A similar but even simpler reasoning combined with Theorem 2.1 in [K] gives the following.

3.4. COROLLARY. *Let* $p: I^{n+1} \to I$ *be projection onto the first factor. Then there exists a continuum* $A \subset I^{n+1}$ *such that*

(i) $p(A) = I$,
(ii) $\dim A = n$,
(iii) $Y \subset A \;\&\; p(Y) \supset C \Rightarrow \dim Y = n$.

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Characterizing strong countable-dimensionality in terms of Baire category

by

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Abstract. The main result of this paper is the following theorem: a metrizable compactum X is a countable union of finite-dimensional compacta (i.e. X is strongly countable-dimensional) if and only if for almost every continuous mapping $f: X \to I^{\omega}$ into the Hilbert cube I^{ω} we have $f(X) \cap Q^{\aleph_0} = \emptyset$, where $Q^{\aleph_0}$ is the product of the rationals. We give also a characterization of strongly countable-dimensional compacta in terms of the Baire category in the space of cuts defined by Hurewicz.

1. Introduction. In this paper we show that metrizable compacta which are countable unions of finite-dimensional compacta (i.e. strongly countable-dimensional compacta) can be characterized in terms of Baire category in the function spaces or the spaces of cuts introduced by Hurewicz. Let us describe the results in some more details.

Let us recall that X is *countable-dimensional* if X is the union of countably many zero-dimensional subspaces. A theorem of Nagata (see [8], Theorem V.5 and its Corollary) states that the subspace N_{ω} of the Hilbert cube I^{ω}, consisting of all points having only finitely many rational coordinates, is universal for countable-dimensional metrizable separable spaces X, i.e. any such X can be embedded in N_{ω}. In [9], we strengthened this result to the effect that the embeddings of a metrizable separable countable-dimensional space X into N_{ω} form a dense subset in the function space $C(X, I^{\omega})$ of all continuous mappings of X into I^{ω}, endowed with the sup-metric. However, in contrast to the classical finite-dimensional case, generally, the set of embeddings is not residual.

More specifically, the main result of this paper is that the set of embeddings of a metrizable compactum X into N_{ω} is residual (equivalently, is of the second category) in $C(X, I^{\omega})$ if and only if X is strongly countable-dimensional. Natural examples of countable-dimensional compacta which are not strongly countable-dimensional can be found in [1], Chapter 10, § 3 or [3], Example 1.12.

This theorem provides a characterization of strongly countable-dimensional compacta in terms of the function space. An "internal" characterization of strongly countable-dimensional compacta in terms of the Baire category in the space of cuts defined by Hurewicz is given in Section 5. This result corresponds also to a characterization of countable-dimensional spaces by sequences of partitions given by Nagata (see Section 5 for details).

Finally, we extend in Section 4 some results from Section 3 to the case of non-separable completely metrizable spaces.

2. Terminology and notation. Our terminology follows [2] and [8].

2.1. By the *dimension* we understand the covering dimension dim. By a *compactum* we mean a compact metric space. A space X is *countable-dimensional* (*strongly countable-dimensional*) if X is the union of countably many zero-dimensional (finite-dimensional and closed) subspaces.

2.2. By I we denote the unit interval $[0, 1]$, by I^ω — the Hilbert cube, by Q — the set of rationals in I and by $Q^{\aleph_0}$ — the countable power of Q. If τ is a cardinal number, then $S(\tau)$ denotes the hedgehog space of spininess τ (see [8], Definition VI.6). We consider I^ω with a fixed metric $\varrho(x, y) = \left(\sum_{i=1}^{\infty} \frac{1}{2^i}|x_i-y_i|^2\right)^{1/2}$, where $x = (x_i)_{i=1}^{\infty}$, $y = (y_i)_{i=2}^{\infty}$ and I^n with a fixed metric $\varrho_n(x, y) = \left(\sum_{i=1}^{n} \frac{1}{2^i}|x_i-y_i|^2\right)^{1/2}$, where $x = (x_i)_{i=1}^{n}$, $y = (y_i)_{i=1}^{n}$. By $p_n\colon I^\omega \to I^n$ we denote the projection onto the first n coordinates. In the countable power $S(\tau)^{\aleph_0}$ we consider the fixed metric $\sigma(x, y) = \left(\sum_{i=1}^{\infty} \frac{1}{2^i}\varrho(x_i, y_i)^2\right)^{1/2}$, where $x = (x_i)_{i=1}^{\infty}$, $y = (y_i)_{i=1}^{\infty} \in S(\tau)^{\aleph_0}$ and ϱ is a metric defined in Definition VI.6 in [8].

