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Homotopy separators and mappings into cubes *

by

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Abstract. For some mappings, the homotopy separators are defined and studied. For mappings into cubes, some noteworthy homotopy separators are constructed (Th. 3.1). Applications to dimension theory are given.

0. Introduction. This paper is a continuation of the author's work [K]. For convenience of the reader, all necessary definitions from [K] are recalled in the following section. All spaces under discussion are assumed to be metrizable. Therefore all cartesian products of manifolds (of positive dimensions) involve at most countably many factors. For a given mapping f from a space X into a product of manifolds one can define separators of f to be certain subsets of X. This was done in the cited work. Here we study the most natural separators of f of the form $g^{-1}(y)$, where g is a reasonably defined modification of f (within the homotopy class of f). They are called homotopy separators of f. It will be shown that for mappings into products of cells (= homeomorphs of cubes) the family of homotopy separators has remarkable properties which fail for mappings into other products of manifolds. The results have been used in constructions of interesting subcontinua of the cubes I^n presented in the last section of this paper.

1. Terminology and auxiliary lemmas. By a *manifold* we mean a compact connected topological manifold of dimension $\geqslant 1$. A manifold will be denoted by the letter M, usually with subscripts. Then ∂M or $\dot{M}$ denotes the boundary of M, and $\mathring{M}$ the interior of M.

Let X be a space and consider two mappings $f, g\colon X \to M$. Then g is said to be an *admissible deformation* of f provided that there is a homotopy $H\colon (X, f^{-1}(\partial M)) \times I \to (M, \partial M)$ such that $H_0 = f$ and $H_1 = g$. If in addition H can be chosen so that $H(x, t) = f(x)$ for each $x \in f^{-1}(\partial M)$ then g is called a ∂-*deformation* of f.

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More generally, a mapping $(g_j)\colon X \to \prod_{j\in J} M_j$ is called an admissible (∂-)deformation of $(f_j)\colon X \to \prod_{j\in J} M_j$ provided each coordinate mapping $g_j\colon X \to M_j$ is an admissible (∂-)deformation of $f_j\colon X \to M_j$. The mapping $f = (f_j)$ is said to be *essential* provided that every admissible deformation of f is surjective. Every subset Y of X such that $f|Y\colon Y \to \prod_{j\in J} M_j$ is essential is called a *membrane* of f. By a *separator* of f we mean every subset of X whose complement in X is not a membrane of f. Note that the empty set is a separator of f iff f is not essential. Note also that every set of the form $g^{-1}(y)$, where g is an admissible deformation of f, is a separator of f. These special separators of f will be called *homotopy separators* of f. If g is a ∂-deformation of f and $y \in \prod_{j\in J} \mathring{M}_j$ then $g^{-1}(y)$ is said to be an *internal homotopy separator of* f.

The following lemma directly follows from Theorem 1.6 in [K].

1.1. Lemma. *Every open separator of a mapping $f\colon X \to \prod_{j\in J} M_j$ contains an internal homotopy separator of f.*

1.2. Corollary. *If $f\colon X \to M$ and Y is a membrane (separator) of f then $Y \setminus \operatorname{int} f^{-1}(\partial M)$ is a membrane (separator) of f as well* (see [K], Prop. 5.9).

Given a mapping $(f_j)\colon X \to \prod_{j\in J} M_j$, a mapping $(g_j)\colon X \to \prod_{j\in J} M_j$ is said to be an *expansive deformation* of (f_j) provided (g_j) is a ∂-deformation of (f_j) such that $f_j^{-1}(\partial M_j) \subset \operatorname{int} g_j^{-1}(\partial M_j)$ for each $j \in J$.

1.3. Lemma. *Let $f\colon X \to Q$ be a mapping into an n-cell, let $\bar{f}$ be an expansive deformation of f, and let S be a separator of $\bar{f}$. Then there exists an internal homotopy separator S' of f contained in S.*

Proof. Since every separator contains a closed separator (see [K], Th. 1.10), we may assume that S is closed. By 1.2 the set $S' = S \setminus \operatorname{int} \bar{f}^{-1}(\partial Q)$ is a closed separator of $\bar{f}$. It follows that S' is a closed separator of f. Since S' is disjoint from $f^{-1}(\partial Q)$ there is a mapping $r\colon X \setminus S' \to \partial Q$ such that $r(x) = f(x)$ for each $x \in f^{-1}(\partial Q)$. Also there is a mapping $s\colon X \to I$ such that $s(f^{-1}(\partial Q)) \subset (1)$ and $s^{-1}(0) = S'$. Identifying Q with the cone $\dot{Q} \times I / \dot{Q} \times (0)$, with $v = \dot{Q} \times (0)$ being the vertex and $z = [z, 1]$ for each $z \in \dot{Q}$, define a mapping $g\colon X \to Q$ by the formula

$$g(x) = \begin{cases} v & \text{for } x \in S', \\ [r(x), s(x)] & \text{for } x \notin S'. \end{cases}$$

As g is a ∂-deformation of f and $v \in \mathring{Q}$ the set $g^{-1}(v) = S'$ is the desired separator.

