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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Rational spaces and the property of universality

by

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Abstract. The main result of this paper is the following: In the family of all rational metrizable spaces, there exists a universal element T. Moreover, this element has the so-called property of finite intersection for the subfamily of all rational continua, that is, for every rational continuum X there exists a fixed homeomorphism i_X of X into T such that if Y and Z are non-homeomorphic rational continua then the set $i_Y(Y) \cap i_Z(Z)$ is finite.

Introduction. A Hausdorff space is said to be *rational* (resp. *rim-finite*) if it has a countable basis of open sets with countable (resp. finite) boundaries. Obviously, every rim-finite space is regular and every regular rational space is one-dimensional and, hence, is contained topologically in 3-dimensional Euclidean space. In this paper a "rational space" means a rational metrizable space.

Nöbeling (see [5]) proved that in the family of all rim-finite spaces, in the family of all rim-finite compact spaces and in the family of all rim-finite continua there does not exist a universal element.

Also, it is well known (see [4], Vol. II, § 51. IV) that in the family of all rational compact spaces and in the family of all rational continua does not exist a universal element.

Recall that a rational space X has *rim-type* $\leqslant \alpha$, where α is an ordinal number, if it has a basis B of open sets such that the α-derivative (see [4], V. I, § 24. IV) of the boundary of every element of B is empty.

In [2], it is proved that in the family of all (locally connected) rational compact spaces having rim-type $\leqslant \alpha$ and in the family of all (locally connected) rational continua having rim-type $\leqslant \alpha$ there does not exist a universal element. Also, this is true for the family of all (locally connected) rational spaces having a basis of open sets such that the boundary of every element is a compact space whose α-derivative is empty.

A space X is said to be *scattered* if every non-empty subset of X contains an isolated point. A rational space X is said to be *rim-scattered* if it has a basis of open sets with scattered boundaries. Since for every rim-scattered space X there exists a countable ordinal number α such that the rim-type of $X \leqslant \alpha$, in the family of all rim-scattered spaces there does not exists a universal element.

In Section I of this paper we give a notion of *r-partition* of a metric space and prove that a space is a rational space if and only if it is homeomorphic to an *r-partition* of a subspace of the Cantor ternary set with the quotient topology.

In Section II we construct some special topological spaces which are used in Section III where we give the main result: in the family of all rational spaces, there exists a universal element T. Moreover, this element has the so-called *property of finite intersection* for the subfamily of all rational continua, that is, for every rational continuum X there exists a fixed homeomorphism i_X of X into T such that if X and Y are non-homeomorphic rational continua then the set $i_X(X)\cap i_Y(Y)$ is finite.

I. r-partitions of subsets of Cantor's set.

1. By L_n, $n = 1, 2, \ldots$, we denote the set of all ordered n-tuples $i_1 \ldots i_n$ where $i_k = 0$ or $1, k = 1, \ldots, n$. Set $L_0 = \{\emptyset\}$, $L^n = \bigcup_{k=0}^{n} L_k$ and $L = \bigcup_{n=0}^{\infty} L_n$. For $n = 0$, by $i_1 \ldots i_n$ we denote the element $\emptyset$ of L. We say that the element $i_1 \ldots i_n$ of L is a *part* of the element $j_1 \ldots j_m$ if either $n = 0$ or $1 \leqslant n \leqslant m$ and $i_k = j_k$ for every $k \leqslant n$. The elements of L are also denoted by $\bar{\imath}, \bar{\jmath}, \bar{\imath}_1$, etc. If $\bar{\imath} = i_1 \ldots i_n \in L$, then by $\bar{\imath}0$ (resp. $\bar{\imath}1$) we denote the element $i_1 \ldots i_n0$ (resp. $i_1 \ldots i_n1$) of L.

By Λ_n, $n = 1, 2, \ldots$, we denote the set of all ordered n-tuples $i_1 \ldots i_n$, where i_k, $k = 1, \ldots, n$, is a positive integer. Set $\Lambda_0 = \{\emptyset\}$ and $\Lambda = \bigcup_{n=0}^{\infty} \Lambda_n$. The elements of Λ we denote by $\bar{\alpha}, \bar{\beta}$, etc. Let $\bar{\alpha} \in \Lambda_n$, $\bar{\beta} \in \Lambda_m$, $\bar{\alpha} = i_1 \ldots i_n$, $\bar{\beta} = j_1 \ldots j_m$. We write $\bar{\beta} \geqslant \bar{\alpha}$ if either $\bar{\alpha} = \emptyset$ or $1 \leqslant n \leqslant m$ and $i_k = j_k$ for every $k \leqslant n$. Obviously, if $\bar{\alpha}, \bar{\beta} \in \Lambda_n$ and $\bar{\beta} \geqslant \bar{\alpha}$ then $\bar{\beta} = \bar{\alpha}$. Also for every $\bar{\alpha} \in \Lambda_n$ the set of all elements $\bar{\beta} \in \Lambda_{n+1}$ such that $\bar{\beta} \geqslant \bar{\alpha}$ is countable.

By C we denote the Cantor ternary set. By $C_{\bar{\imath}}$, where $\bar{\imath} = i_1 \ldots i_n \in L$, $n \geqslant 1$, we denote the set of all points of C for which the kth digit in the ternary expansion, $k = 1, \ldots, n$, coincides with 0 if $i_k = 0$ and with 2 if $i_k = 1$. Also, set $C_\emptyset = C$. We remind that the sets $C_{\bar{\imath}}$, $\bar{\imath} \in L$, constitute a basis for C. For every point $a \in C$ and for every integer $n \geqslant 0$, by $\bar{\imath}\ (a, n)$ we denote the element $\bar{\imath} \in L_n$ for which $a \in C_{\bar{\imath}}$. Obviously, this element is uniquely determined. For every subset F of C and for every integer $n = 0, 1, 2, \ldots$ by $\mathrm{st}(F, n)$ (it is called the *n-star of F in C*) we denote the union of all sets $C_{\bar{\imath}}$, where $\bar{\imath} \in L_n$, such that $C_{\bar{\imath}} \cap F \neq \emptyset$.

For a subset Q of a space X by $\bar{Q}$, $\mathrm{Int}\, Q$, $\mathrm{Fr}\, Q$ and $|Q|$ we denote the closure, the interior, the boundary, and the power of Q, respectively.

2. Let X be a metric rational space.

LEMMA 1. *There exists a basis $B = \{U_i,\ i = 1, 2, \ldots\}$ of open sets of X such that*:

(1) $U_i = \mathrm{Int}\, \bar{U}_i$,

(2) $|\mathrm{Fr}\, U_i| \leqslant \aleph_0$ *and*

(3) *for every $x \in \mathrm{Fr}\, U_i$, the number of elements of B the boundary of which contains x is finite.*

Proof. Let $\{V_i,\ i = 1, 2, \ldots\}$ be a basis of X such that $|\mathrm{Fr}\, V_i| \leqslant \aleph_0$ for every $i = 1, 2, \ldots$ The elements of basis B we construct by induction such that:

(1) $V_i \subseteq U_i$, (2) $\mathrm{diam}\, U_i < 2 \mathrm{diam}\, V_i$, (3) $|\mathrm{Fr}\, U_i| \leqslant \aleph_0$, (4) $U_i = \mathrm{Int}\, \bar{U}_i$ and (5) there exists a numbering $\{x_{i1}, x_{i2}, \ldots\}$ of the set $\mathrm{Fr}\, U_i$ such that $x_{kj} \notin \mathrm{Fr}\, U_i$ if $k \leqslant i-1$, $j \leqslant i-1$, $i > 1$.

Set $U_1 = \mathrm{Int}\, \bar{V}_1$ and let $\{x_{11}, x_{12}, \ldots\}$ be an arbitrary numbering of the set $\mathrm{Fr}\, U_1$.

Suppose that for every $i < n$ we have constructed the set U_i and a numbering $\{x_{i1}, x_{i2}, \ldots\}$ of the set $\mathrm{Fr}\, U_i$ such that properties (1)–(5) above are satisfied for $i < n$.

We shall construct the set U_n and a numbering $x_{n1}, x_{n2}, \ldots$ of the set $\mathrm{Fr}\, U_n$ such that properties (1)–(5) are satisfied for every $i < n+1$.

For every point x_{ij}, $i \leqslant n-1$, $j \leqslant n-1$, by $O_{x_{ij}}$ we denote an open set such that:

(1) $O_{x_{ij}} = \emptyset$ if $x_{ij} \notin \mathrm{Fr}\, V_n$,

(2) $x_{ij} \in O_{x_{ij}}$ if $x_{ij} \in \mathrm{Fr}\, V_n$,

(3) $|\mathrm{Fr}\, O_{x_{ij}}| \leqslant \aleph_0$,

(4) $\mathrm{diam}\, O_{x_{ij}} \leqslant \frac{1}{2} \mathrm{diam}\, V_n$ and

(5) if $x_{lt} \neq x_{ij}$, $l \leqslant n-1$, $t \leqslant n-1$ then $x_{lt} \notin \mathrm{Fr}\, O_{x_{ij}}$.

Set $U_n = \mathrm{Int}(\bar{V}_n \cup \bigcup_{\substack{i \leqslant n-1 \\ j \leqslant n-1}} \bar{O}_{x_{ij}})$. Obviously,

(1) $V_n \subseteq U_n$,

(2) $\mathrm{diam}\, U_n < 2 \mathrm{diam}\, V_n$,

(3) $|\mathrm{Fr}\, U_n| \leqslant \aleph_0$,

(4) $U_n = \mathrm{Int}\, \bar{U}_n$ and

(5) $x_{ij} \notin \mathrm{Fr}\, U_n$ if $i < n$, $j < n$. Let $\{x_{n1}, x_{n2}, \ldots\}$ be an arbitrary numbering of the set $\mathrm{Fr}\, U_n$.

Thus, we may assume that the set U_i is constructed for every $i = 1, 2, \ldots$ It is clear that the system $B = \{U_i,\ i = 1, 2, \ldots\}$ is the required basis. The proof of the lemma is complete.

3. We say that an upper semi-continuous partition (see [3], Ch. 3) D of a space X is an *r-partition* if every element of D is a non-empty finite set and the set of all elements of D consisting of at least two points is countable.

Let X be a rational space and $B = \{U_i\colon i = 1, 2, \ldots\}$ a basis of X with the properties mentioned in Lemma 1.