2.3. We denote by N_ω (respectively, $K_\infty(\tau)$) the Nagata's countable-dimensional universal space of weight $\aleph_0$ (respectively, of weight τ) consisting of all points in I^ω (respectively, in $S(\tau)^{\aleph_0}$) having only finitely many rational coordinates. By N_n^ω we denote the set of all points in I^ω, having at most n rational coordinates, and by $K_n(\tau)$ — the set of all points in $S(\tau)^{\aleph_0}$, having at most n rational coordinates different from 0.

2.4. If x is a point in a metric space X with a fixed metric ϱ then for $\varepsilon > 0$ the symbol $B(x, \varepsilon)(\bar{B}(x, \varepsilon))$ denotes the open (closed) ball with a center x of radius ε. If $A \subset X$, then $B(A, \varepsilon)$ is an open ball of radius ε around A,

$$\operatorname{diam} A = \sup\{\varrho(x, y)\colon x, y \in A\}$$

is the diameter of A and $\bar{A}$ denotes the closure of A.

2.5. By B^n we denote the closed unit ball in I^n, by S^{n-1} — the $n-1$-dimensional sphere being the boundary of B^n and by ∂I^n — the boundary of I^n. A mapping f of X into B^n (into I^n) is *essential* (see [8], Definition III.5 and III.3.E) if and only if every continuous mapping g of X into B^n (into I^n) which coincides with f on $f^{-1}(S^{n-1})$ (on $f^{-1}(\partial I^n)$) satisfies $g(X) = B^n$ ($g(X) = I^n$). Recall that $\dim X \geqslant n$ if and only if there exists an essential mapping $f\colon X \to I^n$ (see [8], Theorem III.5).

A family $\{(A_i, B_i)\}_{i=1}^n$ of pairs of closed disjoint subsets of X is called *essential* if for every family $\{F_i\}_{i=1}^n$ of closed subsets of X such that F_i is a partition between A_i and B_i we have $\bigcap_{i=1}^n F_i \neq \emptyset$. By a theorem on partitions (see [2], Theorem 1.7.9), $\dim X \geqslant n$ if and only if there exists a family $\{(A_i, B_i)\}_{i=1}^n$ which is essential in X.

2.6. A set $A \subset X$ is of the *first category* in X, if it is the union of countably many nowhere-dense subsets of X. A set is of the *second category* if it is not of the first category. A subset A of X is *residual*, if $X \setminus A$ is a first category set in X (in a complete space X, A is residual if and only if it contains a dense G_δ-subset of X).

2.7. Given a space X and a metric space Y with a fixed bounded metric ϱ, we denote by $C(X, Y)$ the space of all continuous mappings of X into Y endowed with the metric $d(f, g) = \sup\{\varrho(f(x), g(x))\colon x \in X\}$; note that if ϱ is a complete metric in X, then d is a complete metric in $C(X, Y)$.

2.8. We say that a two-element collection $\{K, L\}$ of closed subsets of a space X is a *cut* in X if $\overline{X \setminus K} = L$ and $\overline{X \setminus L} = K$ (see [4]). We use the symbol $K|L$ to denote that $\{K, L\}$ is a cut. We say that a cut $K|L$ in X *separates* a pair (A, B) of disjoint closed subsets of X if A is contained in one of the sets K, L, the set B is contained in the other one and the intersection $K \cap L$ is disjoint from both A and B. Note that if $K|L$ separates (A, B), then $K \cap L$ is a partition between A and B. A sequence $\{K_i|L_i\}_{i=1}^{\infty}$ of cuts in a space X is point-finite if for every $x \in X$ there are at most finitely many indices i with $x \in K_i \cap L_i$.

2.9. The *Hurewicz space* $S(X)$ of cuts of a space X (see [4]) is the set of all cuts in the space X with the topology defined by a base consisting of all the sets of the form $S_X(A, B) = \{K|L\colon K|L$ separates a pair $(A, B)\}$, where A and B are disjoint closed subsets of X. In $S_X(A, B)$ we will consider a topology of a subspace of the space $S(X)$. Hurewicz proved [4] that if X is a compactum then $S(X)$ (and every $S_X(A, B)$) is a completely metrizable space.

3. Residuality of the set of embeddings into N_ω characterizes strongly countable-dimensional compacta. In this section we will prove the following characterization theorem:

3.1. THEOREM. *For a compactum X the following conditions are equivalent*:

(i) *the set $\mathcal{H} = \{h \in C(X, I^\omega)\colon h$ is a homeomorphic embedding and $h(X) \subset N_\omega\}$ is residual in $C(X, I^\omega)$*,

(ii) *the set $\mathcal{F} = \{h \in C(X, I^\omega)\colon h(X) \cap Q^\infty = \emptyset\}$ is of second category in $C(X, I^\omega)$*,

(iii) *X is strongly countable-dimensional.*

The proof of this theorem will be proceeded by a proposition. The proposition will also be applied in Section 4 and is stated in a more general form than needed here.

