*. If $T\cap X$ is dense in T, then every point x in $T\cap X$ is a c-point in $T\cap X$, but this is impossible in light of 2.3 and our hypothesis. Thus $T\cap X$ is not dense in T. This implies that $|T\cap(\beta X-X)|\geqslant 2^c$, so there are at least 2^c free prime (even maximal) closed filters on X. This completes the proof.

Proof of Theorem 1.3. If X is not countably compact, then X has at least 2^c free prime closed filters by 2.1; so we assume that X is countably compact. Let $\mathscr{U}$ be an inflexible open cover of X of regular, uncountable cardinality $\varkappa$. If $\varkappa\geqslant\aleph_2$ then by 2.2, we are done; so we assume that $\varkappa=\aleph_1$. As in the proof of 2.1 we construct a right separated set $Y=\{y_\alpha: \alpha<\varkappa\}$ which has no complete accumulation point, but this time, by using 2.3, we select each y_α so that there are at least 2^c prime closed filters in X converging to y_α. Let Z be a dense, discrete subset of Y. If $|Z|=\aleph_1$, we are done by 2.1; so we assume that Z is countable. Put $Z=\{z_n: n<\omega\}$. Since the closure of Z is not compact, there exists an ultrafilter u on Z which is free on X. For each $n<\omega$, let $\{\mathscr{P}(n,\alpha): \alpha<2^c\}$ be a family of distinct prime closed filters converging to z_n. For each $\alpha<2^c$, define

$$\mathscr{F}(\alpha) = \{F\subset X: F \text{ is closed, and } \{n<\omega: F\in\mathscr{P}(n,\alpha)\}\in u\}.$$

Clearly each $\mathscr{F}(\alpha)$ is a free prime closed filter on X. To see that they are distinct, let $\{W_n: n<\omega\}$ be a family of pairwise disjoint, open sets such that $W_n\cap Z=\{z_n\}$ for all $n<\omega$. If $\alpha\neq\beta$, then for all $n<\omega$, pick

$$F_n\in\big(\mathscr{P}(n,\alpha)-\mathscr{P}(n,\beta)\big)\cup\big(\mathscr{P}(n,\beta)-\mathscr{P}(n,\alpha)\big).$$

If $A=\{n<\omega: F_n\in\mathscr{P}(n,\alpha)\}\in u$, then $\mathrm{cl}_X(\bigcup\{F_n: n\in A\})\in\mathscr{F}(\alpha)-\mathscr{F}(\beta)$; so they are distinct.

We now proceed towards the proof of Theorem 1.5 with the following two lemmas.

2.5. LEMMA. *Every free maximal closed filter on a space X contains a free chain.*

Proof. Let $\mathscr{M}$ be a free maximal closed filter on X, and let $M_0\in\mathscr{M}$. Assume that we have constructed sets $M_\alpha\in\mathscr{M}$ for all $\alpha<\gamma$ such that if $\alpha<\beta<\gamma$, then M_β is a proper subset of M_α. Then $\mathscr{C}=\{M_\alpha: \alpha<\gamma\}$ is a chain contained in $\mathscr{M}$. If $\bigcap\mathscr{C}=\emptyset$, we are done; so we assume that $\bigcap\mathscr{C}\neq\emptyset$. If $\bigcap\mathscr{C}\notin\mathscr{M}$, then, by maximality, there exists $M\in\mathscr{M}$ such that $(\bigcap\mathscr{C})\cap M=\emptyset$. Hence $\{M\cap M_\alpha: \alpha<\gamma\}$ is a free chain contained in $\mathscr{M}$. If $\bigcap\mathscr{C}\in\mathscr{M}$, then since $\bigcap\mathscr{M}=\emptyset$, we can find an element of $\mathscr{M}$ which is a proper subset of $\mathscr{C}$ and continue the induction.

2.6. LEMMA. *If X is a non-compact T_2-space, then X has at least two free prime closed filters.*

Proof. Since there exists a closed subset of X which is not H-closed [4, 17L(3)], we may assume that X is not H-closed. Thus there exists a free maximal open filter on X such that $\mathcal{O}\subset u$. Then $\mathscr{F}(u)$ is a free prime closed filter on X. If $\mathscr{F}(u)$ is not a maximal closed filter, then it is contained in a maximal closed filter; so we have two free prime closed filters on X. We assume, therefore that $\mathscr{F}(u)$ is maximal. Now this implies that for every $F\in\mathscr{F}(u)$, F^0 (interior) is not empty (if $F^0=\emptyset$, then $X-F$ is open and dense, hence is in the maximal open filter $\mathcal{O}$, but this is impossible since F and $X-F$ are not both in u). Let $\mathscr{C}$ be a free chain contained in $\mathscr{F}(u)$. We may assume that $\mathscr{C}=\{C_\alpha: \alpha<\varkappa\}$, $\varkappa$ is regular, and if $\alpha<\beta<\varkappa$, then $C_\beta\subset C_\alpha$. By transfinite induction, construct points $x_\alpha\in X$ and ordinals $\tau_\alpha<\varkappa$ such that

