Proof. Proposition 4.1 and Theorem 4.4. By Corollary 3.7 we may assume that if F is an annulus it is 2-sided. If F is a 1-sided Moebius band, let W be a regular neighborhood of F in M with $W \cap \iota W = \emptyset$. Then $A = \partial W - (W \cap \partial M)$ must be an essential annulus (since $\partial(W \cap \partial M)$ consists of two 1-spheres it is incompressible. If it is boundary parallel, M must be a solid torus and ∂M is not incompressible). Thus A is 2-sided and $A \cap \iota A = \emptyset$. ■

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On the Cauchy equation modulo Z

by

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Abstract. Assume that X is a real linear topological space (which always is assumed to be Hausdorff) and let $f\colon X \to \mathbf{R}$ be a function such that

$$f(x+y)-f(x)-f(y) \in \mathbf{Z}$$

for all $x, y \in X$. Some conditions are established under which f has the form $g+k$, where g is a continuous linear functional on the space X and the function k takes integer values only. An application to the Cauchy equation

$$f(x+y) = f(x)+f(y)$$

for functions acting between linear topological spaces is also given.

Let a function $f\colon \mathbf{R} \to \mathbf{R}$ be given and assume that

$$f(x+y)-f(x)-f(y) \in \mathbf{Z} \tag{1}$$

for all $x, y \in \mathbf{R}$, where $\mathbf{Z}$ denotes the set of all integers. As follows from an example of G. Godini [6, Example 2], it is not generally true that such a function f must be of the form $g+k$ where g is an additive function and k takes integers values only. However, the following theorem has been proved in paper [1]:

THEOREM 1. *If the Cauchy difference $f(x+y)-f(x)-f(y)$, as a function of two real variables, is Lebesgue measurable and takes integer values only, then there exists an additive function $g\colon \mathbf{R} \to \mathbf{R}$ and a Lebesgue measurable function $k\colon \mathbf{R} \to \mathbf{Z}$ such that*

$$f = g+k. \tag{2}$$

In the present paper, the following theorem will be shown:

THEOREM 2. *Assume that X is a real linear topological space.*
If a function $f\colon X \to \mathbf{R}$ satisfies condition (1) *for all $x, y \in X$ and there exists a set $E \subset X$ such that*

$$0 \in \operatorname{Int}(E-E) \tag{3}$$

and

$$f(x) \in Z + (-\tfrac{1}{6}, \tfrac{1}{6}) \tag{4}$$

for every $x \in E$, then there exist a continuous linear functional g: $X \to R$ and a function k: $X \to Z$ such that $f = g + k$.

Remark 1. It is a well-known theorem of H. Steinhaus ([13, Théorème VIII]; cf. also [11, Theorem 4.8]) which says that any Lebesgue measurable set E of positive Lebesgue measure fulfils condition (3). In fact, it is true in a much more general setting. Namely, any Christensen measurable subset E of an abelian Polish group which is not Christensen zero set fulfils condition (3) (cf. [5] and [2, Theorem 2] or [3, Theorem 7.3]). It is also well known (cf. [7, Difference Theorem 10.4]) that any subset E of a linear topological space which is of the second category and satisfies the condition of Baire fulfils condition (3).

Before passing to the proof of Theorem 2 we are going to show that a version of a theorem of M. R. Mehdi [10, Theorem 5] and Theorem 1 may be obtained using Theorem 2. Our version of Mehdi's theorem reads as follows.

COROLLARY 1. *Assume that X is a real linear topological space and let Y be a real locally convex linear topological space.*

If f: $X \to Y$ is an additive function and there exists a set $E \subset X$ such that condition (3) *is satisfied and $f(E)$ is a bounded subset of the space Y, then f is a bounded linear operator.*

Proof. Let us fix arbitrarily a continuous functional Λ: $Y \to R$. Then $\Lambda \circ f$ is an additive functional bounded on a set E fulfilling condition (3), and so there exists a positive real number c such that

$$|c\Lambda[f(x)]| < \tfrac{1}{6}$$

for every $x \in E$. Hence and from Theorem 2 we infer that

$$c\Lambda \circ f = g + k$$

where g: $X \to R$ is a continuous linear functional and the function k takes integer values only The function k: $X \to Z$ being additive must be the zero function. This shows that

$$\Lambda \circ f = \frac{1}{c} g ,$$

and so $\Lambda \circ f$ is a continuous linear functional. Hence we infer that f is a linear operator and, since each weakly bounded subset of the space Y is bounded (cf., e.g., [12, Theorem 3.18]), f is a bounded linear operator.