Set $\mathrm{Fr}\, B = \mathrm{Fr}\, U_1 \cup \mathrm{Fr}\, U_2 \cup \ldots$, $A_0^i = \bar{U}_i$ and $A_1^i = X \setminus U_i$, $i = 1, 2, \ldots$ For every $i_1 \ldots i_n \in L$, we set $X_{i_1 \ldots i_n} = A_{i_1}^1 \cap \ldots \cap A_{i_n}^n$. We now construct a subset $S(X)$ of C and a map $q(X) = q$ of $S(X)$ into X. The point a of C belongs to $S(X)$ if and only if $X_{\bar{i}(a,1)} \cap X_{\bar{i}(a,2)} \cap \ldots \neq \emptyset$. We observe that, for every point $a \in S(X)$, the set $X_{\bar{i}(a,1)} \cap X_{\bar{i}(a,2)} \cap \ldots$ is a singleton. If $\{x\} = X_{\bar{i}(a,1)} \cap X_{\bar{i}(a,2)} \cap \ldots$ then we set $q(a) = x$.

LEMMA 2. *The following propositions are true*:

(1) *If* $x \in X \setminus \operatorname{Fr} B$ *then* $q^{-1}(x)$ *is a singleton.*

(2) *If* $x \in \operatorname{Fr} B$ *then the number of elements of the set* $q^{-1}(x)$ *is a finite number which is larger than or equal to two.*

(3) *The map* q *is continuous.*

(4) *The map* q *is closed.*

(5) *The partition* $D(X) = \{q^{-1}(x)\colon x \in X\}$ *of* $S(X)$ *is an r-partition.*

Proof. By the definition of the map q we have that if $a \in C_{\bar{i}} \cap S(X)$, $\bar{i} \in L_n$, $n \geqslant 1$, then $q(a) \in X_{\bar{i}}$. We also observe that if $x \in X_{\bar{i}}$ then $q^{-1}(x) \cap C_{\bar{i}} \neq \emptyset$. Indeed, let $x \in X_{\bar{i}}$, where $\bar{i} = i_1 \ldots i_n$. Then, for every $1 \leqslant k \leqslant n$, we have $x \in A^k_{i_k}$. For every $k > n$, let i_k be 0 or 1 such that $x \in A^k_{i_k}$. Denote by $\bar{j}(m)$, $m = 1, 2, \ldots$, the element $i_1 \ldots i_m$ of L_m. Then, $x \in X_{\bar{j}(m)}$. Obviously, $\bar{j}(n) = \bar{i}$ and if $m < t$ then $\bar{j}(m)$ is a part of $\bar{j}(t)$. Hence $\bigcap_{m=1}^{\infty} C_{\bar{j}(m)} \neq \emptyset$. Let $\{a\} = \bigcap_{m=1}^{\infty} C_{\bar{j}(m)}$. It is clear that $q(a) = x$ and $a \in C_{\bar{j}(n)} = C_{\bar{i}}$. Now we prove the propositions of the lemma.

(1) Let $x \in X \setminus \operatorname{Fr} B$. By the above, $q^{-1}(x) \neq \emptyset$. Let $a, b \in q^{-1}(x)$ and $a \neq b$. Then there exist an integer n and elements $\bar{i}, \bar{j}$ of L_n, $\bar{i} \neq \bar{j}$ such that $a \in C_{\bar{i}}$ and $b \in C_{\bar{j}}$. Hence, $x = q(a) \in X_{\bar{i}}$ and $x = q(b) \in X_{\bar{j}}$. If $\bar{i} = i_1 \ldots i_n$ and $\bar{j} = j_1 \ldots j_n$ then there exists $k \leqslant n$ such that $i_k \neq j_k$.

This means that $x \in A^k_0$ and $x \in A^k_1$. Hence $x \in \operatorname{Fr} U_k$ which is a contradiction.

(2) Let $x \in \operatorname{Fr} B$. Then there exists k such that $x \in \operatorname{Fr} U_k$. Hence, $x \in A^k_0$ and $x \in A^k_1$. This means that there exist two different elements $\bar{i}$ and $\bar{j}$ of L_k such that $x \in X_{\bar{i}} \cap X_{\bar{j}}$. Hence $q^{-1}(x) \cap C_{\bar{i}} \neq \emptyset$ and $q^{-1}(x) \cap C_{\bar{j}} \neq \emptyset$. Consequently, the number of elements of the set $q^{-1}(x)$ is larger than or equal to two. We show that the set $q^{-1}(x)$ is finite. There exists an integer n such that $x \notin \operatorname{Fr} U_k$ for every $k > n$. This follows by the properties of the basis B. We prove that for every $\bar{i} \in L_n$ the set $q^{-1}(x) \cap C_{\bar{i}}$ is a singleton or empty. Indeed, if $a, b \in q^{-1}(x) \cap C_{\bar{i}}$, $\bar{i} \in L_n$, $a \neq b$, then there exist elements $\bar{j}(1)$ and $\bar{j}(2)$ of L_m, $\bar{j}(1) \neq \bar{j}(2)$, $m > n$, such that $a \in C_{\bar{j}(1)}$, $b \in C_{\bar{j}(2)}$, $C_{\bar{j}(1)} \subseteq C_{\bar{i}}$, $C_{\bar{j}(2)} \subseteq C_{\bar{i}}$. Hence, $x = q(a) \in X_{\bar{j}(1)}$ and $x = q(b) \in X_{\bar{j}(2)}$. If $\bar{j}(1) = j_1 \ldots j_m$ and $\bar{j}(2) = j'_1 \ldots j'_m$ then, since $\bar{j}(1) \neq \bar{j}(2)$ and $\bar{i}$ is a part of $\bar{j}(1)$ and $\bar{j}(2)$, there exists $n < k \leqslant m$ such that $j_k \neq j'_k$. This means that $x \in A^k_0 \cap A^k_1$, that is, $x \in \operatorname{Fr} U_k$ which is a contradiction.

(3) Let $q(a) = x$ and U be an open neighbourhood of x in X. There exist an integer n and $\bar{i} \in L_n$ such that $x \in U_n \subseteq \overline{U}_n \subseteq U$ and $a \in C_{\bar{i}}$. If $\bar{i} = i_1 \ldots i_n$ then $i_n = 0$, because $x \in A^n_0$ and $x = q(a) \in X_{\bar{i}}$. Hence $X_{\bar{i}} \subseteq \overline{U}_n \subseteq U$. Since $q(C_{\bar{i}} \cap S(X)) \subseteq X_{\bar{i}}$ and the set $C_{\bar{i}} \cap S(X)$ is an open neighbourhood of a in $S(X)$, the map q is continuous.

(4) In order to prove that the map q is closed, it is sufficient to prove that if $x \in X \setminus q(F)$, where F is an closed subset of $S(X)$, then there exists a neighbourhood U of x such that $q^{-1}(U) \subseteq S(X) \setminus F$.

Let F be a closed subset of $S(X)$, $x \in X \setminus q(F)$ and $\{\bar{i}^n_1, \ldots, \bar{i}^n_m\}$ be the set of all elements $\bar{i}$ of L_n for which $C_{\bar{i}} \cap q^{-1}(x) \neq \emptyset$. Since $q^{-1}(x)$ is finite and the set F is closed, we can assume that n is so large that $C_{\bar{i}^n_k} \cap F = \emptyset$ for every $k = 1, \ldots, m$. Obviously, $x \in X_{\bar{i}}$, $\bar{i} \in L_n$ if and only if $\bar{i} \in \{\bar{i}^n_1, \ldots, \bar{i}^n_m\}$.

Let U be an open neighbourhood of x in X such that if $x \notin X_{\bar{i}}$, $\bar{i} \in L_n$ then $U \cap X_{\bar{i}} = \emptyset$. We prove that $q^{-1}(U) \cap F = \emptyset$.

Let $y \in U$. If $q^{-1}(y) \cap C_{\bar{i}} \neq \emptyset$ $\bar{i} \in L_n$ then $y \in X_{\bar{i}}$. By the choice of U, $x \in X_{\bar{i}}$ and $\bar{i} \in \{\bar{i}^n_1, \ldots, \bar{i}^n_m\}$. Thus, $q^{-1}(y) \subseteq \bigcup_{k=1}^{m} C_{\bar{i}^n_k}$ and $q^{-1}(U) \subseteq \bigcap_{k=1}^{m} C_{\bar{i}^n_k}$, that is, $q^{-1}(U) \cap F = \emptyset$. Hence, the map q is closed.

(5) By propositions (1) and (2) of the lemma, every element of the partition $D(X)$ is finite. If the element $d = q^{-1}(x)$ contains at least two points of $S(X)$ then by proposition (2) of the lemma $x \in \operatorname{Fr} B$. Hence, the set of all these elements of $D(X)$ is countable.

Since the map q is closed, the partition $D(X)$ is upper semi-continuous (see, for example, [3], Ch. 3, 12). Thus, the partition $D(X)$ is an r-partition of $S(X)$. The proof of the lemma is complete.

4. THEOREM 1. *A space is a rational space if and only if it is homeomorphic to an r-partition of a subspace of the Cantor ternary set with the quotient topology.*

Proof. Let X be a rational space. We may assume that X is a metric space. Consider the subset $S(X)$ of C, the map $q(X) = q$ of $S(X)$ onto X and the partition $D(X)$ of $S(X)$ constructed in Section 3 with respect to a basis $B = \{U_i\colon i = 1, 2, \ldots\}$ of X having the properties of Lemma 1.

From Lemma 2 it follows that $D(X)$ is an r-partition and that q is closed. Let p be the projection of $S(X)$ onto $D(X)$ and i the map of $D(X)$ onto X for which $i \circ p = q$. Obviously, the map i is uniquely determined. Since p and q are continuous and closed, the map i also is continuous and closed. Hence, i is a homeomorphism, because it is "one-to-one".

Conversely, let X be homeomorphic to an r-partition D of a subset S of C with the quotient topology. Let p be the projection of S onto D. Since D is an r-partition, the map p is perfect (that is, it is closed and the pre-image of each point is compact). Since S is a regular space with countable basis, the space D also is a regular space with countable basis (see [3], Ch. 5, Theorem 20).

We show that D is a rational space. Let $d \in D$ and U be an open neighbourhood of d in D. It is necessary to see that there exists an open set V of D with a countable boundary such that $d \in V \subseteq U$.

Obviously, for every $n = 0, 1, 2, \ldots$, the sets $\operatorname{st}(d, n) \cap S$ and $S \setminus \operatorname{st}(d, n)$ are open subsets of S. For every open subset Q of S, by $U(Q)$ we denote the union of all elements of D which are contained in Q. Obviously, $U(Q)$ is an open subset of D. Hence, $U(\mathrm{st}(d, n) \cap S)$ and $U(S \setminus \mathrm{st}(d, n))$ are open subsets of D. The boundary $\mathrm{Fr}\, U(\mathrm{st}(d, n) \cap S)$ of the set $U(\mathrm{st}(d, n) \cap S)$ in D is contained in the set

$$D \setminus \big(U(\mathrm{st}(d, n) \cap S) \cup U(S \setminus \mathrm{st}(d, n))\big).$$

This set consists of the elements d' of D for which $d' \cap (\mathrm{st}(d, n) \cap S) \neq 0$ and $d' \cap (S \setminus \mathrm{st}(d, n)) \neq \emptyset$. It means that d' contains at least two points. Hence the set $\mathrm{Fr}\, U(\mathrm{st}(d, n) \cap S)$ is finite or countable. Obviously, if n is a sufficiently large integer, then the set $V = U(\mathrm{st}(d, n) \cap \leqslant S)$ is contained in U.

