Let N be a cardinal number. We say that a Hausdorff space X is a *N-space* iff there exists a basis B of open sets of X such that

(1) $|B| \leqslant N$ and

(2) for every element $U \in B$ we have $|\mathrm{Fr}\, U| \leqslant N$.

PROBLEM 4. Is there a universal element in the family of all regular (resp. Hausdorff) N-spaces?

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On symmetric products

by

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Abstract. Two notions $X(n)$ and $\mathrm{SP}\, X_G^n$ of symmetric products of a Hausdorff compact space X are studied. The n-fold symmetric product $X(n)$ is a subspace of the hyperspace 2^X of subsets of X containing at most n points. For a group G of permutation of a set of n elements, the n-fold G-symmetric product $\mathrm{SP}_G^n\, X$ is the orbit space of the permutation action of G on the n-fold cartesian product X^n of X. It is proved that some shape properties are invariants under the operation of these products. An example shows that the fixed point property is not such an invariant (this is the negative answer to the Borsuk and Ulam problem [1]). Examples of the symmetric product of some one-dimensional continua are considered.

1. Introduction. In the paper, compact Hausdorff spaces are considered. For a space X, let 2^X denote the space of closed subsets of X with the Vietoris finite topology. For a metric space one can get the same topology by using the Hausdorff metric. The n-fold symmetric product $X(n)$ of the space X is the subspace of 2^X of subsets of X containing at most n points ([1]). The space $X(n)$ can be obtained ([6], [18]) as a quotient space of the cartesian product X^n with the following relation: two points $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in X^n$ are equivalent if the sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are equal. Denote the natural projection by $\pi_n\colon X^n \to X(n)$.

Let G be a group of permutations of a set of n elements. The n-fold G-symmetric product $\mathrm{SP}_G^n X$ ([17], [8], [5]) of a space X is the orbit space of the permutation action of G on the cartesian product X^n of X. Let $\pi_G^n\colon X^n \to \mathrm{SP}_G^n X$ denote the identification map. Thus $\pi_G^n(x_1, \ldots, x_n) = \pi_G^n(y_1, \ldots, y_n)$ iff for some $g \in G$ $y_i = x_{g(i)}$ for $i = 1, \ldots, n$. If G is the group of all permutations of a set of n elements then SP_G^n is denoted by SP^n. It is easy to see that $\mathrm{SP}^2 X = X(2)$ for any space X.

Suppose that π is one of the maps π_n or π_G^n. Let $f\colon X \to Y$ be a map. The map $\times f\colon X^n \to X^n$ defined by $f(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$ preserves fibers of the map π. Hence we can define the map $\pi(f)\colon \pi(X^n) \to \pi(Y^n)$ such that the diagram

$$\begin{array}{ccc} X^n & \xrightarrow{\ f\ } & Y^n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \pi(X^n) & \xrightarrow{\ \pi(f)\ } & \pi(Y^n) \end{array}$$

commutes. If f is continuous, then $\pi(f)$ is continuous ([7]). If maps $f, g\colon X \to Y$ are homotopic then $\pi(f), \pi(g)\colon \pi(X^n) \to \pi(Y^n)$ are homotopic too. If $f\colon X \to Y$ and $g\colon Y \to Z$ then $\pi(gf) = \pi(g) \circ \pi(f)$. If X is the inverse limit of an inverse system $\{X_\alpha, p_\alpha^\beta, L\}$ then $\pi(X^n) = \varprojlim \{\pi(X_\alpha^n), \pi(p_\alpha^\beta), L\}$ ([5], [10]).

Borsuk and Ulam asked ([1]) what topological properties are preserved under the operation of taking a space X to its n-fold symmetric product. Some answers to this questions are in [1], [5], [6], [7], [10]. In particular, Jaworowski proved ([7], [8], see also [5]) that

(1.1) *if a compactum X is* ANR, *then $X(n)$ and $SP_G^n X$ are* ANR (the uncompact case was proved by Nguyen To Nhu [15]).