3.2. Proposition. *Let X be a complete metric space, which is not strongly countable-dimensional. Then every G_δ-subset of $C(X, I^\omega)$ which is dense in some open subset of $C(X, I^\omega)$ contains a mapping $f \in C(X, I^\omega)$ such that $f(X) \cap Q^\infty \neq \emptyset$.*

First let us recall the following well-known lemma.

3.3. Lemma (see [1], Ch. IV, § 6, Main Lemma to Theorem 14 or [8], III.2.A). *Let f: $X \to I^n$ be a continuous mapping such that $f(X) \subset B^n = \bar{B}(y, \varepsilon)$ for some $y \in I^n$ and $\varepsilon > 0$ and the mapping f: $X \to B^n$ is essential. Then there exists $\delta > 0$ such that for every g: $X \to I^n$ satisfying $\varrho_n(g(x), f(x)) < \delta$ for every $x \in X$ we have $g(X) \ni y$.*

3.4. Remark. Let us recall that given a mapping f of X into a metric space Y, a point $y \in f(X)$ is called a *stable value of f* if there exists $\delta > 0$ such that for every mapping g: $X \to Y$ such that $\varrho(g(x), f(x)) < \delta$ for every $x \in X$, $g(X) \ni y$ (cf. [6], Ch. VI.1 and [8], Ch.III.1). Lemma 3.3 states that if f is an essential mapping of X to the closed n-dimensional ball B^n, then the center of B^n is a stable value of f.

We will apply Lemma 3.3 in a special situation described in the next lemma.

3.5. Lemma. *For $(x_1, x_2, \ldots) \in I^\omega$ let $B^n = \bar{B}((x_1, \ldots, x_n), \varepsilon) \subset I^n$ and $\tilde{B}^n = B^n \times (x_{n+1}, x_{n+2}, \ldots) \subset I^\omega$ be n-dimensional balls. Suppose that f: $X \to I^\omega$ is a mapping such that for some $Y \subset X$ we have $f(Y) \subset \tilde{B}^n$ and the mapping $f|Y$: $Y \to \tilde{B}^n$ is essential. Then there exists $\delta > 0$ such that for every $g: X \to I^\omega$ satisfying $d(g, f) < \delta$ we have $p_n \circ g(Y) \ni (x_1, \ldots, x_n)$.*

Proof. Put $f' = p_n \circ f|Y$; then $f'(Y) \subset B^n$ and the mapping f': $Y \to B^n$ is essential. Take $\delta > 0$ satisfying the assertion of Lemma 3.3 for f' and let g: $X \to I^\omega$ be such that $d(g, f) < \delta$. Put $g' = p_n \circ g|Y$; then $\varrho_n(g'(x), f'(x)) < \delta$ for every $x \in Y$ and hence $g'(Y) = p_n \circ g(Y) \ni (x_1, \ldots, x_n)$.

Proof of Proposition 3.2. Let $\mathscr{G}$ be a G_δ-subset of $C(X, I^\omega)$ which is dense in some open subset $\mathscr{U}$ of $C(X, I^\omega)$. We will show that $\mathscr{G}$ contains a mapping h such that $h(X) \cap Q^\infty \neq \emptyset$. Let $\mathscr{G} \supset \bigcap_{i=1}^{\infty} \mathscr{G}_i$, where $\mathscr{G}_1 \supset \mathscr{G}_2 \supset \ldots$ is a sequence of open and dense in $\mathscr{U}$ subsets of $C(X, I^\omega)$. Since X is not strongly countable-dimensional, by the standard kernel construction (cf. [12]) it contains a non-empty closed subset Y such that each non-empty open subset of Y is infinite-dimensional. By induction we will construct a sequence $\{\varepsilon_i\}_{i=1}^\infty$ of positive real numbers, a sequence $\{h_i\}_{i=1}^\infty$ of mappings of X into I^ω, a sequence $\{q_i\}_{i=1}^\infty$ of rational numbers from I and a sequence $\{X_i\}_{i=1}^\infty$ of closed subsets of Y such that for every $i \in N$ the following conditions are satisfied:

(1) $\varepsilon_i < 2^{-i}$ and $\varepsilon_i < \varepsilon_{i-1}$ for $i > 1$,

(2) $B(h_i, \varepsilon_i) \subset \mathscr{G}_i$ and $\bar{B}(h_i, \varepsilon_i) \subset B(h_{i-1}, \varepsilon_{i-1})$ for $i > 1$,

(3) $\operatorname{diam} X_i < 2^{-i}$ and $X_i \subset B(X_{i-1}, 2^{-i})$ for $i > 1$, and

(4) for every $g \in B(h_i, \varepsilon_i)$ we have $(q_1, \ldots, q_i) \in p_i \circ g(X_i)$ (where p_i: $I^\omega \to I^i$ is the projection).