The property of mappings into cubes described in Lemma 1.3 is exceptional — it is not valid even for mappings into the circle. Observe that an expansive deformation of a mapping f into a product of closed manifolds is simply a mapping homotopic to f.

1.4. Example. Let S^1 be the unit circle in the complex plane C. Define

$$X = \{z \in C\colon z = (1+1/t)e^{2\pi i t},\ t \in [1, \infty)\} \cup \{1\}$$

$$f\colon X \to S^1, \quad f(z) = \frac{z}{|z|}, \quad S = \{1\}.$$

Then S is a separator of every mapping homotopic to f but no (internal) homotopy separator of f is contained in S.

1.5. Corollary. *Let $f\colon X \to \prod_{j\in J} Q_j$ be a mapping into a product of cells and let Y be a subset of X intersecting every internal homotopy separator of f. Then Y is a membrane of every expansive deformation of f.*

Proof. Suppose there is an expansive deformation $\bar{f} = (\bar{f}_j)$ of f such that Y is not a membrane of $\bar{f}$. By Lemma 1.5 in [K], for each $j \in J$, there is a separator S_j of $\bar{f}_j$ such that $\bigcap_{j\in J} S_j \subset X \setminus Y$. By 1.3 there exist internal homotopy separators $S'_j \subset S_j$ of f_j, $j \in J$. Then $\bigcap_{j\in J} S'_j$ is an internal homotopy separator of f disjoint from Y. This contradiction completes the proof.

1.6. Remark. Keeping the notation of Example 1.4 and setting $Y = X \setminus S$ shows that in general 1.5 is not valid for mappings into other products of manifolds.

2. Cross-sections of homotopy separators. It is well known that for a given space X we have $\dim X \geq n$ iff X admits an essential mapping into the n-cube I^n Let us adopt the following result (see [A-P], p. 531) as a definition of strongly infinite-dimensional spaces, briefly: SID. A space X is SID iff it admits an essential mapping into the Hilbert cube $I^\infty = I \times I \times \ldots$. Spaces which are not SID are called weakly infinite-dimensional, briefly: WID.

The aim of this section is to show that the sets intersecting all internal homotopy separators of a given mapping must have dimensions at least as large as the dimension of the target space.

2.1. Theorem. *Let $f\colon X \to \prod_{j\in J} M_j$ and let Y be a subset of X. If either*

(i) *J is finite and $\dim Y < \dim \prod_{j\in J} M_j$, or*

(ii) *J is infinite and Y is* WID,

then there exists an internal homotopy separator of f disjoint from Y.

Proof. For each $j \in J$ let $Q_j \subset M_j$ be a closed cell with $\dim Q_j = \dim M_j$ and let $n = \Sigma \dim M_j$ ($n = \infty$ if J is infinite). Let $X_0 = f^{-1}(\prod_{j\in J} Q_j) = \bigcap_{j\in J} f_j^{-1}(Q_j)$ and let $Y_0 = Y \cap X_0$. Since Y_0 is a closed subset of Y it follows that $\dim Y_0 < n$ (for $n = \infty$ this means that Y_0 is WID). It follows that there is no essential mapping from Y_0 onto I^n. Let $\varphi\colon X_0 \to \prod_{j\in J} Q_j$ be given by $\varphi(x) = f(x)$ for each $x \in X_0$. By 1.5 there exist a ∂-deformation $\psi\colon X_0 \to \prod_{j\in J} Q_j$ of φ and a point $v \in \prod_{j\in J} \mathring{Q}_j$ such that

$$Y_0 \cap \psi^{-1}(v) = \emptyset.$$

Since $X_0 \cap f_j^{-1}(\partial Q_j) = \varphi_j^{-1}(\partial Q_j)$ we infer that $\psi_j(x) = \varphi_j(x) = f_j(x)$ for each $x \in X_0 \cap f_j^{-1}(\partial Q_j)$. Since the sets are closed in X, it follows that there is a well-defined map

$$\overline{\psi}_j\colon X_0 \cup f_j^{-1}(\partial Q_j) \to Q_j$$

given by

$$\overline{\psi}_j(x) = \begin{cases} f_j(x) & \text{for } x \in f_j^{-1}(\partial Q_j), \\ \psi_j(x) & \text{for } x \in X_0. \end{cases}$$