The invariance under the operation of symmetric product of some shape properties was studied by Kodama, Spież and Watanabe ([10]). They proved that if X is either ASR or ANSR or movable or uniform movable then $X(n)$ is such too. They also showed that if $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ then $\mathrm{Sh}(X(n)) = \mathrm{Sh}(Y(n))$ and if $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ then $\mathrm{Sh}(X(n)) \geqslant \mathrm{Sh}(Y(n))$.

2. Shape properties of symmetric products. For notation of shape theory see [3], [13], [14]. Let $\underline{X} = \{X_\alpha, p_\alpha^\beta, L\}$ be an inverse system of $X_\alpha \in$ ANR with projections $p_\alpha^\beta\colon X_\beta \to X_\alpha$, $\alpha < \beta$, $\alpha, \beta \in L$, where L is a direct set. $\underline{X}$ is movable ([13]) provided every $\alpha \in L$ admits a $\beta \in L$, $\beta \geqslant \alpha$, such that each $\gamma \in L$, $\gamma \geqslant \alpha$, admits a map $r\colon X_\beta \to X_\gamma$ satisfying

$$p_\alpha^\gamma \circ r \simeq p_\alpha^\beta. \tag{2.1}$$

A compact space X is movable if X is an inverse limit of a movable inverse system.

(2.2) LEMMA ([5], [10]). *If $X = \varprojlim \underline{X}$ where $\underline{X} = \{X_\alpha, p_\alpha^\beta, L\}$ is an inverse system of compact spaces and G is a group of permutation of a set of n elements, then*

$$X(n) = \varprojlim \{X_\alpha(n), \pi_n(p_\alpha^\beta), L\},$$

$$PS_G^n X = \varprojlim \{PS_G^n X_\alpha, \pi_G^n(p_\alpha^\beta), L\}.$$

(2.3) THEOREM. *If a compact space X is movable, then $SP_G^n X$ is movable for every group G of permutations of a set of n elements.*

Proof. Let X be an inverse limit of a movable ANR-system $\underline{X} = \{X_\alpha, p_\alpha^\beta, L\}$. By Lemma (2.2), $SP_G^n X$ is an inverse limit of an inverse system $\underline{Y} = \{SP_G^n X_\alpha, \pi_G^n(p_\alpha^\beta), L\}$ which by (1.1) is an ANR-system. Let $\alpha \in L$. Since $\underline{X}$ is movable, there is $\beta \in L$, $\beta \geqslant \alpha$ such that for every $\gamma \in L$, $\gamma \geqslant \alpha$ there is a map $r\colon X_\beta \to X_\gamma$ satisfying (2.1). Hence a map $\pi_G^n(r)\colon SP_G^n X_\beta \to SP_G^n X_\gamma$ satisfies the condition

$$\pi_G^n(p_\alpha^\gamma) \circ \pi_G^n(r) \simeq \pi_G^n(p_\alpha^\beta).$$

Thus $\underline{Y}$ and so $SP_G^n X$ are movable.

By a similar argument as in [10], where it is done for the n-fold symmetric product $X(n)$, one can prove the following two theorems for any positive integer n and a group G of permutations of a set of n elements.

(2.4) THEOREM. *Let X and Y be compact. If $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ then*

$$\mathrm{Sh}(SP_G^n X) \leqslant \mathrm{Sh}(SP_G^n Y)$$

and if $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ then $\mathrm{Sh}(SP_G^n X) = \mathrm{Sh}(SP_G^n Y)$.