Thus, the space D and, hence, the space X is a rational space. The proof of the theorem is complete.

II. A special constructions of rational spaces.

1. By $C(1)$ we denote the set of all points of C which are the end points of the components of the set $[0, 1] \setminus C$. Let $y(0), y(1), y(2), \ldots$ be the sequence the elements of which are all finite subsets of $C(1)$ which are not singletons. We assume that $y(0) = \emptyset$ and if $n \neq m$ then $y(n) \neq y(m)$.

In this section, by A we denote a set of pairs (S, D) where S is a non-empty subset of C and D is an r-partition of S. It is possible that $S_1 = S_2$ and $D_1 = D_2$ for two different elements (S_1, D_1) and (S_2, D_2) of A.

By $A(S)$ we denote the set of all subsets S of C such that there exists a pair (S, D) which belongs to A. If (S_1, D_1) and (S_2, D_2) are different elements of A then S_1 and S_2 we consider as different elements of $A(S)$.

Let $(S, D) \in A$. By $D(1)$ we denote the subset of D which consists of all non-degenerate elements of D. For every subset Q of $D(1)$, by Q^* we denote the union of all elements of Q.

We suppose that for every element $\bar{\imath} \in L$ there exists a subset $A(\bar{\imath})$ of A such that:

(1) $A(\emptyset) = A$,

(2) $A(\bar{\imath}) \cap A(\bar{\jmath}) = \emptyset$ if $\bar{\imath}, \bar{\jmath} \in L_k$, $k \geqslant 0$, $\bar{\imath} \neq \bar{\jmath}$,

(3) $A(\bar{\imath}) = A(\bar{\imath}0) \cup A(\bar{\imath}1)$ and

(4) for every $(S_1, D_1), (S_2, D_2) \in A$, $(S_1, D_1) \neq (S_2, D_2)$ there exist an integer $k \geqslant 0$ and elements $\bar{\imath}, \bar{\jmath} \in L_k$, $\bar{\imath} \neq \bar{\jmath}$, such that $(S_1, D_1) \in A(\bar{\imath})$ and $(S_2, D_2) \in A(\bar{\jmath})$. Obviously, the above supposition is true if and only if the power of A is less than or equal to continuum.

We say that elements (S_1, D_1) and (S_2, D_2) of A are *n-equivalent*, $n = 0, 1, 2, \ldots$ and write $(S_1, D_1) \overset{n}{\sim} (S_2, D_2)$ if

(1) there exists an element $\bar{\imath} \in L_n$ such that $(S_1, D_1), (S_2, D_2) \in A(\bar{\imath})$ and

(2) for every element $d \in D_1$ there exists an element $d' \in D_2$ such that $\mathrm{st}(d, n) = \mathrm{st}(d', n)$ and conversely, that is, for every element d of D_2 there exists an element d' of D_1 such that $\mathrm{st}(d, n) = \mathrm{st}(d', n)$.

By $A(n)$, $n = 0, 1, 2, \ldots$, we denote the set of all n-equivalence classes of A. It is easy to see that the set $A(n)$ is finite.

Consider the set $C \times A(S)$. In this set we define a subset $J(A)$ as follows: the element (a, S) of $C \times A(S)$ belongs to $J(A)$ if and only if $a \in S$.

2. Lemma 3. *For every integer $k = 0, 1, 2, \ldots$, for every $\bar{\alpha} \in \Lambda_k$ and for every $\bar{\gamma} \in \Lambda_q$ where $q \leqslant k$ there exist a subset $A(\bar{\alpha})$ of $A(S)$ and subsets $x(\bar{\alpha})$ and $U(x(\bar{\gamma}), k)$ (it is possible that $A(\bar{\alpha}) = \emptyset$, $x(\bar{\alpha}) = \emptyset$ and $U(x(\bar{\gamma}), k) = \emptyset$ for some $\bar{\alpha}$, $\bar{\gamma}$ and k) of $J(A)$ such that:*

(1) $A(\emptyset) = A(S)$.

(2) If $\bar{\alpha}_1, \bar{\alpha}_2 \in \Lambda_k$ and $\bar{\alpha}_1 \neq \bar{\alpha}_2$ then $A(\bar{\alpha}_1) \cap A(\bar{\alpha}_2) = \emptyset$.

(3) If $\bar{\gamma} \in \Lambda_{k-1}$, $k \geqslant 1$, then $A(\bar{\gamma}) = \bigcup_{\bar{\gamma} \leqslant \bar{\alpha}, \bar{\alpha} \in \Lambda_k} A(\bar{\alpha})$.

(4) The set $A(\bar{\alpha})$ is contained in some element of the set $A(k)$ and if (S_1, D_1), $(S_2, D_2) \in A$ and $S_1, S_2 \in A(\bar{\alpha})$ then either $y(k) \in D_1$ and $y(k) \in D_2$, or $y(k) \notin D_1$ and $y(k) \notin D_2$.

(5) The set $x(\bar{\alpha})$ is non-empty iff there exists $(S, D) \in A$ such that $S \in A(\bar{\alpha})$ and $y(k) \in D$.

(6) If $x(\bar{\alpha}) \neq \emptyset$ then $x(\bar{\alpha}) = (y(k) \times A(\bar{\alpha})) \cap J(A)$.

(7) The set $U(x(\bar{\gamma}), k)$ is non-empty iff $x(\bar{\gamma}) \neq \emptyset$.

(8) If $x(\bar{\gamma}) \neq \emptyset$ then there exists an integer $t(\bar{\gamma}, \bar{\alpha}) \geqslant k$ such that

(i) $i(\bar{\gamma}, \bar{\alpha}) \geqslant t(\bar{\gamma}, \bar{\alpha}_1)$ if $\bar{\alpha} \geqslant \bar{\alpha}_1 \geqslant \bar{\gamma}$,

(ii) if $\bar{\alpha}_1, \bar{\alpha}_2, \ldots \in \Lambda_k$, $\bar{\alpha}_i \neq \alpha_j$, $i \neq j$, $A(\bar{\alpha}_i) \neq \emptyset$, then $\lim t(\bar{\gamma}, \bar{\alpha}_i) = \infty$ and

(iii) $U(x(\bar{\gamma}), k) = \bigcup_{\bar{\gamma} \leqslant \bar{\alpha}, \bar{\alpha} \in \Lambda_k} \big((\mathrm{st}(y(q), t(\bar{\gamma}, \bar{\alpha})) \times A(\bar{\alpha})) \cap J(A)\big)$.

(9) If $(S, D) \in A$, $S \in A(\bar{\gamma})$, $\bar{\gamma} \in \Lambda_q$, $q \leqslant k-1$, $k \geqslant 1$, $d \in D$ and

$$(d \times \{S\}) \cap U(x(\bar{\gamma}), k) \neq \emptyset$$

then $d \times \{S\} \subseteq U(x(\bar{\gamma}), k-1)$.

Proof. We prove the lemma by induction on the integer k.

Let $k = 0$. We set $A(\emptyset) = A(S)$, $x(\emptyset) = \emptyset$ and $U(x(\emptyset), 0) = \emptyset$. Obviously, properties (1), (2), (4)–(8) are satisfied for $k = 0$.

Suppose that the sets $A(\bar{\alpha})$, $x(\bar{\alpha})$ and $U(x(\bar{\gamma}), k)$ are constructed for every $k < m$ such that properties (1)–(9) are true if $k < m$.

We construct the sets $A(\bar{\alpha})$, $x(\bar{\alpha})$ for every $\bar{\alpha} \in \Lambda_m$ and the set $U(x(\bar{\gamma}), m)$ for every $\bar{\gamma} \in \Lambda_q$, $q \leqslant m$ such that properties (1)–(9) of the lemma are true if $k \leqslant m$.

Let $\bar{\beta}$ be an arbitrary element of Λ_{m-1}. Consider the set $A(\bar{\beta})$. If $A(\bar{\beta}) = \emptyset$ then we set $A(\bar{\alpha}) = \emptyset$ for every $\bar{\alpha} \in \Lambda_m$, $\bar{\alpha} \geqslant \bar{\beta}$.

Let $A(\bar{\beta}) \neq \emptyset$, $(S, D) \in A$ and $S \in A(\bar{\beta})$. Since D is upper semicontinuous, there exists an integer $t \geqslant m$ such that for every $\bar{\gamma} \in \Lambda_q$, $0 \leqslant q \leqslant m-1$, $\bar{\gamma} \leqslant \bar{\beta}$ (hence, $A(\bar{\beta}) \subseteq A(\bar{\gamma})$) if $y(q) \in D$ (hence, $x(\bar{\gamma}) \neq \emptyset$) $d \in D$, $d \cap \mathrm{st}(y(q), t) \neq \emptyset$, then $d \subseteq \mathrm{st}(y(k), t(\bar{\gamma}, \bar{\beta}))$. If there exists an integer q, $0 \leqslant q \leqslant m-1$, for which $y(q) \in D$, then by $t((S, D), \bar{\beta})$ we denote the minimum of the above integers t.

If for every q, $0 \leqslant q \leqslant m-1$, we have $y(q) \notin D$ then we set $t((S, D), \bar{\beta}) = m$.

Divide the set $A(\bar{\beta})$ into equivalence classes as follows: Let $(S_1, D_1), (S_2, D_2) \in A$ where $S_1, S_2 \in A(\bar{\beta})$. We say that S_1 and S_2 belong to an *equivalence class* of $A(\bar{\beta})$

if and only if $t((S_1, D_1), \bar{\beta}) = t((S_2, D_2), \bar{\beta})$, the elements S_1, S_2 belong to an element of the set $A(m)$ and, either $y(m) \in D_1$, $y(m) \in D_2$ or $y(m) \notin D_1$, $y(m) \notin D_2$. Obviously, the number of equivalence classes of the set $A(\bar{\beta})$ is finite or countable.

Hence, there is an one-to-one correspondence between these equivalence classes and a subset of elements $\bar{\alpha}$ of Λ_m for which $\bar{\alpha} \geqslant \bar{\beta}$. By $A(\bar{\alpha})$ we denote the equivalence class which corresponds to the element $\bar{\alpha}$ of Λ_m. If there is no equivalence class which corresponds to the element $\bar{\alpha}$ of Λ_m, $\bar{\alpha} \geqslant \bar{\beta}$, then we set $A(\bar{\alpha}) = \emptyset$.