Since $X_0 \cup f_j^{-1}(\partial Q_j)$ is a closed subset of $f_j^{-1}(Q_j)$, the mapping $\overline{\psi}_j$ extends to a mapping

$$\overline{\overline{\psi}}_j\colon f_j^{-1}(Q_j) \to Q_j.$$

Finally, for each $j \in J$, let $g_j\colon X \to M_j$ be a mapping given by the formula

$$g_j(x) = \begin{cases} f_j(x) & \text{for } x \notin f_j^{-1}(Q_j), \\ \overline{\overline{\psi}}_j(x) & \text{for } x \in f_j^{-1}(Q_j). \end{cases}$$

Then $g = (g_j)$ is a ∂-deformation of f. Moreover, $Y \cap g^{-1}(v) = Y_0 \cap \psi^{-1}(v) = \emptyset$, which completes the proof.

3. Singular homotopy separators.

The essential ideas used in the proof of the following result come from a paper by R. Pol [P].

3.1. THEOREM. *Let $(f_Q, f_R)\colon X \to Q_J \times R$ be a mapping, where X is compact, Q_J is a product of cells and R is an arbitrary space. Let $Z \subset R$ be a set which admits a continuous surjection onto the Cantor set. Then for every expansive deformation $\tilde{f}_Q$ of $f_Q\colon X \to Q_J$ there exists an internal homotopy separator S of f_Q such that*
(*) *$Y \subset S \wedge f_R(Y) \supset Z \Rightarrow Y$ is a membrane of $\tilde{f}_Q$.*

Proof. There exists an expansive deformation $g_Q\colon X \to Q_J$ of f_Q such that $\tilde{f}_Q$ is an expansive deformation of g_Q.

Let Φ denote the set of all ∂-deformations of g_Q and let $a \in \mathring{Q}_J$ (if $Q_J = \prod_{j\in J} Q_j$ then we denote $\mathring{Q}_J = \prod_{j\in J} \mathring{Q}_j$) be any point. It follows from 1.5 that for every $Y \subset X$ we have

(1) $Y \cap \varphi^{-1}(a) \neq \emptyset$ for each $\varphi \in \Phi \Rightarrow Y$ is a membrane of $\tilde{f}_Q$.

Since $\Phi \subset Q_J^X$, and the space Q_J^X in separable and metrizable, there exist a set $Z_0 \subset Z$ and a continuous surjection

$$Z_0 \ni z \to \psi_z \in \Phi.$$

Let $\psi\colon f_R^{-1}(Z_0) \to Q_J$ be given by

$$\psi(x) = \varphi_{f_R(x)}(x),$$

i.e. ψ acts on $f_R^{-1}(z)$ as φ_z. If follows that ψ is continuous.

Consider $\psi^{-1}(a)$. Since $\psi^{-1}(a) \cap f_R^{-1}(z) = \varphi_z^{-1}(a) \cap f_R^{-1}(z)$ for each $z \in Z_0$, by (1) we have

(2) $Y \subset \psi^{-1}(a) \wedge f_R(Y) \supset Z_0 \Rightarrow Y$ is a membrane of $\tilde{f}_Q$.

Observe that $f_R^{-1}(Z_0) \setminus \psi^{-1}(a)$ is not a membrane of g_Q since ψ is a ∂-deformation of $g_Q | f_R^{-1}(Z_0)$. From 1.5 we infer that there is an internal homotopy separator S of f_Q disjoint from $f_R^{-1}(Z_0) \setminus \psi^{-1}(a)$. Thus $S \cap f_R^{-1}(Z_0) \subset \psi^{-1}(a)$, and (2) implies that S satisfies (*).