(2.5) THEOREM. *If X is an ASR (resp. and* ANSR *or uniform movable) then $SP_G^n X$ is* ASR *(resp.* ANSR *or uniform movable).*

A continuum X is said to be pointed 1-movable ([2], [11]) if for some $x \in X$ the pointed continuum (X, x) is an inverse limit of a pointed ANR-sequence $\{(X_n, x_n), p_n^m\}$ such that for every integer n there is $m \geqslant n$ such that for every pointed map $f\colon (S^1, s_0) \to (X_m, x_m)$, where S^1 is a circle, and every $m' \geqslant n$ there is a pointed map $r\colon (S^1, s_0) \to (X_{m'}, x_{m'})$ with $p_n^m \circ f \simeq p_n^{m'} \circ r$ rel. s_0.

(2.6) THEOREM. *If a continuum X is pointed 1-movable, then $X(n)$ and $SP_G^n X$ are pointed 1-movable.*

Proof. The pointed 1-movability is an invariant under the operation of taking a space X to its cartesian product ([16]) and also is an invariant of continuous mapping of continua [11]. So, if X is pointed 1-movable then X^n and then $\pi_n(X^n) = X(n)$ and $\pi_G^n(X^n) = SP_G^n X$ are pointed 1-movable.

The shape dimension $\mathrm{sd}(X)$ ([13]) of a compact space X is less or equal to k if X is an inverse limit of an inverse ANR-system $\{X_\alpha, p_\alpha^\beta, L\}$ with $\dim X_\alpha \leqslant k$.

(2.7) LEMMA ([6]). *If X is a separable metric space then*

$$\dim X(n) = \dim X^n.$$

Using similar arguments, one can proof that for the X:

(2.8) $\dim SP_G^n X = \dim X^n$.

(2.9) THEOREM. *If X is a compact space and $\mathrm{sd}(X) \leqslant k$ then*

$$\mathrm{sd}(X(n)) \leqslant k\cdot n \quad \text{and} \quad \mathrm{sd}(SP_G^n X) \leqslant k\cdot n.$$

Proof. Let $X = \varprojlim \{X_\alpha, p_\alpha^\beta, L\}$ where $X_\alpha \in$ ANR and $\dim X_\alpha \leqslant k$ for $\alpha \in L$. Since $\dim X_\alpha^n \leqslant k\cdot n$, by Lemmas (2.2), (2.7) and (2.8) $\mathrm{sd}(X(n)) \leqslant k\cdot n$ and $\mathrm{sd}(SP_G^n X) \leqslant k\cdot n$.

3. Examples of symmetric product of some one-dimensional continua. The 2-fold symmetric product of a circle S is a Mobius strip ([1]). We can imagine a circle S as the segment $[0, 1]$ with identified points 0 and 1 and the $S(2)$ as the triangle $\{(a, b);\ 0 \leqslant b \leqslant a \leqslant 1\}$ with identified vertices and points $(a, 0)$ and $(1, a)$ for $0 < a < 1$.

(3.1) The subset $S' = \{\{a, 0\}: a \in S\}$ of the symmetric product $S(2)$ is homeomorphic to S and is a deformation retract of $S(2)$.
The map $h: S \to S' \subset S(2)$ given by $h(a) = [(a, 0)]$ is an embedding. The deformation $H: S(2)\times[0, 1] \to S(2)$ can be defined by

$$H([(a, b)], t) = \begin{cases}[(a+tb, (1-t)b)] & \text{if } a+b \leqslant 1,\\ [(a+t-ta, b-t+ta)] & \text{if } a+b > 1.\end{cases}$$

Let $f: S \to S$ be a map of degree k such that $f(0) = 0$. Then $\pi_2(f)(S') \subset S'$ and $\deg \pi_2(f)|_{S'} = k$, so we have the following homotopies:

(3.2) $$h\circ f \simeq \pi_2(f)\circ h \quad \text{and} \quad h^{-1}\circ H_1\circ \pi_2(f) \simeq f\circ h^{-1}\circ H_1 .$$

(3.3) THEOREM. *If S is a solenoid then* $\mathrm{Sh}(S) = \mathrm{Sh}(S(2))$.