Inductive construction. Take an arbitrary $h_0 \in \mathscr{G}$ and put $\varepsilon_0 = 1$ and $X_0 = Y$. Assume that $m = 1$ or that ε_i, h_i, q_i and X_i are already defined for $i < m$. Since $\mathscr{G}_m$ is open and dense in $\mathscr{U}$, there exists $f \in \mathscr{U}$ and $\eta > 0$ such that

$$B(f, \eta) \subset B(h_{m-1}, \varepsilon_{m-1}) \cap \mathscr{G}_m .$$

By (4) with $i = m-1$ there exists a point $y = (q_1, \ldots, q_{m-1}, x_m, x_{m+1}, x_{m+2}, \ldots) \in f(X_{m-1})$, where $q_1, \ldots, q_{m-1}$ are already defined rational numbers and x_i are some real numbers for $i \geqslant m$. Choose a rational number q_m such that $|x_m - q_m| < \frac{\eta}{4}$ and let

$$y' = (q_1, \ldots, q_m, x_{m+1}, x_{m+2}, \ldots) ;$$

we have then $y' \in B\left(y, \frac{\eta}{4}\right) = U$. Take $x \in f^{-1}(y) \cap X_{m-1}$; since $x \in f^{-1}(U)$, there exists $0 < R < 2^{-i}$ such that $X_m = (x\bar{B}, R) \cap Y \subset f^{-1}(U)$. Denote

$$B^m = \bar{B}\left((q_1, \ldots, q_m), \frac{\eta}{8}\right) \times (x_{m+1}, x_{m+2}, \ldots) ;$$

B^m is a closed m-dimensional ball contained in U. Since every non-empty open subset of Y is infinite-dimensional, we have $\dim X_m \geqslant m$, and hence there exists an essential mapping k: $X_m \to B^m$. Extend k to a mapping h_m: $X \to I^\omega$ such that $h_m|X_m = k$, $h_m(f^{-1}(\bar{U})) \subset \bar{U}$ and $h_m|f^{-1}(I^\omega \setminus U) = f|f^{-1}(I^\omega \setminus U)$. It is easy to see that $d(h_m, f) \leqslant \operatorname{diam} \bar{U} \leqslant \frac{\eta}{2}$. By Lemma 3.5 there exists $\delta_1 > 0$ such that for every g: $X \to I^\omega$ satisfying $d(g, h_m) < \delta_1$ we have $p_m \circ g(X_m) \ni (q_1, \ldots, q_m)$. Take $\delta_2 > 0$ such that $\bar{B}(h_m, \delta_2) \subset B(f, \eta)$ and put $\varepsilon_m = \min(\delta_1, \delta_2, \varepsilon_{i-1}, 2^{-i})$. Then ε_m, h_m, q_m and X_m satisfy conditions (1)–(4) with $i = m$. The inductive construction is complete.

Now, from the completeness of the space $C(X, I^\omega)$ it follows that there exists $h \in \bigcap_{i=1}^{\infty} \bar{B}(h_i, \varepsilon_i)$. By (5) for each $i = 1, 2, \ldots$ there exists a point $y_i \in X_i$ such that $p_i \circ h(y_i) = (q_1, \ldots, q_i)$. By (3), $\{y_i\}_{i=1}^\infty$ is a Cauchy sequence in the complete space X; hence it is convergent to some $y_0 \in X$. We have $h(y_0) = (q_1, q_2, \ldots) \in Q^\infty$, thus $h(X) \cap Q^\infty \neq \emptyset$.

3.6. Remark. Let us notice that Proposition 3.2 remains true if I^ω is replaced by a space $S(\tau)^{\aleph_0}$, where τ is an arbitrary cardinal number, and Q^∞ is replaced by the set A of all points in $S(\tau)^{\aleph_0}$, which have all coordinate rational and different from 0. To see this it suffices to make the following minor changes in the proof of Theorem 3.2: in notation of this proof, we choose $q_i \neq 0$ for $i = 1, 2, \ldots$ and we put $B^m = \bar{B}((q_1, \ldots, q_m), r) \times (x_{m+1}, x_{m+2}, \ldots)$, where $r < \min\left\{\frac{\eta}{8}, |q_1|, \ldots, |q_m|\right\}$; we also use the fact that $S(\tau)^{\aleph_0}$ is an absolute extensor.