Thus, for every $\bar{\alpha} \in \Lambda_m$ we construct the set $A(\bar{\alpha})$. Now, we construct the set $x(\bar{\alpha})$, $\bar{\alpha} \in \Lambda_m$. If there exist $(S, D) \in A$ such that $S \in A(\bar{\alpha})$ and $y(m) \in D$ then we set $x(\bar{\alpha}) = (y(m) \times A(\bar{\alpha})) \cap J(A)$. If for every $(S, D) \in A$, $S \in A(\bar{\alpha})$ we have $y(m) \notin D$ then we set $x(\bar{\alpha}) = \emptyset$.

Finally, construct the set $U(x(\bar{\gamma}), m)$, $\bar{\gamma} \in \Lambda_q$, $q \leqslant m$. If $x(\bar{\gamma}) = \emptyset$ then we set $U(x(\bar{\gamma}), m) = \emptyset$. If $x(\bar{\gamma}) \neq \emptyset$ and $q = m$ then we set

$$U(x(\bar{\gamma}), m) = (\operatorname{st}(y(m), m) \times A(\bar{\gamma})) \cap J(A) \quad \text{and } t(\bar{\gamma}, \bar{\gamma}) = m .$$

Let $x(\bar{\gamma}) \neq \emptyset$ and $q < m$. Let $\bar{\alpha} \in \Lambda_m$, $\bar{\gamma} \leqslant \bar{\alpha}$, $A(\bar{\alpha}) \neq \emptyset$. There is a uniquely determined element $\bar{\beta}$ of Λ_{m-1} such that $\bar{\beta} \leqslant \bar{\alpha}$, and hence $A(\bar{\alpha}) \subseteq A(\bar{\beta}) \neq \emptyset$. If $(S, D) \in A$, $S \in A(\bar{\alpha})$ then we set $t(\bar{\gamma}, \bar{\alpha}) = \max\{t((S, D), \bar{\beta}), t(\bar{\gamma}, \bar{\beta})\}$. By the construction of the set $A(\bar{\alpha})$, the number $t(\bar{\gamma}, \bar{\alpha})$ does not depend on the element (S, D). If $A(\bar{\alpha}) = \emptyset$ then we set $t(\bar{\gamma}, \bar{\alpha}) = t(\bar{\gamma}, \bar{\beta}) + 1$.

We set

$$U(x(\bar{\gamma}), m) = \bigcup_{\bar{\gamma} \leqslant \bar{\alpha}, \bar{\alpha} \in \Lambda_m} ((\operatorname{st}(y(q), t(\bar{\gamma}, \bar{\alpha})) \times A(\bar{\alpha})) \cap J(A)) .$$

It is easy to see that the properties (1)–(6) for $k = m$ follow by the construction of the sets $A(\bar{\alpha})$ and $x(\bar{\alpha})$, $\bar{\alpha} \in \Lambda_m$.

We prove that property (7) is true for $k = m$. If $x(\bar{\gamma}) = \emptyset$ then by the construction we have $U(x(\bar{\gamma}), m) = \emptyset$. If $q \leqslant m$, $x(\bar{\gamma}) \neq \emptyset$, $\bar{\gamma} \in \Lambda_q$, then there exists $(S, D) \in A$, $S \in A(\bar{\gamma})$ such that $y(q) \in D$. There exists an element $\bar{\alpha} \in \Lambda_m$ such that $S \in A(\bar{\alpha})$. Then

$$(\operatorname{st}(y(k), t(\bar{\gamma}, \bar{\alpha})) \times A(\bar{\alpha})) \cap J(A) \neq \emptyset ,$$

and hence

$$U(x(\bar{\gamma}), m) \neq \emptyset .$$

Property (8) for $k = m$ follows by the construction of the set $U(x(\bar{\gamma}), m)$ and number $t(\bar{\gamma}, \bar{\alpha})$. We observe that if $\bar{\gamma} \in \Lambda_m$ then

$$U(x(\bar{\gamma}), m) = (\operatorname{st}(y(m), m) \times A(\bar{\gamma})) \cap J(A) = \bigcup_{\bar{\gamma} \leqslant \bar{\alpha}, \bar{\alpha} \in \Lambda_m} (\operatorname{st}(y(m), m) \times A(\bar{\alpha})) \cap J(A) .$$

Now, we prove property (9) if $k = m$. Let $(S, D) \in A$, $S \in A(\bar{\gamma})$, $\bar{\gamma} \in \Lambda_q$, $q \leqslant m-1$, $d \in D$ and $(d \times \{S\}) \cap U(x(\bar{\gamma}), m) \neq \emptyset$. Obviously, $U(x(\bar{\gamma}), m) \neq \emptyset$. By property (7), $x(\bar{\gamma}) \neq \emptyset$. By properties (4) and (5), $y(q) \in D$. There exists an element $\bar{\alpha} \in \Lambda_m$ such that $S \in A(\bar{\alpha}) \subseteq A(\bar{\gamma})$.

By property (8), $(d \times \{S\}) \cap ((\operatorname{st}(y(q), t(\bar{\gamma}, \bar{\alpha})) \times A(\bar{\alpha})) \cap J(A)) \neq \emptyset$. This means that $d \cap \operatorname{st}(y(q), t(\bar{\gamma}, \bar{\alpha})) \neq \emptyset$. Let $A(\bar{\alpha}) \subseteq A(\bar{\beta})$, $\bar{\beta} \in \Lambda_{m-1}$. Then, $t(\bar{\gamma}, \bar{\alpha}) \leqslant t((S, D), \bar{\beta})$.

By the definition of the number $t((S, D), \bar{\beta})$ we have $d \subseteq \operatorname{st}(y(q), t(\bar{\gamma}, \bar{\beta}))$. Hence, $d \times \{S\} \subseteq (\operatorname{st}(y(q), t(\bar{\gamma}, \bar{\beta})) \times \{S\}) \cap J(A) \subseteq U(x(\bar{\gamma}), m-1)$. The proof of the lemma is complete.

3. By $T(A)$ we denote the set whose elements are all sets $x(\bar{\alpha}) \neq \emptyset$, $\bar{\alpha} \in \Lambda_k$, $k = 0, 1, 2, \ldots$ and all singletons $\{x\}$, where x belongs to $J(A)$ and does not belong to any set $x(\bar{\alpha})$. We observe that different sets $x(\bar{\alpha})$ are disjoint.

By $U(A)$ we denote the set consisting of:

(1) all sets $U(x(\bar{\alpha}), n)$ for every $x(\bar{\alpha})$ and for every $n = 0, 1, 2, \ldots$, and

(2) all sets of the form $((\bigcup_{\bar{i} \in l_k} C_{\bar{i}}) \times A(\bar{\alpha})) \cap J(A)$, for every subset l_k of L_k and for every $\bar{\alpha} \in \Lambda$.

If $U \in U(A)$ then by $O(U)$ we denote the set of all points d of $T(A)$ which are contained in the set U. If U has the form $U(x(\bar{\alpha}), n)$ then we set $O(x(\bar{\alpha}), n) = O(U)$. If $U = ((\bigcup_{\bar{i} \in l_k} C_{\bar{i}}) \times A(\bar{\alpha})) \cap J(A)$ then we set $O((\bigcup_{\bar{i} \in l_k} C_{\bar{i}}) \times A(\bar{\alpha})) = O(U)$.

By $O(A)$ we denote the set of all sets of the form $O(U)$, $U \in U(A)$.

In the further the closure of a set Q is also denoted by $\operatorname{cl}(Q)$.

Lemma 4. *The set $O(A)$ forms a basis of open sets for a topology on $T(A)$.*

Proof. It is sufficient to prove that:

(1) if $d \in T(A)$ then there exists an element O of $O(A)$ such that $d \in O$ and

(2) if $d \in O_1 \cap O_2$, where $O_1, O_2 \in O(A)$ then there exists an element $O \in O(A)$ for which $d \in O \subseteq O_1 \cap O_2$.

Let $d \in T(A)$. If $d = x(\bar{\alpha})$, $\bar{\alpha} \in \Lambda_k$, then $d \subseteq U(x(\bar{\alpha}), n)$ for every $n \geqslant k$. Hence, $d \in O(x(\bar{\alpha}), n) \in O(A)$. If $d = \{x\}$, $x = (a, S)$ then $d \subseteq C_{\bar{i}} \times A(\bar{\alpha})$ where $a \in C_{\bar{i}}$ and $S \in A(\bar{\alpha})$. Hence, $d \in O(C_{\bar{i}} \times A(\bar{\alpha})) \in O(A)$.

Let $d \in O_1, O_2$, where $O_1, O_2 \in O(A)$. First, suppose that $d = \{x\}$, where $x = (a, S)$. Then there exist $\bar{i}_1, \bar{i}_2 \in L$ and $\bar{\alpha}_1, \bar{\alpha}_2 \in \Lambda$ such that

$$d \in O(C_{\bar{i}_1} \times A(\bar{\alpha}_1)) \subseteq O_1$$

and $d \in O(C_{\bar{i}_2} \times A(\bar{\alpha}_2)) \subseteq O_2$. Since $a \in C_{\bar{i}_1} \cap C_{\bar{i}_2}$ and $S \in A(\bar{\alpha}_1) \cap A(\bar{\alpha}_2)$, there exist $\bar{i} \in L$ and $\bar{\alpha} \in \Lambda$ such that $a \in C_{\bar{i}} \subseteq C_{\bar{i}_1} \cap C_{\bar{i}_2}$ and $S \in A(\bar{\alpha}) \subseteq A(\bar{\alpha}_1) \cap A(\bar{\alpha}_2)$. If we set $O = O(C_{\bar{i}} \times A(\bar{\alpha}))$ then we have

$$d \in O = O(C_{\bar{i}} \times A(\bar{\alpha})) \subseteq O(C_{\bar{i}_1} \times A(\bar{\alpha}_1)) \cap O(C_{\bar{i}_2} \times A(\bar{\alpha}_2)) \subseteq O_1 \cap O_2 .$$

Now consider the case where $d = x(\bar{\alpha})$, $\bar{\alpha} \in \Lambda_k$. We prove that there exists an integer $n(1) \geqslant 0$ such that $O(x(\bar{\alpha}), n(1)) \subseteq O_1$. Indeed, let O_1 have the form $O((\bigcup_{\bar{i} \in l_n} C_{\bar{i}}) \times A(\bar{\delta}))$. Then, since $x(\bar{\alpha}) \in O_1$, we have $y(k) \subseteq \bigcup_{\bar{i} \in l_n} C_{\bar{i}}$ and $A(\bar{\alpha}) \subseteq A(\bar{\delta})$. There exists an integer $m \geqslant k$ such that $\operatorname{st}(y(k), m) \subseteq \bigcup_{\bar{i} \in l_n} C_{\bar{i}}$. Let $n(1)$ be an arbitrary integer larger than m. Consider the set $U(x(\bar{\alpha}), n(1))$. Since $A(\bar{\alpha}) \subseteq A(\bar{\delta})$ and $t(\bar{\alpha}, \bar{\beta}) \geqslant n(1)$ for every $\bar{\beta} \in \Lambda_{n(1)}$, we have $U(x(\bar{\alpha}), n(1)) \subseteq (\bigcup_{\bar{i} \in l_n} C_{\bar{i}}) \times A(\bar{\delta})$, that is, $O(x(\bar{\alpha}), n(1)) \subseteq O((\bigcup_{\bar{i} \in l_n} C_{\bar{i}}) \times A(\bar{\delta})) = O_1$.