(1) $\alpha\leqslant\tau_\alpha$ and $x_\alpha\in C_\alpha$,

(2) $x_\beta\notin C_{\tau_\alpha}$ for all $\beta<\alpha$, and

(3) $\alpha<\beta$ implies $\tau_\alpha<\tau_\beta$.

Then $Z=\{x_\alpha: \alpha<\varkappa\}$ is a discrete subset of X and has no complete accumulation

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point in X (else $\bigcap\{C_\alpha: \alpha<\varkappa\}\neq\emptyset$). Thus X has indeed at least two free prime closed filters (by 2.1).

Proof of Theorem 1.5. Suppose that X is a non-compact T_2-space which has only countably many free prime closed filters. Since X must be countably compact (by 2.1) every maximal closed filter on X is closed under the operation of taking countable intersections. Let $\{\mathcal{M}_i: i<\omega\}$ list all free maximal closed filters on X, and let $\{\mathcal{P}_n: n<\omega\}$ list all free prime, non-maximal, closed filters on X (if any). Since distinct maximal filters contain disjoint sets, there exists

$$M_0\in\mathcal{M}_0-\bigcup\{\mathcal{M}_i: 0<i<\omega\}.$$

For every $n<\omega$, pick $M_n\in\mathcal{M}_0-\mathcal{P}_n$, and put $F=\bigcap\{M_n: n<\omega\}$. Then $F\in\mathcal{M}_0$. By 2.6, there exist two free prime closed filters, say $\mathcal{A}$ and $\mathcal{B}$, on the non-compact subspace F. Define

$$\mathcal{A}^*=\{H\subset X: H \text{ is closed and } H\cap F\in\mathcal{A}\}, \text{ and}$$
$$\mathcal{B}^*=\{H\subset X: H \text{ is closed and } H\cap F\in\mathcal{B}\}.$$

Then $\mathcal{A}^*$ and $\mathcal{B}^*$ are distinct free prime closed filters on X, and are contained in maximal closed filters. Now M_0 is a member of both $\mathcal{A}^*$ and $\mathcal{B}^*$; so $\mathcal{A}^*$ and $\mathcal{B}^*\subset\mathcal{M}_0$. Say that $\mathcal{B}^*\neq\mathcal{M}_0$. Thus $\mathcal{B}^*=\mathcal{P}_n$ for some $n<\omega$, but this is impossible since $M_n\in\mathcal{B}^*$. This completes the proof.

Proof of 1.6. We need to construct a T_1-space which has a unique free prime closed filter. For the set, take $X=\omega_1$. For a base for the topology take

$$\{\alpha-F: \alpha<\omega_1 \text{ and } F \text{ is a finite subset of } \omega_1\}.$$

Note that if H is closed and infinite in this topology, then H contains a final segment of ω_1, and if α is the first ordinal such that $[\alpha,\omega_1)\subset H$, then $H=S\cup[\alpha,\omega_1)$, where S is either a finite set or an increasing sequence of ordinals converging to α in the order topology on ω_1. From this one can show that the set $\mathcal{P}$ of all closed, non-compact subsets of X is the unique prime closed filter on X.

In conclusion, we remark that our techniques show (in ZFC) that certain special classes of non-compact, completely regular T_2-spaces spaces have 2^c free prime closed filters. One such class is the class of all finally $\aleph_2$-compact spaces (i.e., spaces in which every open cover has a subcover of cardinality strictly less than $\aleph_2$). In such spaces, every inflexible open cover has cardinality $\leqslant\aleph_1$, so in the proof of Theorem 1.3, we do not have to call on 2.2, and therefore we get 2^c free prime closed filters. Another such class is the class of spaces having countable spread (i.e., every discrete subset is countable). In such spaces we start the proof with any inflexible open cover, and its corresponding right separated set Y. Then the discrete subset Z of Y is countable; so the last part of the proof of Theorem 1.3 shows that we get 2^c free prime closed filters in this case too.

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