Proof of Theorem 3.1. Since $\mathscr{H}\subset\mathscr{F}$, (i) $\Rightarrow$ (ii). The proof of the implication (iii) $\Rightarrow$ (i) follows from the classical embedding theorems: if $X=\bigcup_{n=1}^{\infty}F_n$, where $F_n=\overline{F}_n$ and $\dim X_n\leqslant n$ for $n=1,2,\ldots$, then the set $\mathscr{F}_n=\{h\in C(X,I^{\omega})\colon h$ is a homeomorphic embedding and $h(F_n)\subset N_n^{\omega}\}$ is residual in $C(X,I^{\omega})$ for every $n=1,2,\ldots$ (see [7], § 45, VII, Remark (ii) after Theorem 1 and § 44, VI. Theorem 2). Thus the set $\mathscr{H}$, containing $\bigcap_{n=1}^{\infty}\mathscr{F}_n$, is residual in $C(X,I^{\omega})$. It remains to prove that (ii) $\Rightarrow$ (iii). Suppose that X is not strongly countable-dimensional. We will show that $\mathscr{F}$ is a first category set. First, let us observe that the set $\mathscr{F}$ is coanalytic (see [7], § 39 for this notion). Indeed, $C(X,I^{\omega})\setminus\mathscr{F}=\{f\in C(X,I^{\omega})\colon f(X)\cap\cap Q^{\infty}\neq\emptyset\}=p_1\big(\{(f,t)\in C(X,I^{\omega})\times I^{\omega}\colon t\in Q^{\infty}, t\in f(X)\}\big)=p_1\big((C(X,I^{\omega})\times Q^{\infty})\cap\cap\{(f,t)\colon t\in f(t)\}\big)$, where $p_1\colon C(X,I^{\omega})\times I^{\omega}\to C(X,I^{\omega})$ is the projection. Thus $C(X,I^{\omega})\setminus\mathscr{F}$ is a continuous image of a Borel set in a complete space $C(X,I^{\omega})\times I^{\omega}$, and hence it is analytic. Now, by Proposition 3.2, for every open set U in $C(X,I^{\omega})$ the set $U\cap\mathscr{F}$ is not residual in U, i.e. the set $U\setminus\mathscr{F}$ is not of the first category. Since the set $C(X,I^{\omega})\setminus\mathscr{F}$, being analytic, has the Baire property and is not of the first category at any point of $C(X,I^{\omega})$, it follows that its complement $\mathscr{F}$ is a first category set (see [7], § 11, IV, Corollary 2). This ends the proof.

4. Some extensions to the non-separable case. Theorem 3.1 can be generalized to non-separable spaces in two ways: we can consider the set of continuous mappings of a given space X into N_{ω} instead of embeddings or we can consider the set of embeddings into the Nagata universal space $K_{\infty}(\tau)$, where τ is the weight of X. In this way we obtain the following two theorems.

4.1. THEOREM. *For a complete metric space, the following conditions are equivalent:*

(i) *the set* $\mathscr{F}=\{f\in C(X,I^{\omega})\colon f(X)\subset N_{\omega}\}$ *is residual in* $C(X,I^{\omega})$,

(ii) *the set* $\mathscr{F}'=\{f\in C(X,I^{\omega})\colon f(X)\cap Q^{\infty}=\emptyset\}$ *is residual in* $C(X,I^{\omega})$,

(iii) X *is strongly countable-dimensional.*

4.2. THEOREM. *For a complete metric space of weight* $\tau\geqslant\aleph_0$, *the following conditions are equivalent:*

(i) *the set* $\mathscr{H}=\{h\in C(X,S(\tau)^{\aleph_0})\colon h$ *is a homeomorphic embedding and* $h(X)\subset K_{\infty}(\tau)\}$ *is residual in* $C(X,S(\tau)^{\aleph_0})$,

(ii) *the set* $\mathscr{H}'=\{h\in C(X,S(\tau)^{\aleph_0})\colon h$ *is a homeomorphic embedding and* $h(X)\cap A=\emptyset\}$, *where A is the set of all points in* $S(\tau)^{\aleph_0}$ *which have all coordinates rational and different from* 0, *is residual in* $C(X,S(\tau)^{\aleph_0})$,

(iii) X *is strongly countable-dimensional.*

To prove the implication (iii) $\Rightarrow$ (i) in the theorem, we will need the following proposition.

4.3. PROPOSITION. *Let F be a closed subspace of a normal space X and* $\dim F\leqslant n$. *Then*

(i) *the set* $\mathscr{F}=\{f\in C(X,I^{\omega})\colon \overline{f(F)}\subset N_n^{\omega}\}$ *is residual in* $C(X,I^{\omega})$, *and*

(ii) *the set* $\mathscr{G}=\{f\in C(X,S(\tau)^{\aleph_0})\colon \overline{f(F)}\subset K_n(\tau)\}$ *is residual in* $C(X,S(\tau)^{\aleph_0})$ *for any cardinal number* τ.