Proof. The S is an inverse limit of an inverse sequence of circles $\{S_m, p^n_m\}$. By Lemma (2.2), $S(2) = \varprojlim\{S_m(2), \pi_2(p^n_m)\}$. By condition (3.2), $\underline{h} = \{h_n\}$, where $h_n = h$ for $n = 1, 2, \ldots$, is a shape equivalence between S and $S(2)$.

It is easy to prove the following lemma.

(3.4) LEMMA. *Let a space X be a wedge $Y \vee Z$ of spaces Y and Z where $Y \cap Z = \{x_0\}$. The 2-fold product $X(2)$ is a union of the spaces $Y(2)$, $Z(2)$ and $Y\times Z$ lying in such a way that $Y(2) \cap Y\times Z = \{\{y, x_0\};\ y \in Y\}$, $Y(2) \cap Z(2) = \{x_0\}$ and $Z(2) \cap Y\times Z = \{\{x_0, z\};\ z \in Z\}$.*

Mardešić ([12]) proved that Case-Chamberlin curve C ([4]) is non-movable but its suspension has a trivial shape. The curve C is an inverse limit of an inverse sequence $\{C_n, p^n_m\}$ where, for every n, C_n is the wedge of two circles S and S' and $p^{n+1}_n = p: S \vee S' \to S \vee S'$ is such that for generators a and b of the fudamental group $\pi_1(S \vee S')$, $p_\#(a) = aba^{-1}b^{-1}$ and $p_\#(b) = a^2b^2a^{-2}b^{-2}$.

(3.5) THEOREM. *The 2-fold product $C(2)$ of the Case-Chamberlin curve C has a trivial shape.*

Proof. By Lemma (3.4) $(S \vee S')(2) = S(2) \cup S'(2) \cup S\times S'$ and by (3.1) $S\times S'$ is a deformation retract of $(S \vee S')(2)$. Let a' and b' be generators of the group $\pi_1(S\times S')$. The map $\pi_2(p): (S \vee S')(2) \to (S \vee S')(2)$ induces on the fundamental groups the homomorphism $\pi_2(p)_\#$ which maps a' to $a' b' a'^{-1} b'^{-1}$ and b' to $a'^2 b'^2 a'^{-2} b'^{-2}$. The fundamental group of $S\times S'$ is comutative, so $\pi_2(p)_\#$ is the zero homeomorphism. Since the second homotopy group of $S\times S'$ is trivial, p is null-homotopic. Thus the shape of $C(2)$ is trivial

4. Fixed point property. Borsuk nad Ulam ([1], p. 878, problem (β)) asked if the fixed point property is an invariant under the operation of the symmetric product. We give the negative answer to this question.

(4.1) LEMMA. *Let X be a contractible compactum and $x_0 \in X$. If there is a contraction $H: X\times[0, 1] \to X$ such that $H_0 = \mathrm{id}_X$, $H(x_0, t) = H(x, 1) = x_0$ for $t \in [0, 1]$, $x \in X$ and, for every $0 \leqslant t < t' \leqslant 1$, $H_t^{-1}(\{x_0\}) \subset \operatorname{int} H_{t'}^{-1}(\{x_0\})$, then $X' = \{\{x_0, x\} \in X(2);\ x \in X\}$ is a retract of $X(2)$.*

Proof. For every $x \in X$, let $t(x) = \inf\{t;\ H(x, t) = x_0\}$. Since

$$H_t^{-1}(\{x_0\}) \subset \operatorname{int} H_{t'}^{-1}(\{x_0\})$$

for $t < t'$, the map $t: X \to [0, 1]$ is continuous.

We define $r: X(2) \to X'$ by

$$r(\{x, y\}) = \begin{cases}\{x_0, H(y, t(x))\} & \text{if } t(x) \leqslant t(y),\\ \{x_0, H(x, t(y))\} & \text{if } t(x) \geqslant t(y).\end{cases}$$

The map r is well defined since if $t(x) = t(y)$ then $H(y, t(x)) = H(x, t(y)) = x_0$. Both $H(x, t(y))$ and $H(y, t(x))$ are continuous, thus r is continuous. Since $t(x_0) = 0$ and $H(x, 0) = x$, it follows that $r(\{x_0, x\}) = \{x_0, x\}$.