Let O_1 have the form $O(x(\bar\beta_1), n_1)$, $\bar\beta_1 \in \Lambda_{k_1}$. Since $O_1 \neq \emptyset$, we have $x(\bar\beta_1) \neq \emptyset$. If $x(\bar\alpha) = x(\bar\beta_1)$ then, setting $n(1) = n_1$, we have $d \in O(x(\bar\alpha), n(1)) \subseteq O(x(\bar\beta_1), n_1) = O_1$. Hence, we may suppose that $x(\bar\alpha) \neq x(\bar\beta_1)$. We show that $y(k) \cap y(k_1) = \emptyset$. Let $(S, D) \in A$ where $S \in A(\bar\alpha)$. Then $y(k) \in D$. Since $x(\bar\alpha) \in O(x(\bar\beta_1), n_1)$, we have $A(\bar\alpha) \subseteq A(\bar\beta_1)$. Hence, $S \in A(\bar\beta_1)$ and $y(k_1) \in D$. If $y(k) = y(k_1)$ then $k = k_1$ and $\bar\alpha \neq \bar\beta_1$, because in the opposite case $x(\bar\alpha) = x(\bar\beta_1)$. From this it follows that $A(\bar\alpha) \cap A(\bar\beta_1) = \emptyset$ which is a contradiction. Hence, $y(k) \neq y(k_1)$. Since $y(k)$ and $y(k_1)$ are distinct elements of the partition D, $y(k) \cap y(k_1) = \emptyset$. There exists an integer $t_0 \geqslant 0$ such that $\operatorname{st}(y(k_1), t_0) \cap y(k) = \emptyset$.

Let $S \in A(\bar\alpha)$. Since $A(\bar\alpha) \subseteq A(\bar\beta_1)$, there exists a uniquely determined element $\bar\gamma \in \Lambda_{n_1}$, $\bar\gamma \geqslant \bar\beta_1$ such that $S \in A(\bar\gamma)$. Since $x(\bar\alpha) \subseteq U(x(\bar\beta_1), n_1)$ and $y(k) \times \{S\} \subseteq x(\bar\alpha)$, we have $y(k) \times \{S\} \subseteq U(x(\bar\beta_1), n_1)$. By the structure of the set $U(x(\bar\beta_1), n_1)$ (see Property (8) of Lemma 3)) we have $y(k) \times \{S\} \subseteq \operatorname{st}(y(k_1), t(\bar\beta_1, \bar\gamma)) \times A(\bar\gamma)$. Hence, $y(k) \subseteq \operatorname{st}(y(k_1), t(\bar\beta_1, \bar\gamma))$. This means that $t(\bar\beta_1, \bar\gamma) \leqslant t_0$. Let

$$t = \max_{\bar\gamma}\{t(\bar\beta_1, \bar\gamma)\colon A(\bar\gamma) \cap A(\bar\alpha) \neq \emptyset\}.$$

Then $t \leqslant t_0$ and $\operatorname{st}(y(k_1), t) \supseteq y(k)$. There exists an integer $n \geqslant 0$ such that $\operatorname{st}(y(k), n) \subseteq \operatorname{st}(y(k_1), t)$. Hence, for every $\bar\gamma \geqslant \bar\beta_1$ for which $A(\bar\gamma) \cap A(\bar\alpha) \neq \emptyset$ we have $\operatorname{st}(y(k), n) \subseteq \operatorname{st}(y(k_1), t(\bar\beta_1, \bar\gamma))$ and, consequently

$$\operatorname{st}(y(k), n) \times A(\bar\gamma) \subseteq \operatorname{st}(y(k_1), t(\bar\beta_1, \bar\gamma)) \times A(\bar\gamma).$$

From this it follows that $d = x(\bar\alpha) \in O(\operatorname{st}(y(k), n) \times A(\bar\alpha)) \subseteq O(x(\bar\beta_1), n_1)$.

By the preceding there exists an integer $n(1)$ such that

$$O(x(\bar\alpha), n(1)) \subseteq O(\operatorname{st}(y(k), n) \times A(\bar\alpha)) \subseteq O(x(\bar\beta_1), n_1) = O_1.$$

Similarly, there exists an integer $n(2) \geqslant 0$ such that $O(x(\bar\alpha), n(2)) \subseteq O_2$. Let $n(0) = \max\{n(1), n(2)\}$ and set $O = O(x(\bar\alpha), n(0))$. Then,

$$d \in O = O(x(\bar\alpha), n(0)) \subseteq O(x(\bar\alpha), n(1)) \cap O(x(\bar\alpha), n(2)) \subseteq O_1 \cap O_2.$$

The proof of the lemma is complete.

In the remainder of this paper we let $T(A)$ have the topology having $O(A)$ for a basis of open sets. Obviously, the set $O(A)$ is finite or countable.

LEMMA 5. *The space $T(A)$ is a Hausdorff rational space.*

Proof. We prove that the space $T(A)$ is a Hausdorff space. Let $d_1, d_2 \in T(A)$ and $d_1 \neq d_2$. Consider the cases:

(1) $d_1 = \{(a_1, S_1)\}$, $d_2 = \{(a_2, S_2)\}$,

(2) $d_1 = \{(a_1, S_1)\}$, $d_2 = x(\bar\alpha)$, $\bar\alpha \in \Lambda_k$ and

(3) $d_1 = x(\bar\alpha_1)$, $\bar\alpha_1 \in \Lambda_{k_1}$ and $d_2 = x(\bar\alpha_2)$, $\bar\alpha_2 \in \Lambda_{k_2}$.

In the first case, either $a_1 \neq a_2$, or $S_1 \neq S_2$. If $a_1 \neq a_2$ (resp. $S_1 \neq S_2$) then there exist an integer $n \geqslant 0$ and elements $\bar\imath, \bar\jmath \in L_n$, $\bar\imath \neq \bar\jmath$ (resp. $\bar\alpha_1, \bar\alpha_2 \in \Lambda_n$, $\bar\alpha_1 \neq \bar\alpha_2$) such that $a_1 \in C_{\bar\imath}$ and $a_2 \in C_{\bar\jmath}$ (resp. $S_1 \in A(\bar\alpha_1)$ and $S_2 \in A(\bar\alpha_2)$). Set

$$U_1 = (C_{\bar\imath} \times A(\emptyset)) \cap J(A)$$

and $U_2 = (C_{\bar\jmath} \times A(\emptyset)) \cap J(A)$ (resp. $U_1 = (C_\emptyset \times A(\bar\alpha_1)) \cap J(A)$ and

$$U_2 = (C_\emptyset \times A(\bar\alpha_2)) \cap J(A).$$

Then $d_1 \in O(U_1)$, $d_2 \in O(U_2)$ and $O(U_1) \cap O(U_2) = \emptyset$.

In the second case, either $S_1 \notin A(\bar\alpha)$, or $a_1 \notin y(k)$. Let $S_1 \notin A(\bar\alpha)$ and $\bar\alpha_1$ be element of Λ_k for which $S \in A(\bar\alpha_1)$. Setting $U_1 = (C_\emptyset \times A(\bar\alpha_1)) \cap J(A)$ and $U_2 = (C_\emptyset \times A(\bar\alpha)) \cap J(A)$, we have $d_1 \in O(U_1)$, $d_2 \in O(U_2)$ and $O(U_1) \cap O(U_2) \neq \emptyset$.

Let $a_1 \notin y(k)$ and $n \geqslant 0$ be an integer for which $\operatorname{st}(\{a_1\}, n) \cap \operatorname{st}(y(k), n) = \emptyset$. Setting $U_1 = (\operatorname{st}(\{a_1\}, n) \times A(\emptyset)) \cap J(A)$ and $U_2 = (\operatorname{st}(y(k), n) \times A(\emptyset)) \cap J(A)$, we have $d_1 \in O(U_1)$, $d_2 \in O(U_2)$ and $O(U_1) \cap O(U_2) = \emptyset$.

Finally, in the third case, either $y(k_1) \cap y(k_2) = \emptyset$, or $y(k_1) \cap y(k_2) \neq \emptyset$ and $A(\bar\alpha_1) \cap A(\bar\alpha_2) = \emptyset$. If $y(k_1) \cap y(k_2) = \emptyset$ then there exists an integer $n \geqslant 0$ such that $\operatorname{st}(y(k_1)), n) \cap \operatorname{st}(y(k_2), n) = \emptyset$.

Set $U_1 = (\operatorname{st}(y(k_1), n) \times A(\emptyset)) \cap J(A)$ and $U_2 = (\operatorname{st}(y(k_2), n) \times A(\emptyset)) \cap J(A)$. Then, $d_1 \in O(U_1)$, $d_2 \in O(U_2)$ and $O(U_1) \cap O(U_2) = \emptyset$.

Let $A(\bar\alpha_1) \cap A(\bar\alpha_2) = \emptyset$. Setting $U_1 = (C_\emptyset \times A(\bar\alpha_1)) \cap J(A)$ and

$$U_2 = (C_\emptyset \times A(\bar\alpha_2)) \cap J(A),$$

we have $d_1 \in O(U_1)$, $d_2 \in O(U_2)$ and $O(U_1) \cap O(U_2) = \emptyset$. Thus, the space $T(A)$ is a Hausdorff space.

We prove that the space $T(A)$ is a rational space. For this it is sufficient to prove that if $U \in U(A)$ then $\operatorname{Fr} O(U) \subseteq \{d \in T(A)\colon U \cap d \neq \emptyset,\ (J(A) \setminus U) \cap d \neq \emptyset\}$. Let $d \in \operatorname{Fr} O(U)$. Then, $d \notin O(U)$, that is, $d \notin U$ and, hence, $(J(A) \setminus U) \cap d \neq \emptyset$.

Let $U = ((\bigcup_{\bar\imath \in l_n} C_{\bar\imath}) \times A(\bar\alpha)) \cap J(A)$, where l_n is an subset of L_n and $\bar\alpha \in \Lambda_k$. Let $d \in T(A)$. We show that if $d \cap U = \emptyset$ then there exists an element U_1 of $U(A)$ such that $d \in O(U_1)$ and $O(U_1) \cap O(U) = \emptyset$. Indeed, if $d = \{x\}$, $x = (a, S)$ then either $a \notin \bigcup_{\bar\imath \in l_n} C_{\bar\imath}$ or $a \in \bigcup_{\bar\imath \in l_n} C_{\bar\imath}$ and $S \in A(\bar\beta)$, where $\bar\beta \in \Lambda_k$, $\bar\beta \neq \bar\alpha$. In the first case, let $\bar\jmath \in L_n$ and $a \in C_{\bar\jmath}$. Setting $U_1 = (C_{\bar\jmath} \times A(\emptyset)) \cap J(A)$, we have $d \in O(U_1)$ and $O(U_1) \cap O(U) = \emptyset$. In the second case, let $U_1 = (C_\emptyset \times A(\bar\beta)) \cap J(A)$. Obviously, $d \in O(U_1)$ and $O(U_1) \cap O(U) = \emptyset$. Thus, in both cases $d \notin \operatorname{Fr} O(U)$.

