[14] W. L. C. Sargent, *A descriptive definition of Cèsaro–Perron integrals*, Proc. London Math. Soc. (2), 47 (1941), 212–247.

[15] — *On generalized derivatives and Cèsaro–Denjoy integral*, Proc. London Math. Soc. (2), 52 (1951), 365–376.

[16] S. Verblunsky, *On the Peano derivatives*, Proc. London Math. Soc. (3), 22 (1971), 313–324.

[17] A. Zygmund, *Trigonometric series*, Cambridge 1968.

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Two examples concerning small intrinsic isometries

by

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Abstract. Borsuk, and Olędzki and Spież, have given examples in certain cases of arc length preserving embeddings (intrinsic isometries) whose images have arbitrarily small diameters. We give two additional examples, one negative and one positive, and raise several specific questions concerning further possibilities.

Introduction. Basic definitions and properties are in [1], [2], and [3]. If X is a metric space with metric ϱ, then the *intrinsic metric* on X induced by ϱ is given by $\varrho_X(x, y) =$ least upper bound {length (L): L is an arc in X containing x and y}. For ϱ_X to be defined it is necessary that each two points of X lie in some arc of finite length; for ϱ_X to induce the same topology as ϱ, it is necessary and sufficient that, for each $x \in X$ and each $\varepsilon > 0$, there is a neighborhood U of x in X, such that for each $y \in U$ there is an arc L of length $< \varepsilon$ in U containing both x and y. Spaces for which the two metrics are compatible are called *geometrically acceptable* (GA). All sets we consider will be GA. A mapping of X onto Y is an *intrinsic isometry* if it is a isometry with respect to the intrinsic metrics; or equivalently, if it preserves all arc lengths (Borsuk [2]). A mapping f of X into Y is an *intrinsic embedding* if $f\colon X \to f(X)$ is an intrinsic isometry; here the intrinsic metric on $f(X)$ is defined using arcs in $f(X)$.

We will say that X is *intrinsically small in* Y if, for each $\varepsilon > 0$, there is an intrinsic embedding $f\colon X \to Y$ such that $f(X)$ has diameter $< \varepsilon$ in the original metric of Y. The three previously known results concerning intrinsically small spaces are:

(1) E^n is intrinsically small in E^{n+1} (Olędzki and Spież [6]; Borsuk [1] earlier obtained E^{2n});

(2) Each 1-dimensional polytope in an E^n or in Hilbert space is intrinsically small in E^3 (Borsuk [1]);

(3) No subset of E^n containing an open set is intrinsically small in E^n (Borsuk [2]).

We will add two more examples: a certain compact 1-dimensional subset of E^2 is *not* intrinsically small in E^2; and bounded cylinders in E^3 are intrinsically small in E^3.

EXAMPLE 1. A certain plane Sierpiński curve.

Let D denote the unit disk $I\times I$, where $I=[0,1]$. Let $\{D_i\colon i=1,2,\ldots\}$ be a countably infinite collection of pairwise disjoint circular disks in $\operatorname{int}(D)$, with C_i the boundary of D_i and d_i its diameter, satisfying:

(i) $\bigcup\limits_{i=1}^{\infty} D_i$ is dense in D,

(ii) $\varrho(C_0, C_1) > 4/10$,

(iii) $\sum\limits_{i=1}^{\infty} \pi d_i < 1/10$. Then $X = D - \bigcup\limits_{i=1}^{\infty}\operatorname{int}(D_i)$ is the desired example. That X is GA follows from the fact, that, for each two points x and y of X,

$$\varrho(x,y)\leqslant \varrho_x(x,y)\leqslant (\pi/2)\varrho(x,y);$$

the latter inequality comes from replacing any part of the segment xy which passes through a D_i by the shorter arc in $\operatorname{bdy}(D_i)$ with the same endpoints. X is a Sierpiński curve. Note that any arc in X which intersects both C_0 and C_1 must have length $>4/10$.