Proof. The special case of Proposition 4.3 when $F=X$ was proved in [10] — see the proofs of Proposition 3.3 and 3.5 in [10] — and the general case can be obtained by a slight modification of these proofs. For example, to prove (ii) we modify the proof of Proposition 3.3 of [10] by putting

$$\mathscr{F}(K,J)=\{f\in C(X,S(\tau)^{\aleph_0})\colon \varrho(f(F),F(K,J))>0\}.$$

Consequently, in the proof that $\mathscr{F}(K,J)$ is dense in $C(X,S(\tau)^{\aleph_0})$ we modify the construction of the function g_t in the following way: first we extend the mapping $f|f^{-1}(S_t)\cap F\colon f^{-1}(S_t)\cap F\to S_t$ to a mapping $g_t'\colon f^{-1}(K_t)\cap F\to S_t$ (using the theorem on extending mapping to spheres) and next we extend the mapping g_t' (using a theorem of Tietze) to a mapping $g_t\colon f^{-1}(K_t)\to K_t$ in such a way that $g_t|f^{-1}(S_t)=f|f^{-1}(S_t)$.

Immediately from Proposition 4.3 we obtain the following.

4.4. COROLLARY. *If Y is a strongly countable-dimensional closed subspace of a normal space X, then*

(i) *the set* $\mathscr{F}=\{f\in C(X,I^{\omega})\colon f(Y)\subset N_{\omega}\}$ *is residual in* $C(X,I^{\omega})$ *and*

(ii) *the set* $\mathscr{G}=\{f\in C(X,S(\tau)^{\aleph_0})\colon f(Y)\subset K_{\infty}(\tau)\}$ *is residual in* $C(X,S(\tau)^{\aleph_0})$ *for every cardinal number* τ.

Proof of Theorem 4.1. Since $\mathscr{F}\subset\mathscr{F}'$, (i) $\Rightarrow$ (ii). To prove that (ii) $\Rightarrow$ (iii) suppose that X is a complete metric space which is not strongly countable-dimensional. By Proposition 3.2, the set $\mathscr{F}'=\{f\in C(X,I^{\omega})\colon f(X)\cap Q^{\infty}=\emptyset\}$ does not contain any dense G_{δ}-subset of $C(X,I^{\omega})$, and hence $\mathscr{F}'$ is not residual. The implication (iii) $\Rightarrow$ (i) follows from Corollary 4.4 and is true for every normal space X.

Proof of Theorem 4.2. The implication (i) $\Rightarrow$ (ii) is obvious and the implication (ii) $\Rightarrow$ (iii) follows from Remark 3.6. Finally, (iii) implies (i) by Corollary 4.4. (ii).

We will end this section by a remark concerning n-dimensional spaces.

4.5. Remark. For a normal space X, the following conditions are equivalent:

(i) $\dim X\leqslant n$,

(ii) the set $\mathscr{F}=\{f\in C(X,I^{n+1})\colon \overline{f(X)}\subset N_n^{n+1}\}$ is residual in $C(X,I^{n+1})$, where $N_n^{n+1}=\{x\in I^{n+1}\colon x$ has $\leqslant n$ rational coordinates$\}$,

(iii) the set $\mathcal{F}' = \{f \in C(X, I^{n+1})\colon f(X) \cap (0, 0, \ldots, 0) = \emptyset\}$ is dense in $C(X, I^{n+1})$

Proof. The implication (i) $\Rightarrow$ (ii) follows from the proof of Propositon 3.3 in [10] and the implication (ii) $\Rightarrow$ (iii) is obvious. To prove that (iii) $\Rightarrow$ (i) suppose that $\dim X > n$. Then there exists an essential mapping $f\colon X \to I^{n+1}$. Then by Lemma 3.3 there exists $\varepsilon > 0$ such that for every $g\colon X \to I^{n+1}$ satisfying $d(f, g) < \varepsilon$ we have $g(X) \ni (0, 0, \ldots)$. Thus the set $\mathcal{F}'$ is not dense in $C(X, I^{n+1})$.

Note that the implication (iii) $\Rightarrow$ (i) was proved by Hurewicz in [5] for compact metric spaces (cf. [2], Problem 1.9.C).

5. Characterizing strongly countable-dimensional compacta by the Baire category in the Hurewicz's space of cuts. By a theorem of Nagata (see [8], Theorem VI.2) a metrizable space X is countable-dimensional if and only if for every sequence $\{(A_i, B_i)\}_{i=1}^{\infty}$ of pairs of disjoint closed subsets of X there exist closed sets $F_1, F_2, \ldots$ such that F_i is a partition between A_i and B_i and the family $\{F_i\}_{i=1}^{\infty}$ is point-finite. It is clear that the partitions in this theorem can be replaced by the cuts (see Section 2.8 for this notion).

The following theorem gives a corresponding characterization of the strongly countable-dimensional compacta.

5.1. Theorem. *For a compactum X the following conditions are equivalent:*

(i) *X is strongly countable-dimensional,*

(ii) *for every sequence $\{(A_i, B_i)\}_{i=1}^{\infty}$ of pairs of disjoint closed subsets of X the set C point-finite sequences of cuts $\{K_i|L_i\}_{i=1}^{\infty}$ is residual in the product $\prod_i S_X(A_i, B_i)$ of the Hurewicz's spaces of cuts separating A_i and B_i in X.*

The condition (ii) was distinguished by R. Pol in [11]. As observed in [11], Remark after Definition 3.1, the condition that C is residual is equivalent to the condition that C is of the second category.