Now, let $d = x(\bar\alpha_1)$, $\bar\alpha_1 \in \Lambda_{k_1}$. If $A(\bar\alpha_1) \cap A(\bar\alpha) = \emptyset$ then we set

$$U_1 = (C_\emptyset \times A(\bar\alpha_1)) \cap J(A).$$

If $A(\bar\alpha_1) \cap A(\bar\alpha) \neq \emptyset$ then $y(k)_1) \cap (\bigcup_{\bar\imath \in l_n} C_{\bar\imath}) = \emptyset$. Hence, there exists an integer $t \geqslant 0$ such that $\operatorname{st}(y(k_1), t) \cap (\bigcup_{\bar\imath \in l_n} C_{\bar\imath}) = \emptyset$. Set $U_1 = (\operatorname{st}(y(k_1), t) \times A(\bar\alpha_1)) \cap J(A)$. In both cases we have that $d \in O(U_1)$ and $O(U_1) \cap O(U) = \emptyset$, that is, $d \notin \operatorname{Fr} O(U)$.

Suppose that $U = U(x(\bar\alpha), n)$ where $\bar\alpha \in \Lambda_k$ and $n \geqslant k$. Let $d \in T(A)$ and $d \cap U = \emptyset$. First, let $d = \{x\}$, $x = (a, S)$. If $S \notin A(\bar\alpha)$ then there exists an element $\bar\beta \in \Lambda_k$, $\bar\beta \neq \bar\alpha$ such that $S \in A(\bar\beta)$. In this case we set $U_1 = (C_\emptyset \times A(\bar\beta)) \cap J(A)$.

If $S \in A(\bar{\alpha})$ then there exists $\bar{\gamma} \in \Lambda_n$ such that $S \in A(\bar{\gamma})$. In this case

$$a \notin \mathrm{st}(y(k), t(\bar{\alpha}, \bar{\gamma})) .$$

Let $\bar{j} \in L_{t(\bar{\alpha}, \bar{\gamma})}$ and $a \in C_{\bar{j}}$. Obviously, $C_{\bar{j}} \cap \mathrm{st}(y(k), t(\bar{\alpha}, \bar{\gamma})) = \emptyset$. Set

$$U_1 = (C_{\bar{j}} \times A(\bar{\gamma})) \cap J(A) .$$

In both cases $d \in O(U_1)$ and $O(U_1) \cap O(U) = \emptyset$, that is, $d \notin \mathrm{Fr}\, O(U)$.

Now, let $d = x(\bar{\alpha}_1)$, $\bar{\alpha}_1 \in \Lambda_{k_1}$. If $A(\bar{\alpha}_1) \cap A(\bar{\alpha}) = \emptyset$ then we set

$$U_1 = (C_{\emptyset} \times A(\bar{\alpha})) \cap J(A) .$$

If $A(\bar{\alpha}_1) \cap A(\bar{\alpha}) \neq \emptyset$ then it is easy to prove that for every $\bar{\beta} \in \Lambda_n$ for which

$$A(\bar{\beta}) \cap A(\bar{\alpha}_1) \neq \emptyset,\ \bar{\beta} \geqslant \bar{\alpha}$$

we have $\mathrm{st}(y(k_1), t(\bar{\alpha}, \bar{\beta})) \cap \mathrm{st}(y(k), t(\bar{\alpha}, \bar{\beta})) = \emptyset$. Let

$$t = \min_{\bar{\beta}} \{t(\bar{\alpha}, \bar{\beta}):\ \bar{\beta} \geqslant \bar{\alpha},\ \bar{\beta} \in \Lambda_n \text{ and } A(\bar{\beta}) \cap A(\bar{\alpha}) = \emptyset\} .$$

We have $\mathrm{st}(y(k_1), t) \cap \mathrm{st}(y(k), t) = \emptyset$. Set $U_1 = (\mathrm{st}(y(k_1), t) \times A(\bar{\alpha}_1)) \cap J(A)$. In both cases $d \in O(U_1)$ and $O(U_1) \cap O(U) = \emptyset$, that is, $d \notin \mathrm{Fr}\, O(U)$.

Thus, if $d \in \mathrm{Fr}\, O(U)$ then $d \cap U \neq \emptyset$ and $(J(A) \setminus U) \cap d \neq \emptyset$. The proof of the lemma is complete.

5. Lemma 6. *The space $T(A)$ is a regular space.*

Proof. Let $d \in T(A)$ and $O(U)$ be a neighbourhood of d. We shall find an element $U_1 \in U(A)$ such that $d \in O(U_1) \subseteq \mathrm{cl}(O(U_1)) \subseteq O(U)$.

Consider the cases:

(1) $d = \{(a, S)\}$, $U = ((\bigcup_{\bar{i} \in l_k} C_{\bar{i}}) \times A(\bar{\alpha})) \cap J(A)$, where $l_k \subseteq L_k$, $\bar{\alpha} \in \Lambda_n$,

(2) $d = (a, S)$, $U = U(x(a), n)$, $\bar{\alpha} \in \Lambda_k$, $k \leqslant n$.

(3) $d = x(\bar{\alpha})$, $\bar{\alpha} \in \Lambda_k$, $U = U(x(\bar{\alpha}), n)$, $n \geqslant k$,

(4) $d = x(\bar{\alpha}_1)$, $\bar{\alpha}_1 \in \Lambda_{n_1}$, $U = ((\bigcup_{\bar{i} \in l_k} C_{\bar{i}}) \times A(\bar{\alpha}_2)) \cap J(A)$, where $\bar{\alpha}_2 \in \Lambda_{n_2}$, $l_k \subseteq L_k$ and

(5) $d = x(\bar{\alpha}_1)$, $\bar{\alpha}_1 \in \Lambda_{n_1}$, $U = U(x(\bar{\alpha}_2), n)$, $\bar{\alpha}_2 \in \Lambda_{n_2}$, $n \geqslant n_2$, $x(\bar{\alpha}_1) \neq x(\bar{\alpha}_2)$.

In the first case we have $a \in \bigcup_{\bar{i} \in l_k} C_{\bar{i}}$ and $S \in A(\bar{\alpha})$. We may assume without loss of generality that $l_k = \{\bar{i}\}$. Consider the pair $(S, D) \in A$. S nce D is an upper semi-continuous partition, there exists an integer $k_1 \geqslant 0$ such that if

$$d'' \in D,\ d'' \cap \mathrm{st}(\{a\}, k_1) \neq \emptyset$$

then $d'' \subseteq C_{\bar{i}}$. There exists an integer $k_2 \geqslant 0$ such that $\mathrm{st}(\{a\}, k_2) \cap y(t) = \emptyset$ for every set $y(t)$, $0 \leqslant t \leqslant n$, for which $a \notin y(t)$. Let $m = \max\{k_1, k_2, n\}$, $\bar{j} \in L_m$, $a \in C_{\bar{j}}$, $\bar{\alpha}_1 \in \Lambda_m$ and $S \in A(\bar{\alpha}_1)$. Set $U_1 = (C_{\bar{j}} \times A(\bar{\alpha}_1)) \cap J(A)$. We shall prove that $d \in O(U_1) \subseteq \mathrm{cl}(O(U_1)) \subset O(U)$. Obviously, $d \in O(U_1) \subseteq O(U)$.

Let $d' \in \mathrm{cl}(O(U_1))$. By the proof of Lemma 5 we have $d' \cap U_1 \neq \emptyset$. If d' is a singleton then $d' \in O(U_1) \subseteq O(U)$. Hence, we may assume that $d' = x(\bar{\beta})$, $\bar{\beta} \in \Lambda_{n_1}$. Consider the case $n_1 \leqslant n$. Obviously, $A(\bar{\alpha}) \subseteq A(\bar{\beta})$. Let $a \notin y(n_1)$ then, by the choice of the element $\bar{j}$ of Λ_m, we have $y(n_1) \cap C_{\bar{j}} = \emptyset$, that is, $\mathrm{st}(y(n_1), m) \cap C_{\bar{j}} = \emptyset$. Set $U_2 = ((\mathrm{st}(y(n_1), m) \times A(\bar{\beta})) \cap J(A)$. Then, $d' \in O(U_2)$ and $O(U_2) \cap O(U_1) = \emptyset$, that is, $d' \notin \mathrm{cl}(O(U_1))$. Suppose that $a \in y(n_1)$. Since $\{(a, S)\}$ is an element of $T(A)$ and the different elements of $T(A)$ are disjoint, we have $\{(a, S)\} = x(\bar{\beta})$ which is a contradiction, because $y(n_1)$, and hence $x(\bar{\beta})$ is not a singleton.

Now, suppose that $n_1 > n$. Obviously, $A(\bar{\beta}) \subseteq A(\bar{\alpha})$. We prove that $y(n_1) \subseteq C_{\bar{i}}$. If $y(n_1) \nsubseteq C_{\bar{i}}$ then $\mathrm{st}(y(n_1), m) \nsubseteq C_{\bar{i}}$. Since $d \cap U_1 \neq \emptyset$, we have that $y(n_1) \cap C_{\bar{j}} \neq \emptyset$, that is, $C_{\bar{j}} \subseteq \mathrm{st}(y(n_1), m)$. Also, $A(\bar{\beta}) \cap A(\bar{\alpha}_1) \neq \emptyset$. Let $(S_1, D_1) \in A$ and $S_1 \in A(\bar{\beta}) \cap A(\bar{\alpha}_1)$. Then the sets S and S_1 belong to an element of the set $A(m)$.

Since $y(n_1)$ is an element of D_1, there exists an element d'' of D such that $\mathrm{st}(d'', m) = \mathrm{st}(y(n_1), m)$. This means that $d'' \cap C_{\bar{j}} \neq \emptyset$; hence $d'' \cap \mathrm{st}(\{a\}, k_1) \neq \emptyset$ and $d'' \nsubseteq C_{\bar{i}}$ which is a contradiction. Hence, $y(n_1) \subseteq C_{\bar{i}}$ and $d' \subseteq U$, that is, $d' \in O(U)$.

In the second case, there exists an element $\bar{\beta} \in \Lambda_n$ such that $(a, S) \in (\mathrm{st}(y(k), t(\bar{\alpha}, \bar{\beta})) \times A(\bar{\beta})) \cap J(A) \subseteq U$, that is, $d \in O(U') \subseteq O(U)$, where $U' = (\mathrm{st}(y(k), t(\bar{\alpha}, \bar{\beta})) \times \times A(\bar{\beta})) \cap J(A)$. Hence the case follows by the first case.