THEOREM 1. *The space of Example 1 is a compact 1-dimensional planar* GA *set that is not intrinsically small in the plane.*

Proof. Suppose X were intrinsically small in the plane: let $f\colon X\to E^2$ be an intrinsic embedding with $\operatorname{diam}(f(X))<1/10$. If L is any line in E^2 which intersects the interiors of all complementary domains of $f(C_0)\cup f(C_1)$, then $\pi(f(C_0))\cap\pi(f(C_1))$ contains an interval of positive length, say δ, where π is the orthogonal projection parallel to L of E^2 onto some $E^1\subset E^2$. Choose n so large that $\sum\limits_{i=n+1}^{\infty}$ length $(\pi(fC_i)))<\delta$; and choose $p\in E^1$ so that $\pi^{-1}(p)$ intersects $f(C_0)$ and $f(C_1)$, and does not intersect $f(C_i)$ if $i>n$. Let A be a smallest arc in $\pi^{-1}(p)$ having one endpoint in $f(C_0)$ and the other in $f(C_1)$.

Length$(A)<1/10$, since its endpoints are in $f(X)$. If $A\subset f(X)$, then $f^{-1}(A)$ would be an arc in X from C_0 to C_1 of length $<1/10$ (since intrinsic isometries and their inverses preserve arc length [2]), which would be a contradiction. A will be modified to produce a similar contradiction.

Note that the set of boundary curves of the complimentary domains of X, $\{C_i\colon i=0,1,2,\ldots\}$, is exactly the set of simple closed curves in X which do not separate X. Since f is an embedding, $f(X)$ is a Sierpiński curve (Whyburn [7]); hence $\{f(C_i)\colon i=0,1,2,\ldots\}$, being the simple closed curves of $f(X)$ which do not separate $f(X)$, is also the set of boundary curves of the complimentary domains of $f(X)$. [Whether or not $f(C_0)$ is the "outer boundary" is immaterial.] Let

$$\mathscr{C}=\{f(C_i)\colon A\cap f(C_i)\neq\emptyset\},$$

let U be the component of $E^2-\bigcup\{f(C_i)\colon f(C_i)\in\mathscr{C}\}$ whose closure contains $f(X)$, and let $\mathscr{A}$ be the set of the closures of the components of $A\cap U$.

If $H\in\mathscr{A}$, then H is a non-degenerate subsegment of $A\cap f(X)$, and each of its endpoints lies in an element of $\mathscr{C}$. They must lie in different elements of $\mathscr{C}$; for if both were in the same $f(C_i)$, then H would be shorter than either subarc of $f(C_i)$ connecting them, and $f^{-1}(H)$ would be an arc in X containing two points of C_i and shorter than either subarc of C_i between them. But this is contrary to the construction of X. Hence each $H\in\mathscr{A}$ has its endpoints on different elements of $\mathscr{A}$ and has length $\geqslant$ the minimum distance between elements of $\mathscr{C}$; since $\mathscr{C}$ is finite, $\mathscr{A}$ is finite also.

Let $\varphi\colon [0,1]\to A$ be a parameterization with $\varphi(0)=A\cap f(C_0)$ and $\varphi(1)=A\cap f(C_1)$. Let α_j,β_j, $0\leqslant j\leqslant k$, be the points of $[0,1]$ such that $\{\varphi([\alpha_j,\beta_j])\colon 0\leqslant j\leqslant k\}=\mathscr{A}$, and $0=\alpha_0<\beta_0\leqslant\alpha_1<\ldots\leqslant\alpha_k<\beta_k=1$. Definie a path $\varphi'\colon [0,1]\to\mathscr{C}^*\cup\mathscr{A}^*$ [where '*' denotes 'union of all the elements of'] as follows:

Case 1. $\varphi'=\varphi$ on $[\alpha_j,\beta_j]$, $0\leqslant j\leqslant k$.