The implication (i) $\Rightarrow$ (ii) was proved in [11], Corollary 3.4. We will prove that (ii) $\Rightarrow$ (i), which answers Question 3.5 of [11].

Proof of the implication (ii) $\Rightarrow$ (i). First let us notice that if X satisfies (ii) then every closed subspace Y of X satisfies (ii). This follows from a reasoning given in the proof of Lemma 3.3 in [11]. Indeed, let $S_{X,Y} \subset S(X)$ and

$$f\colon \prod_i (S_{X,Y} \cap S_X(A_i, B_i)) \to \prod_i S_Y(A_i, B_i)$$

be defined as in [11], Lemma 3.3. Then, using the fact that $S_{X,Y}$ is a dense G_δ-subset of $S(X)$ and that the mapping f is open and onto, it can easily be verified that if the set of point-finite sequences of cuts in $S' = \prod_{i=1}^{\infty} S_Y(A_i, B_i)$ is of the first category in S' then the set of point-finite sequences of cuts in $S = \prod_i S_X(A_i, B_i)$ is of the first category in S.

Suppose now that X satisfies (ii) but is not strongly countable-dimensional. Then X contains a closed subspace Y such that each open non-empty subspace of Y is infinite-dimensional. By the remark made above we can assume without loss of generality that X has the same property as Y.

Let $\{(C_i, D_i)\}_{i=1}^{\infty}$ be a family of pairs of disjoint closed subsets of X such that for every pair (A, B) of closed disjoint subsets of X the inclusions $A \subset C_i$ and $B \subset D_i$ hold for infinitely many indices i. Since X satisfies (ii), the set of point-finite sequences of cuts $\{K_i|L_i\}_{i=1}^{\infty} \in S = \prod_i S_X(C_i, D_i)$ is residual in S, i.e. it contains a dense G_δ-set $G = \bigcap_{n=1}^{\infty} U_n$, where U_n is open and dense in S and $U_{n+1} \subset U_n$.

Since the space S is completely metrizable, we can choose a complete metric ϱ in S. By $\operatorname{diam} A$ we will denote the diamater of $A \subset S$ with respect to ϱ.

By induction we will construct for every $n = 1, 2, \ldots$: an index $i_n \in N$, a finite subset $N_n \subset N$ and a family $\{(A_i^n, B_i^n)\}_{i=1}^{\infty}$ of pairs of disjoint closed subsets of X such that

(1) $i_n \notin N_{n-1}$ and $N_n \supset N_{n-1} \cup \{i_1, \ldots, i_{n-1}\}$ for $n > 1$,

(2) $S_X(A_i^n, B_i^n) \subset S_X(C_i, D_i)$ and $(A_i^n, B_i^n) = (C_i, D_i)$ for $i \notin N_n$,

(3) if $V_n = \prod_{i \in N} S_X(A_i^n, B_i^n) \subset S$, then $V_n \subset U_n$, $\operatorname{diam} V_n < \frac{1}{n}$ and $\overline{V}_n \subset V_{n-1}$ for $n > 1$, and

(4) the family $\mathcal{A}_n = \{(A_{i_j}^n, B_{i_j}^n)\}_{j=1}^{n}$ is essential.

Note that by (4) we have

(5) for every $\{K_i|L_i\}_{i=1}^{\infty} \in V_n$, $\bigcap_{j=1}^{n} (K_{i_j} \cap L_{i_j}) \neq \emptyset$.

The inductive construction. Since $\dim X \geqslant 1$, there exists an essential mapping $f_1\colon X \to I$. There exists $i_1 \in N$ such that $f_1^{-1}(0) \subset C_{i_1}$ and $f_1^{-1}(1) \subset D_{i_1}$. Since the set U_1 is open and dense in S, it contains an open basic subset V_1 of the form $V_1 = \prod_{i \in N} S_X(A_i^1, B_i^1)$, where $(A_i^1, B_i^1) = (C_i, D_i)$ for $i \notin N_1$, where N_1 is some finite subset of N containing i_1. We can assume that $\operatorname{diam} V_1 < 1$. Since $S_X(A_{i_1}^1, B_{i_1}^1) \subset S_X(C_{i_1}, D_{i_1}) \subset S_X(f_1^{-1}(0), f_1^{-1}(1))$, the family $\mathcal{A}_1 = \{(A_{i_1}^1, B_{i_1}^1)\}$ is essential.

Suppose that i_k, N_k and families $\{(A_i^k, B_i^k)\}_{i=1}^{\infty}$ satisfying (1)–(4) are constructed for $k \leqslant n$.