Consider the third case. There exists an integer $t \geqslant n$ such that if $0 \leqslant m \leqslant n$ and $y(k) \cap y(m) = \emptyset$ then $\mathrm{st}(y(k), t) \cap y(m) = \emptyset$. Set $U_1 = U(x(\bar{\alpha}), t+1)$. Let $d' \in \mathrm{cl}(O(U_1))$. Then $d' \cap U_1 \neq \emptyset$. Let $(S, D) \in A$, $\bar{\beta} \in \Lambda_n$, $\bar{\gamma} \in \Lambda_{t+1}$, $S \in A(\bar{\gamma})$, $A(\bar{\gamma}) \subseteq A(\bar{\beta})$ and $(a, S) \in d' \cap U_1$. If $d' = \{(a, S)\}$ then, obviously $d' \in O(U)$. Let $d' = x(\bar{\delta})$, $\bar{\delta} \in \Lambda_q$. By the structure of the set U_1 (see property (8) of Lemma 3) we have $a \in \mathrm{st}(y(k), t)$, and hence $\mathrm{st}(y(k), t) \cap y(q) \neq \emptyset$. Obviously, $y(k), y(q) \in D$. Hence either $q = k$, or $q > n$. In the first case, $d' \in O(U)$. In the second case, since $S \in A(\bar{\delta})$, we have $A(\bar{\delta}) \subseteq A(\bar{\beta})$. On the other hand,

$$(y(q) \times \{S\}) \cap U(x(\bar{\alpha}), t+1) \neq \emptyset .$$

By property (9) of Lemma 3 we have $y(q) \times \{S\} \subseteq U(x(\bar{\alpha}), t)$. This means that $y(q) \subseteq \mathrm{st}(y(k), t(\bar{\alpha}, \bar{\gamma}_1)) \subseteq \mathrm{st}(y(k), t(\bar{\alpha}, \bar{\beta}))$ where $\bar{\gamma}_1 \in \Lambda_t$ and $\bar{\beta} \leqslant \bar{\gamma}_1 \leqslant \bar{\gamma}$. Hence,

$$x(\bar{\delta}) = (y(q) \times A(\bar{\delta})) \cap J(A) \subseteq (y(q) \times A(\bar{\beta})) \cap J(A)$$
$$\subseteq (\mathrm{st}(y(k), t(\bar{\alpha}, \bar{\beta})) \times A(\bar{\beta})) \cap J(A) \subseteq U(x(\bar{\alpha}), n)$$

that is, $d' \in O(U)$.

In the fourth case we have

$$y(n_1) \subseteq \bigcup_{\bar{i} \in l_k} C_{\bar{i}} \text{ and } A(\bar{\alpha}_1) \subseteq A(\bar{\alpha}_2) .$$

There exists an integer $m \geqslant n_1$ such that $\mathrm{st}(y(n_1), m) \subseteq \bigcup_{\bar{i} \in l_k} C_{\bar{i}}$. Obviously,

$$(\mathrm{st}(y(n_1), m) \times A(\bar{\alpha}_1)) \cap J(A) \subseteq U .$$

Hence, $U(x(\bar{\alpha}_1), m) \subseteq U$. Thus, the case follows from the third case.

Consider the fifth case. Since

$$x(\bar{\alpha}_1) \neq x(\bar{\alpha}_2) \text{ and } A(\bar{\alpha}_1) \cap A(\bar{\alpha}_2) \neq \emptyset,\ y(n_1) \cap y(n_2) = \emptyset.$$

Hence there exists an integer t such that $\mathrm{st}(y(n_2), t) \cap y(n_1) = \emptyset$. If $\bar{\beta} \in \Lambda_n$ and $t(\bar{\alpha}_2, \bar{\beta}) \geqslant t$ then

$$\big((\mathrm{st}(y(n_2), t(\bar{\alpha}_2, \bar{\beta})) \times A(\bar{\beta})) \cap J(A)\big) \cap x(\bar{\alpha}_1) = \emptyset .$$

Hence there exist finitely many elements $\bar{\beta}_1, \ldots, \bar{\beta}_m$ of Λ_{n_2} such that

$$x(\bar{\alpha}_1) \subseteq \bigcup_{i=1}^{m} \big((\mathrm{st}(y(n_2), t(\bar{\alpha}_2\ \bar{\beta}_i)) \times A(\bar{\beta}_i)\big) \cap J(A)) .$$

Moreover $y(n_1) \subseteq \bigcap_{i=1}^{m} \mathrm{st}(y(n_2),\ t(\bar{\alpha}_2, \bar{\beta}_i))$ and $A(\bar{\alpha}_1) \subseteq \bigcup_{i=1}^{m} A(\bar{\beta}_i)$. Let q be an integer such that $\mathrm{st}(y(n_1), q) \subseteq \bigcap_{i=1}^{m} \mathrm{st}(y(n_2),\ t(\bar{\alpha}_2, \bar{\beta}_i))$. Then,

$$d \subseteq \big(\mathrm{st}(y(n_1), q) \times A(\bar{\alpha}_1) \cap J(A)\big) \subseteq \big(\bigcup_{i=1}^{m} \big((\mathrm{st}(y(n_2), t(\bar{\alpha}_2, \bar{\beta}_i)) \times A(\bar{\beta}_i))\big) \cap J(A) \subseteq U .$$

Thus, the case reduces to the fourth case. The proof of the lemma is complete.

Corollary 1. *The space $T(A)$ is a rational metrizable space.*

III. The existence of a universal element.

1. Let S be a subset of C and D an r-partition of S. For every element $\bar{j} \in L_n$, $n = 0, 1, \ldots$, by $D_{\bar{j}}$ we denote the set of all elements d of D for which $d \cap C_{\bar{j}0} \neq \emptyset$ and $d \cap C_{\bar{j}1} \neq \emptyset$ (we consider that $\emptyset 0 = 0$ and $\emptyset 1 = 1$). It is easy to see that $D_{\bar{j}}$ is a closed subset of the quotient space D.

Lemma 7. *For every rational space X there exists a subset $S(X)$ of C and an r-partition $D(X)$ of $S(X)$ such that:*

(1) *the space X is homeomorphic to a subset of the quotient space $D(X)$,*

(2) *$D^*(X)(1) \subseteq C(1)$, and*

(3) *if Y is a rational space and $D(X)(1) = D(Y)(1)$ then $S(X) = S(Y)$ and $D(X) = D(Y)$.*

Proof. By Theorem 1, there exist a subset $S_1(X)$ of C and an r-partition $D_1(X)$ of $S_1(X)$ such that the quotient space $D_1(X)$ is homeomorphic to X.

Since $D_1^*(X)(1)$ is a countable subset of C, there is a homeomorphism (see, for example, [1], Ch. 4, exercise 4.3. H(e)) $i(X)$ of C onto C such that

$$i(X)(D_1^*(X)(1)) \subseteq C(1) .$$

By $D_2(X)$ (resp. $D_2'(X)$) we denote the set of all subsets $i(X)(d)$ of C where $d \in D_1(X)$ (resp. $d \in D_1(X)(1)$). Also, we set $S_2(X) = i(X)(S_1(X))$. It is easy to see that:

(1) $D_2(X)$ is an r-partition of the set $S_2(X)$,

(2) the space X is homeomorphic to the quotient space $D_2(X)$, and

(3) $D_2'(X) = D_2(X)(1)$.

Set $S(X) = C \setminus \bigcup_{\bar{j} \in L} \big(\mathrm{cl}((D_2(X))_{\bar{j}}^*) \setminus D_2(X)_{\bar{j}}^*$.

By $D(X)$ we denote the union of the set $D_2(X)$ and the set of all singletons $\{x\}$ such that $x \in S(X) \setminus S_2(X)$. Obviously, the set $D(X)(1)$ is countable. A straightforward proof shows that $D(X)$ is an upper semi-continuous partition of $S(X)$. Hence $D(X)$ is an r-partition. It is easy to see that $D_2(X)$ is a subspace of $D(X)$ and $D(X)(1) = D_2(X)(1)$. Hence, $D^*(X)(1) \subseteq C(1)$.

Now, let Y be a rational space. Also, let $S(Y)$ be a subset of C and $D(Y)$ an r-partition of $S(Y)$ which are determined for the space Y as above.

Suppose that $D(X)(1) = D(Y)(1)$, that is, the set of all non-degenerate elements of the set $D(X)$ coincides with the set of all non-degenerate elements of the set $D(Y)$. For every element $\bar{j} \in L$, we have $D_{\bar{j}}(X) = D_{\bar{j}}(Y)$. Consequently, $D_2(X)_{\bar{j}} = D_2(Y)_{\bar{j}}$ and $\mathrm{cl}(D_2(X)_{\bar{j}}^*) = \mathrm{cl}(D_2(Y)_{\bar{j}}^*)$. From this and the determination of the sets $S(X)$ and $S(Y)$ we have $S(X) = S(Y)$. Hence, $D(X) = D(Y)$.

2. Let R be a family of spaces. An element T of R is called a *universal* element if T contains topologically every element of R.

Theorem 2. *In the family of all rational metrizable spaces there exists a universal element.*

Proof. For every rational space X, let $S(X)$ be a subset of C and $D(X)$ an r-partition of $S(X)$ which are determined in Lemma 7.

Let A be the family of all pairs $q = (S(X), D(X))$. We consider that if q and $r = (S(Y), D(Y))$ are different elements of A then either $S(X) \neq S(Y)$ or $D(X) \neq D(Y)$, and hence by property (3) of Lemma 7, $D(X)(1) \neq D(Y)(1)$. Therefore, the power of A is less than or equal to continuum and we can suppose that for every $\bar{\imath} \in L$ there exists a subset $A(\bar{\imath})$ of A such that all properties mensioned in II. 1 are satisfied.

Let $T(A)$ be the rational metrizable space constructed in Section II for the family A. We prove that the space $T(A)$ is a universal element for the family of all rational metrizable spaces.

Indeed, let X be a rational space. In the space $T(A)$ consider the set $T(S(X))$ of all elements d of $T(A)$ such that $d \cap ((C \times \{S(X)\}) \cap J(A)) \neq \emptyset$.

We prove that there exists a homeomorphism $p(X) = p$ of $D(X)$ onto $T(S(X))$.

Let $d' \in D(X)$. Then, either $d' = \{a\}$, where $a \in S(X) \setminus D^*(X)(1)$, or $d' \in D(X)(1)$.