Case 2. If $\beta_j<\alpha_{j+1}$, then $\varphi([\beta_j,\alpha_{j+1}])$ intersects only one element of $\mathscr{C}$, say $f(C_i)$. Hence $\varphi(\beta_j)\in f(C_i)$ and $\varphi(\alpha_{j+1})\in f(C_i)$, and there are two arcs in $f(C_i)$ connecting them; define φ' on $[\beta_j,\alpha_{j+1}]$ to be a homeomorphism onto one of these arcs, having length $\leqslant(\pi/2)d_i$.

Thus $\varphi'([0,1])$ is a path in $[A\cap f(X)]\cup\mathscr{C}^*$ from $f(C_0)$ to $f(C_1)$, and it contains an arc B from $f(C_0)$ to $f(C_i)$. Length$(B)<2/10$, since B may be partitioned into a finite collection of subarcs, each of which is either a subsegment of A or a subarc of some $f(C_i)$; the sum of the lengths of the first type is $<1/10$ (since length(A) $<1/10$), and the sum of the lengths of the second type $<1/10$ also (since the sum of the circumferences of all the C_i's $<1/10$). Hence $f^{-1}(B)$ is an arc in X from C_0 to C_1 of length $\leqslant 2/10$, which is impossible. ∎

Similarly, in higher dimensions we have

THEOREM 2. *Each E^n, for $n\geqslant 2$, $n\neq 4$, contains a compact $(n-1)$-dimensional* GA *set that is not intrinsically small in E^n.*

Proof. The example and proof are almost identical to Example 1 and Theorem 1; some differences only will be noted. D will be the unit n-cell I^n, and the D_i's will be round n-cells. That X and $f(X)$ are $(n-1)$-dimensional Sierpiński curves, and that the complimentary domains of $f(X)$ form a null sequence, follow from Cannon's generalization [4] to higher dimensions of Whyburn's characterization [7] of the 2-dimensional Sierpiński curve. (This is also the source of the restriction that $n\neq 4$.) The δ is chosen so that $\pi(f(C_0))\cap\pi(f(C_1))$ contains an $(n-1)$-ball of $(n-1)$-volume $>\delta$. Each $\pi(f(C_i))$ lies in an $(n-1)$-ball of radius $\leqslant(\pi/2)d_i$, and hence has $(n-1)$-volume $\leqslant K_{n-1}d_i^{n-1}$, for some constant K_{n-1}. Since $\sum d_i$ converges, so does $\sum K_{n-1}d_i^{n-1}$; hence some tailend $\bigcup\limits_{i=n+1}^{\infty}\pi(f(C_i))$ will not cover $\pi(f(C_0))\cap\pi(f(C_1))$, and p may be chosen as before.

The rest of the proof follows as in Theorem 1. ∎

EXAMPLE 2. Cylinders in E^3.

THEOREM 3. *Each bounded cylinder over a simple closed curve of finite length in E^3 is intrinsically small in E^3.*

Proof. We may suppose the cylinder is of the form $C\times[0, r]$, where C is a circle in E^2. Let $\varepsilon>0$ be given. The desired intrinsic embedding will be constructed by a sequence of "paper foldings"; functional notation will not be used.

Step 1. Subdivide C into an even number $2n$ of equal subarcs, each of length $<\varepsilon/4$. Let P and Q denote two half-planes containing the z-axis and forming an angle of π/n. Choose two points, a and b, in the xy-plane such that

(i) $a\in P$ and $b\in Q$,

(ii) $d(a, b)=$ the length of each subarc of C,

(iii) b is closer to the z-axis than a is,

(iv) a is closer than $\varepsilon/4$ to the z-axis.

Reflecting the segment ab about the half-planes P and Q in a "kaleidoscope" fashion produces a star-shaped simple closed curve intrinsically isometric to C. Let a' and b' be the points r units directly above a and b respectively; kaleidoscoping the rectangle $abb'a'$ produces a cylinder intrinsically isometric to $C\times[0, r]$. It is contained in the cylinder of radius $\varepsilon/4$ about the z-axis.