Knill ([9]) constructed a contractible continuum B such that B, but not $B\times[0, 1]$, has the fixed point property. B is a subset of the Euclidean space E^3, and $B = R \cup 2D$ where $R = \{(r\cdot\cos M,\ r\cdot\sin M,\ (2-r)\,2^{-M}) \in E^3;\ 1 \leqslant r \leqslant 2 \text{ and } M \geqslant 1\}$ and $2D$ is a disk of radius 2 and a center $0 = (0, 0, 0)$ and lies in the plane $x_3 = 0$.

(4.2) THEOREM. *There is a continuum X with the fixed point property such that the $X(2)$ does not have the fixed point property.*

Proof. Let I be the closed segment $[(0, 0, 0), (0, 0, 1)] \subset E^3$, $X = B \cup I$. X has the fixed point property as a wedge of two spaces with the fixed point property. By Lemma (3.4) $X(2)$ is a union of the spaces $B(2)$, $I(2)$ and $B\times I$ where $B(2) \cap I(2) = \{0\}$, $B(2) \cap B\times I = \{\{b, 0\};\ b \in B\}$ and $I(2) \cap B\times I = \{\{0, t\};\ t \in I\}$. It is easy to see that there is a deformation of B to 0 as in the assumption of Lemma (4.1), so $B(2)$ can be retracted to $B\times I \cup I(2)$. The space $I(2)$ is homeomorphic to a disk ([1]) with the arc $A = \{\{0, t\};\ t \in I\}$ in its boundary; hence $I(2)$ can be retracted to A. Thus the whole space $X(2)$ can be rectracted to $B\times I$. Then $X(2)$ does not have the fixed point property, since its retract does not.

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On functions of bounded n-th variation

by

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Abstract. Following Sargent [15], a definition of bounded nth variation for real valued functions is introduced and it is shown that this definition is equivalent to that of Russell [11]. Various properties of functions of generalized bounded variation are established.

1. Introduction. One approach to get a definition of functions of bounded variation of higher order is based on the concept of higher order divided differences (cf. [9], p. 24). This was followed by Russell ([10], [11]) and others (see, for example, [2]). This method was also followed in [7] to define absolute continuity of higher order. Another approach was due to Sargent ([14], [15]) who introduced the concept of *absolute continuity of higher order* which involved the notion of generalized derivatives. Sargent was concerned with the descriptive definition of the Cèsaro–Denjoy integrals which needed the concept of absolute continuity of higher order. She did not specifically mention bounded variation but her method suggested a definition of bounded variation of higher order. The two approaches are different. Therefore, it is natural to ask if these two approaches have any connection. The purpose of the present paper is to give an answer to this question. Following Sargent [15] (see also [4]) we have introduced two definitions of bounded variation of order n which are analogous to the concept of VB and VB^* of [13], pp. 221–228, and showed that on intervals these definitions are equivalent to that used by Russel [11].

2. Definitions and notation. Let f be defined in some neighbourhood of x. If there are real numbers $\alpha_0(=f(x))$, $\alpha_1 \ldots, \alpha_r$ depending on x but not on h such that

$$f(x+h) = \sum_{i=0}^{r} \alpha_i \frac{h^i}{i!} + o(h^r),$$

then α_r is called the *Peano derivative* of f at x of order r and is denoted by $f_{(r)}(x)$. Clearly, if $f_{(r)}(x)$ exists then $f_{(i)}(x)$ exists for all i, $1 \leqslant i < r$. Also, if the ordinary rth derivative $f^{(r)}(x)$ exists, then $f_{(r)}(x)$ exists and is equal to $f^{(r)}(x)$. The converse is true for $r = 1$ only.