If $d' = \{a\}, a \in S(X) \setminus D^*(X)(1)$, then the set $\{(a, S(X))\}$ is an element of $T(A)$. Indeed, in the opposite case there exist an integer $n \geqslant 0$ and $\bar{\alpha} \in \Lambda_n$ such that $(a, S(X)) \in x(\bar{\alpha}) \neq \emptyset$. This means that $S(X) \in A(\bar{\alpha})$, $y(n) \in D(X)$ and $a \in y(n)$. This is a contradiction because $a \notin D^*(X)(1)$ and $y(n) \subseteq D^*(X)(1)$. Set

$$p(d') = \{(a, S(X))\} .$$

Let $d' \in D(X)(1)$. There exists an integer $n > 0$ such that $d' = y(n)$. Also, there exists a uniquely determined element $\bar{\alpha}$ of Λ_n such that $S(X) \in A(\bar{\alpha})$. We set $p(d') = x(\bar{\alpha})$. Thus, we define the map p of $D(X)$ into $T(S(X))$.

We prove that p is "onto". Let $d \in T(S(X))$. If $d = \{(a, S)\}$, then $S = S(X)$ and $a \in S(X)$. We have that $a \notin D^*(X)(1)$. Indeed, in the opposite case, there exists an integer $n > 0$ such that $y(n) \in D(X)(1)$ and $a \in y(n)$. Let $\bar{\alpha} \in \Lambda_n$ and $S(X) \in A(\bar{\alpha})$. Then $(a, S) = (a, S(X)) \in x(\bar{\alpha})$, which is a contradiction because $x(\bar{\alpha})$ is not a singleton and $d \cap x(\bar{\alpha}) \neq \emptyset$. Hence, $\{a\} \in D(X)$ and $p(\{\alpha\}) = d$.

Let $d = x(\bar{\alpha})$, where $\bar{\alpha} \in \Lambda_n$. Then, $S(X) \in A(\bar{\alpha})$ and $y(n) \in D(X)$. Hence, $p(y(n)) = d$. Thus, p is "onto".

We prove that p is "one-to-one". Let d_1 and d_2 be different elements of $D(X)$. Suppose that $p(d_1) = p(d_2)$. Then, there exists $a \in S(X)$ such that

$$(a, S(X)) \in p(d_1) \cap p(d_2).$$

Hence, $a \in d_1 \cap d_2$ which is a contradiction. Thus, p is "one-to-one".

We prove that p is a homeomorphism. Let $p(d') = d$ and let $O(U)$ be a neighbourhood of d in the space $T(A)$. There exists an element $U_1 \in U(A)$ such that $d \in O(U_1) \subseteq \mathrm{cl}(O(U_1)) \subseteq O(U)$. Let V' be the open subset of $S(X)$ for which $V' \times \{S(X)\} = U_1 \cap (C \times \{S(X)\})$. Let V be the set of all elements of $D(X)$ which are contained in the set V'. Since $D(X)$ is upper semi-continuous, V is an open neighbourhood of d' in $D(X)$. We prove that $p(V) \subseteq O(U)$. Indeed, let $d'' \in V$. Then, $d'' \subseteq V'$. This means that $p(d'') \cap U_1 \neq \emptyset$. Hence, $p(d'') \in \mathrm{cl}(O(U_1))$ and consequently $p(d'') \in O(U)$, that is, the map p is continuous.

Conversely, let $p^{-1}(d) = d'$ and let V be an open neighbourhood of d' in $D(X)$. We may consider that there is an open set V' of $S(X)$ such that the set V consists of all elements of $D(X)$ which are contained in the set V'. Since $D(X)$ is upper semi-continuous, there exists an integer k such that $\mathrm{st}(d', k) \subseteq V'$ and if $d'' \in D(X)$ and $d'' \cap \mathrm{st}(d', k) \neq \emptyset$ then $d'' \subseteq V'$. Let $U = O(\mathrm{st}(d', k) \times A(\emptyset)) \cap T(S(X))$. Obviously, U is an open neighbourhood of d in $T(S(X))$. We prove that $p^{-1}(U) \subseteq V$. Indeed, let $d_1 \in U$. Then $p^{-1}(d_1) \cap \mathrm{st}(d', k) \neq \emptyset$. Hence, $p^{-1}(d_1) \subseteq V'$, and, consequently $p^{-1}(d_1) \in V$, that is, the map p^{-1} is continuous.

Thus, p is a homeomorphism of $D(X)$ onto the subset $T(S(X))$ of $T(A)$. Hence, X is homeomorphic to a subset of $T(A)$, that is, $T(A)$ is a universal element of the family of all rational metrizable spaces. The proof of the theorem is complete.

3. Let R be a family of spaces and R_1 be a subfamily of R. We say that a universal element T of R has the *property of finite intersection* with respect to the subfamily R_1 if for every element $X \in R_1$ there exists a fixed homeomorphism i_X of X into T such that if X and Y are different elements of R_1 then the set $i_X(X) \cap i_Y(Y)$ is finite.

COROLLARY 2. *In the family of all rational spaces there exists a universal element having the property of finite intersection with respect to a given subfamily with power less than or equal to the continuum.*

Proof. Let R be the family of all rational spaces and R_1 be the given subfamily. Let A be the family of pairs $(S(X), D(X))$ mentioned in the proof of Theorem 2.

For every space $X \in R_1$ we consider a pair $(\hat{S}(X), \hat{D}(X))$, where $\hat{S}(X)$ is a subset of C and $\hat{D}(X)$ is an r-partition of $\hat{S}(X)$ such that: the quotient space $\hat{D}(X)$ is homeomorphic to X and $(\hat{D}(X)(1))^* \subseteq C(1)$. By A_1 we denote the set of all pairs $(\hat{S}(X), \hat{D}(X))$, $X \in R_1$. We suppose that if X and Y are two different elements of A_1 (it is possible that X and Y are homeomorphic) then $(\hat{S}(X), \hat{D}(X))$ and $(\hat{S}(Y), \hat{D}(Y))$ are different elements of A_1

Let A' be the free union of the sets A and A_1. Obviously, the power of the set A' is less than or equal to the continuum. Hence we can consider the space $T(A')$ constructed in Section II for the family A'. As in the proof of Theorem 2 we can prove that if the element $(S'(X), D'(X))$ of A' correspond to the element X of R or R_1 then the subset $T(S'(X))$ (see the proof of Theorem 2) of $T(A')$ contains topologically the space X.

In order to prove the corollary, it is sufficient to prove that if X and Y are different elements of A_1 then the set $T(\hat{S}(X)) \cap T(\hat{S}(Y))$ is finite.

Let X and Y are different elements of A_1. Since $\hat{S}(X)$ and $\hat{S}(Y)$ are different elements of $A'(S)$, there exists an integer $n > 0$ and elements $\bar{\alpha}, \bar{\beta} \in \Lambda_n$, $\bar{\alpha} \neq \bar{\beta}$, such that $\hat{S}(X) \in A'(\bar{\alpha})$ and $\hat{S}(Y) \in A'(\bar{\beta})$. From this it follows that if $d \in T(A')$, $(C \times A'(\bar{\alpha})) \cap d \neq \emptyset$ and $d \cap (C \times A'(\bar{\beta})) \neq \emptyset$ then $d = x(\bar{\gamma})$ where $\bar{\gamma} \in \Lambda$, $\bar{\alpha} \geqslant \bar{\gamma}$ and $\bar{\beta} \geqslant \bar{\gamma}$. Obviously, the number of such elements, d is finite. Hence the set $T(\hat{S}(X)) \cap T(\hat{S}(Y))$ is finite. The proof of the corollary is complete.

We observe that the power of R_1 cannot exceed $2^{\aleph_0}$ and that there are $2^{2^{\aleph_0}}$ rational metrizable spaces.

COROLLARY 3. *In the family of all rational spaces there exists a universal element having the property of finite intersection with respect to the subfamily of all rational continua.*

COROLLARY 4. *In the family of all rational spaces there exists a universal element having the property of finite intersection with respect to the family of all closed subspaces of a given rational space.*

4. PROBLEM 1. Is there a universal element in the family of all Hausdorff rational spaces?

PROBLEM 2. Is there a universal element in the family of all plane rational spaces?

PROBLEM 3. Is there a rational space which contains toplogically every rational continuum and which is not a universal element in the family of all rational spaces?

Let N be a cardinal number. We say that a Hausdorff space X is a *N-space* iff there exists a basis B of open sets of X such that

(1) $|B| \leqslant N$ and

(2) for every element $U \in B$ we have $|\mathrm{Fr}\, U| \leqslant N$.

PROBLEM 4. Is there a universal element in the family of all regular (resp. Hausdorff) N-spaces?

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On symmetric products

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Abstract. Two notions $X(n)$ and $\mathrm{SP}\, X_G^n$ of symmetric products of a Hausdorff compact space X are studied. The n-fold symmetric product $X(n)$ is a subspace of the hyperspace 2^X of subsets of X containing at most n points. For a group G of permutation of a set of n elements, the n-fold G-symmetric product $\mathrm{SP}_G^n\, X$ is the orbit space of the permutation action of G on the n-fold cartesian product X^n of X. It is proved that some shape properties are invariants under the operation of these products. An example shows that the fixed point property is not such an invariant (this is the negative answer to the Borsuk and Ulam problem [1]). Examples of the symmetric product of some one-dimensional continua are considered.

1. Introduction. In the paper, compact Hausdorff spaces are considered. For a space X, let 2^X denote the space of closed subsets of X with the Vietoris finite topology. For a metric space one can get the same topology by using the Hausdorff metric. The n-fold symmetric product $X(n)$ of the space X is the subspace of 2^X of subsets of X containing at most n points ([1]). The space $X(n)$ can be obtained ([6], [18]) as a quotient space of the cartesian product X^n with the following relation: two points $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in X^n$ are equivalent if the sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are equal. Denote the natural projection by $\pi_n\colon X^n \to X(n)$.

Let G be a group of permutations of a set of n elements. The n-fold G-symmetric product $\mathrm{SP}_G^n X$ ([17], [8], [5]) of a space X is the orbit space of the permutation action of G on the cartesian product X^n of X. Let $\pi_G^n\colon X^n \to \mathrm{SP}_G^n X$ denote the identification map. Thus $\pi_G^n(x_1, \ldots, x_n) = \pi_G^n(y_1, \ldots, y_n)$ iff for some $g \in G$ $y_i = x_{g(i)}$ for $i = 1, \ldots, n$. If G is the group of all permutations of a set of n elements then SP_G^n is denoted by SP^n. It is easy to see that $\mathrm{SP}^2 X = X(2)$ for any space X.

Suppose that π is one of the maps π_n or π_G^n. Let $f\colon X \to Y$ be a map. The map $\times f\colon X^n \to X^n$ defined by $f(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$ preserves fibers of the map π. Hence we can define the map $\pi(f)\colon \pi(X^n) \to \pi(Y^n)$ such that the diagram

$$\begin{array}{ccc} X^n & \xrightarrow{\ f\ } & Y^n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \pi(X^n) & \xrightarrow{\ \pi(f)\ } & \pi(Y^n) \end{array}$$