Step 2. Rotate the rectangle $abb'a'$ about the edge ab through a small angle, moving $a'b'$ toward the z-axis; this moves a' outside of the angle PQ, and moves b' inside PQ. Pick a point c' between a' and b' close to P such that $c'b'$ misses P. Rotate the triangle $ac'a'$ about its edge ac', moving a' toward the z-axis, until a' lies on the plane P; the entire edge aa' will lie on P, and vertical projection on the figure $abb'c'a'$ is one-to-one.

Step 3. Slide the figure $abb'c'a'$ rigidly toward the z-axis until b' lies on the plane Q, keeping ab in the xy-plane and aa' in the plane P; b will be moved outside of the angle PQ. Pick a point c between a and b close to Q such that ac misses Q. Rotate the triangle cbb' about its edge cb', moving b away from the z-axis, until b lies on the plane Q; the entire edge bb' will lie on Q, and vertical projection on the figure $acbb'c'a'$ will be one-to-one. Kaleidoscoping the figure $acbb'c'a'$ produces a cylinder intrinsically isometric to the original cylinder on which vertical projection is one-to-one.

Step 4. Following [6], we reflect in two "mirrors": the xy-plane will reflect upward, and the plane $z=\varepsilon/2$ will reflect downward. The part of the cylinder with $0\leqslant z\leqslant\varepsilon/2$ remains unchanged; the part with $\varepsilon/2<z<2\varepsilon/2$ is reflected downward; the part with $2\varepsilon/2<z<3\varepsilon/2$ is reflected upward; etc. The resulting folded cylinder has diameter $<\varepsilon/2$ in the vertical direction and also in the horizontal direction; hence its diameter in E^3 is $<\varepsilon$. ■

Questions. Borsuk [3] states the underlying question: are there general embedding theorems for intrinsically small embeddings, analogous perhaps to the embedding

of n-dimensional spaces into E^{2n+1}? Since so little is known, some very elementary questions are of interest.

QUESTION 1. Is every planar Sierpiński curve GA? (Each has non-GA embeddings in E^3.)

QUESTION 2. Is Theorem 1 true for other planar GA Sierpiński curves — some, all, or the standard middle-thirds one?

QUESTION 3. The proof of Theorem 1 is a modification of a proof that a planar disk is not intrinsically small in the plane; Borsuk [2] has shown that each intrinsic isometry of a planar disk in the plane is in fact an isometry. Is this true for some or every planar GA Sierpiński curve?

QUESTION 4. 1-dimensional polyhedra in E^3 are intrinsically small in E^3 (Borsuk [1]); what about universal curves in E^3 (spaces homeomorphic to the space M_1^3 of [5, p. 122]) — are some or all of them intrinsically small in E^3? (Something like M_1^3 would be a possible generalization of Example 1.)

QUESTION 5. The embedding of Theorem 3 is a paper folding — a piecewise isometry on a triangulation of the cylinder. Are there small paper foldings of some other elementary polyhedra in E^3 — a 2-sphere, torus, or a "book with three pages" (three disks with an edge in common)?

QUESTION 6. Are there small intrinsic embeddings of any of these three, paper foldings or not?

References

[1] K. Borsuk, *On intrinsic isometries*, Bull. Acad. Polon. Sci. 29 (1981), 83–90.
[2] — *On homeomorphisms preserving lengths of arcs*, Glas. Mat. 16 (1981), 307–311.
[3] — *Some remarks on intrinsic geometry*, in *General Topology and Modern Analysis* L. F. McAuley and M. M. Rao, eds., Academic Press, New York 1981.
[4] J. W. Cannon, *A positional characterization of the (n-1)-dimensional Sierpiński curve in S^n* ($n\neq 4$), Fund. Math. 79 (1973), 107–112.
[5] R. Engelking, Dimension Theory, North Holland, Amsterdam 1978.
[6] J. Olędzki and S. Spież, Remarks on intrinsic isometries, Fund. Math. 119 (1983), 241–247.
[7] G. T. Whyburn, Topological characterization of the Sierpiński curve, Fund. Math. 45 (1958), 320–324.

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