Proof of Theorem 1. Taking into account the measurability of the Cauchy difference and making use of a theorem of M. Laczkovich [9, Theorem 5], we can

represent the function f as a sum of an additive function a: $R \to R$ and a Lebesgue measurable function m: $R \to R$:

$$f = a + m .$$

Of course, Cauchy differences of the functions f and m are equal. In particular,

$$m(x+y) - m(x) - m(y) \in Z$$

for all $x, y \in R$. Hence and from Lemma 1 given below we infer that there exist a real constant c and a Lebesgue measurable function k: $R \to Z$ such that

$$m(x) = cx + k(x)$$

for every $x \in R$. This shows that (2) holds with the function g: $R \to R$ defined by the formula

$$g(x) = a(x) + cx ,$$

which, of course, is an additive function.

LEMMA 1. *If a Lebesgue measurable function f: $R \to R$ satisfies condition* (1) *for all $x, y \in R$ then there exists a real constant c and a Lebesgue measurable function k: $R \to Z$ such that*

$$f(x) = cx + k(x)$$

for every $x \in R$.

Proof. Let g: $R \to (-\tfrac{1}{2}, \tfrac{1}{2}]$ be the function such that

$$f(x) - g(x) \in Z \tag{5}$$

for every $x \in R$. Then (cf. (1))

$$g(x+y) - g(x) - g(y) \in Z$$

for all $x, y \in R$, from which we infer that also

$$g(x-y) - g(x) + g(y) \in Z \tag{6}$$

for all $x, y \in R$. Moreover, because of the measurability of the function f, the function g is also Lebesgue measurable. Hence each of the sets

$$\left\{x \in R: -\frac{1}{2} + \frac{j}{6} < g(x) \leqslant -\frac{1}{2} + \frac{j+1}{6}\right\}, \quad j \in \{0, 1, \ldots, 5\} ,$$

is Lebesgue measurable. Since the union of these sets gives the set of all reals, one of these sets has a positive Lebesgue measure. Denote it by M and fix an $x_0 \in M$. Then

$$|g(x) - g(x_0)| < \tfrac{1}{6}$$

for every $x \in M$. Hence (cf. also (6) and the definition of the function g)

$$g(x - x_0) = g(x) - g(x_0) \in (-\tfrac{1}{6}, \tfrac{1}{6})$$

for every $x \in M$ and, putting $E = M - x_0$, we get

$$g(x) \in (-\tfrac{1}{6}, \tfrac{1}{6})$$

for every $x \in E$. Making use of Theorem 2 (cf. also Remark 1), we infer that there exist a real constant c and a function $k\colon \mathbf{R} \to \mathbf{Z}$ such that

$$g(x) = cx + k(x)$$

for every $x \in \mathbf{R}$. This (cf. also (5)) ends the proof.

In connection with Lemma 1 let us observe that similarly the following two propositions may be proved (cf. also Remark 1).

PROPOSITION 1. *Assume that X is a separable real F-space.*

If a Christensen measurable function $f\colon X \to \mathbf{R}$ satisfies condition (1) *for all $x, y \in X$ then there exist a continuous linear functional $g\colon X \to \mathbf{R}$ and a Christensen measurable function $k\colon X \to \mathbf{Z}$ such that $f = g + k$.*

PROPOSITION 2. *Assume that X is a real linear topological Baire space.*

If a Baire measurable function $f\colon X \to \mathbf{R}$ satisfies condition (1) *for all $x, y \in X$, then there exist a continuous linear functional $g\colon X \to \mathbf{R}$ and a Baire measurable function $k\colon X \to \mathbf{Z}$ such that $f = g + k$.*

At the end let us pass to a proof of our main theorem.

Proof of Theorem 2. Let

$$U = \operatorname{Int} f^{-1}(\mathbf{Z} + (-\tfrac{1}{3}, \tfrac{1}{3}))$$

and observe that U is a neighbourhood of the origin. In fact (cf. (3) and (4)), we have

$$0 \in \operatorname{Int}(E - E) \subset \operatorname{Int}[f^{-1}(\mathbf{Z} + (-\tfrac{1}{6}, \tfrac{1}{6})) - f^{-1}(\mathbf{Z} + (-\tfrac{1}{6}, \tfrac{1}{6}))] \subset U.$$