Let $M = \bigcup_{j=1}^{n} (A_{i_j}^n \cup B_{i_j}^n)$. Since the family $\mathcal{A}_n$ is essential, the set $X \setminus M$ is non-empty. Hence there exists a non-empty open subset W of X such that $\overline{W} \subset X \setminus M$. Since $\dim W \geqslant n+1$, there exists an essential mapping $g\colon \overline{W} \to I^{n+1}$. For each $j = 1, \ldots, n$ take $h_j\colon M \to I$ such that $h_j(A_{i_j}^n) = 0$ and $h_j(B_{i_j}^n) = 1$ and let $h_{n+1}\colon M \to I$ be such that $h_{n+1}(M) = 0$. Let $h = \{h_j\}_{j=1}^{n+1}\colon M \to I^{n+1}$ be a diagonal mapping. For $j = 1, \ldots, n$ we have then $A_{i_j}^n \subset h^{-1}(E_j)$ and $B_{i_j}^n \subset h^{-1}(F_j)$, where $E_j = \{(x_i)_{i=1}^{n+1} \in I^{n+1}\colon x_j = 0\}$ and $F_j = \{(x_i)_{i=1}^{n+1}\colon x_j = 1\}$ are opposite faces of I^{n+1}. Let $f_{n+1}\colon X \to I^{n+1}$ be a function such that $f_{n+1}|\overline{W} = g$ and $f_{n+1}|M = h$.

Since g is an essential mapping, its extension f_{n+1} is essential. Thus the family $\{(f_{n+1}^{-1}(E_j), f_{n+1}^{-1}(F_j))\}_{j=1}^{n+1}$ is essential (see [1], Ch.5, § 8, Lemma 2). There exists $i_{n+1} \in N \setminus N_n$ such that $f_{n+1}^{-1}(E_{n+1}) \subset C_{i_{n+1}}$ and $f_{n+1}^{-1}(F_{n+1}) \subset D_{i_{n+1}}$. Then also

(6) the family $\{(f_{n+1}^{-1}(E_1), f_{n+1}^{-1}(F_1)), \ldots, (f_{n+1}^{-1}(E_n), f_{n+1}^{-1}(F_n)), (C_{i_{n+1}}, D_{i_{n+1}})\}$ is essential.

Since $f_{n+1}^{-1}(E_j) \supset A_{i_j}^n$ and $f_{n+1}^{-1}(F_j) \supset B_{i_j}^n$ for $j = 1, \ldots, n$, the basic open set

$$V'_{n+1} = \prod_{j=1}^{n} S_X(f_{n+1}^{-1}(E_j), f_{n+1}^{-1}(F_j)) \times \prod_{i \notin \{i_1, \ldots, i_n\}} S_X(A_i^n, B_i^n)$$

is contained in V_n. Since U_{n+1} is open and dense in S, there exists a non-empty open basic in S set V_{n+1} such that $V_{n+1} \subset U_{n+1} \cap V'_{n+1} \subset U_{n+1} \cap V_n$ with diam $V_{n+1} < \frac{1}{n+1}$. The set V_{n+1} is of the form

$$V_{n+1} = \prod_{i \in N} S_X(A_i^{n+1}, B_i^{n+1}), \quad \text{where } S_X(A_i^{n+1}, B_i^{n+1}) = S_X(C_i, D_i)$$

for $i \notin N_{n+1}$, where N_{n+1} is some finite subset of N containing $\{i_1, \ldots, i_{n+1}\} \cup N_n$. Since

$$S_X(A_{i_j}^{n+1}, B_{i_j}^{n+1}) \subset S_X(f_{n+1}^{-1}(E_j), f_{n+1}^{-1}(F_j)) \quad \text{for } j = 1, \ldots, n,$$

and

$$S_X(A_{i_{n+1}}^{n+1}, B_{i_{n+1}}^{n+1}) \subset S_X(C_{i_{n+1}}, D_{i_{n+1}}),$$

it follows by (6) that the family $\mathscr{A}_{n+1} = \{(A_{i_j}^{n+1}, B_{i_j}^{n+1})\}_{j=1}^{n}$ is essential. The inductive construction is completed.

Now, since the space S with a metric ϱ is complete, we have $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$. Let $\{K_i|L_i\}_{i=1}^{\infty} \in \bigcap_{n=1}^{\infty} V_n$; since $V_n \subset U_n$ for every n, $\{K_i|L_i\}_{i=1}^{\infty} \in G$ and hence it is point-finite. On the other hand, by (5), for every $n \in N$ we have $\bigcap_{j=1}^{n} (K_{i_j} \cap L_{i_j}) \neq \emptyset$; hence $\bigcap_{j=1}^{\infty} (K_{i_j} \cap L_{i_j}) \neq \emptyset$, since X is compact. Thus $\{K_i|L_i\}_{i=1}^{\infty}$ is not point-finite, which gives a contradiction.

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