3.2. Remark. Let us make a comment about the set Z in the Theorem. What is really needed from Z is to assure that every separable metric space can be obtained as a continuous image of a subset of Z. The Cantor set C enjoys this property — hence the hypothesis about Z. One might wonder if such a set Z must already contain a copy of C. The answer is "no". We shall show that *there exists a 0-dimensional separable metric space Z which admits a continuous surjection onto C and such that every compact subset of Z is countable.* To this end consider a decomposition of I into two disjoint sets A and B containing no Cantor sets. One of them, say A, has the cardinality $\mathfrak{c}$. Hence there is a surjective function $\alpha\colon A \to C$ (not continuous). Define Z to be the graph of this function, i.e. $Z = \{(s,t) \in A \times C\colon \alpha(s) = t\}$. Restricting the projection $A \times C \to C$ to Z we get a desired set.

Let us adopt the following convention: $\dim X \geqslant \infty$ denotes that X is SID.

3.3. THEOREM. *Let X be a separable metric space with $\dim X \geqslant n+1$, $n = 1, 2, \ldots, \infty$. Then there exists a mapping $f\colon X \to I$ and a closed membrane A of f such that*

(*) $Y \subset A \wedge f(Y) \supset C \Rightarrow \dim Y \geqslant n$.

Proof. By the assumption there exists an essential mapping from X into I^{n+1}. Then it is not difficult to construct a metric compactification X^* of X and a mapping $(g, h)\colon X^* \to I^n \times I$ such that X is a membrane of this mapping. Choose $\tilde{g}\colon X^* \to I^n$ to be an expansive deformation of $g\colon X^* \to I^n$. Then, according to 3.1, there exist an internal homotopy separator S of g such that

(i) $Y \subset S \wedge h(Y) \supset C \Rightarrow Y$ is a membrane of $\tilde{g}$.

Put $A = S \cap X$ and $f = h|X\colon X \to I$. By (i) the condition (*) is satisfied. So, it remains to show that A is a membrane of f. But S is a separator of $g\colon X^* \to I^n$, hence A is a separator of $g|X\colon X \to I^n$. Since $(g|X, h|X)\colon X \to I^n \times I$ is essential the conclusion follows from Theorem 6.1 in [K].

A similar but even simpler reasoning combined with Theorem 2.1 in [K] gives the following.

3.4. Corollary. *Let* $p\colon I^{n+1} \to I$ *be projection onto the first factor. Then there exists a continuum* $A \subset I^{n+1}$ *such that*

(i) $p(A) = I$,
(ii) $\dim A = n$,
(iii) $Y \subset A \;\&\; p(Y) \supset C \Rightarrow \dim Y = n$.

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Characterizing strong countable-dimensionality in terms of Baire category

by

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Abstract. The main result of this paper is the following theorem: a metrizable compactum X is a countable union of finite-dimensional compacta (i.e. X is strongly countable-dimensional) if and only if for almost every continuous mapping $f\colon X \to I^{\omega}$ into the Hilbert cube I^{ω} we have $f(X) \cap Q^{\aleph_0} = \emptyset$, where $Q^{\aleph_0}$ is the product of the rationals. We give also a characterization of strongly countable-dimensional compacta in terms of the Baire category in the space of cuts defined by Hurewicz.

1. Introduction. In this paper we show that metrizable compacta which are countable unions of finite-dimensional compacta (i.e. strongly countable-dimensional compacta) can be characterized in terms of Baire category in the function spaces or the spaces of cuts introduced by Hurewicz. Let us describe the results in some more details.

Let us recall that X is *countable-dimensional* if X is the union of countably many zero-dimensional subspaces. A theorem of Nagata (see [8], Theorem V.5 and its Corollary) states that the subspace N_{ω} of the Hilbert cube I^{ω}, consisting of all points having only finitely many rational coordinates, is universal for countable-dimensional metrizable separable spaces X, i.e. any such X can be embedded in N_{ω}. In [9], we strengthened this result to the effect that the embeddings of a metrizable separable countable-dimensional space X into N_{ω} form a dense subset in the function space $C(X, I^{\omega})$ of all continuous mappings of X into I^{ω}, endowed with the sup-metric. However, in contrast to the classical finite-dimensional case, generally, the set of embeddings is not residual.

More specifically, the main result of this paper is that the set of embeddings of a metrizable compactum X into N_{ω} is residual (equivalently, is of the second category) in $C(X, I^{\omega})$ if and only if X is strongly countable-dimensional. Natural examples of countable-dimensional compacta which are not strongly countable-dimensional can be found in [1], Chapter 10, § 3 or [3], Example 1.12.

This theorem provides a characterization of strongly countable-dimensional compacta in terms of the function space. An "internal" characterization of strongly countable-dimensional compacta in terms of the Baire category in the space of cuts