Moreover, as it follows directly from the definition of the set U, there are functions $k\colon U \to \mathbf{Z}$ and $h\colon U \to (-\tfrac{1}{3}, \tfrac{1}{3})$ such that

$$f(x) = k(x) + h(x) \tag{7}$$

for every $x \in U$. Fix a balanced neighbourhood $W \subset X$ of the origin such that

$$W + W \subset U.$$

Then (cf. (7) and (1))

$$h(x+y) - h(x) - h(y) \in \mathbf{Z}$$

for all $x, y \in W$ and

$$|h(x+y) - h(x) - h(y)| < 3 \cdot \tfrac{1}{3} = 1$$

for all $x, y \in W$. Hence

$$h(x+y) = h(x) + h(y)$$

for all $x, y \in W$. Making use of a theorem of Z. Daróczy and L. Losonczi (cf. [4, Satz 4] or [8, Chapter XIII, § 6, Theorem 1] and observe that it holds in linear topological spaces as well) we infer that there exists (exactly one) an additive function $g\colon X \to \mathbf{R}$ such that

$$h(x) = g(x) \tag{8}$$

for every $x \in W$. Since the set $g(W)$ is bounded, a well-known theorem of F. Bernstein and G. Doetsch (cf. [8, Chapter VI, § 4, Theorem 2] and observe that it holds in linear topological spaces as well) gives us continuity of the function g and so g is a continuous linear functional. We shall show that

$$f(x) - g(x) \in \mathbf{Z}$$

for every $x \in X$ and this will end the proof. Fix an $x \in X$ and let n be a positive integer such that $\frac{x}{n} \in W$. Then (cf. (8), (7) and (1))

$$\begin{aligned} f(x) - g(x) &= f\left(n\frac{x}{n}\right) - g\left(n\frac{x}{n}\right) \\ &= \left[f\left(n\frac{x}{n}\right) - nf\left(\frac{x}{n}\right)\right] + n\left[f\left(\frac{x}{n}\right) - g\left(\frac{x}{n}\right)\right] \\ &= \left[f\left(n\frac{x}{n}\right) - nf\left(\frac{x}{n}\right)\right] + nk\left(\frac{x}{n}\right) \in \mathbf{Z} + \mathbf{Z} = \mathbf{Z}. \end{aligned}$$

Theorem 2 has been proved.

Added in proof. Professor Ludwig Reich kindly informed us about J. G. van der Corput's paper *Goniometrische functies gekarakteriseerd door een functionaalbetrekking*, Euclides 17 (1940), 55–75. In fact by using a theorem given there at p. 64, the proof of our Theorem 2 is very easy (at last in the case $X = \mathbf{R}$), and the condition (4) can be replaced by "$f(x) \in \mathbf{Z} + (-\tfrac{1}{4} + \varepsilon, \tfrac{1}{4} - \varepsilon)$ for some $\varepsilon > 0$".

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Homotopy separators and mappings into cubes *

by

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Abstract. For some mappings, the homotopy separators are defined and studied. For mappings into cubes, some noteworthy homotopy separators are constructed (Th. 3.1). Applications to dimension theory are given.

0. Introduction. This paper is a continuation of the author's work [K]. For convenience of the reader, all necessary definitions from [K] are recalled in the following section. All spaces under discussion are assumed to be metrizable. Therefore all cartesian products of manifolds (of positive dimensions) involve at most countably many factors. For a given mapping f from a space X into a product of manifolds one can define separators of f to be certain subsets of X. This was done in the cited work. Here we study the most natural separators of f of the form $g^{-1}(y)$, where g is a reasonably defined modification of f (within the homotopy class of f). They are called homotopy separators of f. It will be shown that for mappings into products of cells (= homeomorphs of cubes) the family of homotopy separators has remarkable properties which fail for mappings into other products of manifolds. The results have been used in constructions of interesting subcontinua of the cubes I^n presented in the last section of this paper.

1. Terminology and auxiliary lemmas. By a *manifold* we mean a compact connected topological manifold of dimension $\geqslant 1$. A manifold will be denoted by the letter M, usually with subscripts. Then ∂M or $\dot{M}$ denotes the boundary of M, and $\mathring{M}$ the interior of M.

Let X be a space and consider two mappings $f, g\colon X \to M$. Then g is said to be an *admissible deformation* of f provided that there is a homotopy $H\colon (X, f^{-1}(\partial M)) \times I \to (M, \partial M)$ such that $H_0 = f$ and $H_1 = g$. If in addition H can be chosen so that $H(x, t) = f(x)$ for each $x \in f^{-1}(\partial M)$ then g is called a *∂-deformation* of f.

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