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Equivariant surgery on essential annuli and Moebius bands in 3-manifolds with respect to involutions

by

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Abstract. A complete classification is given for equivariant surgery on essential annuli and Moebius bands in 3-manifolds with respect to involutions with at most isolated fixed points.

§ 1. Introduction. Suppose M is a 3-manifold that contains an essential annulus. Equivariant surgery on it is *possible* with respect to an involution $\iota\colon M \to M$ with at most isolated fixed points if there exists an essential annulus A in M with either $A \cap \iota A = \emptyset$ or $\iota A = A$ and with no fixed points of ι on A. We prove that for ∂-incompressible essential annuli equivariant surgery is always possible (Theorem 3.6). In contrast, equivariant surgery on incompressible tori in orientable, closed, irreducible 3-manifolds with respect to orientation preserving involutions without fixed points is not possible in general [3], [6]. An essential annulus A in a 3-manifold M is *∂-incompressible* if the components of ∂A are in different components of ∂M, or if ∂A is contained in an incompressible component of ∂M and M is irreducible (Proposition 3.1).

We also prove a relative version of equivariant surgery on an essential annulus $A_0 \subset M$ with $\partial A_0 \cap \iota\partial A_0 = \emptyset$ (Theorem 3.2).

If M is irreducible and if the components of ∂M that contain ∂A are incompressible, we may assume that the annulus A obtained by equivariant surgery in Theorem 3.6 is 2-sided in M (Propositions 3.4 and 3.5).

Our method of proof applies also to essential Moebius bands. The corresponding theorems are given in § 4.

The usual definition of an *incompressible proper surface* F in a 3-manifold M to be essential requires that there does not exist a homotopy $f_t\colon (F, \partial F) \to (M, \partial M)$, $0 \leqslant t \leqslant 1$, $f_0\colon F \to M$ the inclusion, and $f_1(F) \subset \partial M$. This definition is not suitable for our surgery arguments. It follows from Corollary 2.3 that if M does not contain fake 3-cells, then it can be replaced by the requirement that F is not boundary parallel in M.

The method of proof of the above Theorems 3.2 and 3.6 is to move a given essential annulus by an ε-isotopy such that the new annulus and its image under the involution intersect transversally in 1-spheres and proper 1-cells. We may assume that the 1-spheres do not bound 2-cells and that the 1-cells do not separate. Then either the intersection consists only of 1-spheres decomposing the given annulus into annuli, or it consists only of 1-cells decomposing the given annulus into 2-cells. The annuli or 2-cells are modified and rearranged to give an essential annulus with the desired properties. If the given annulus is decomposed into 2-cells, extra care has to be taken to avoid ending up with a Moebius band.

If we assume that the original annulus A_0 is 2-sided in M and further that its boundary components are not parallel in ∂M, then Theorem 3.2, the relative version of equivariant surgery on annuli, is also a consequence of the generalized loop theorem [7].

In [5], the characterization of those 3-manifolds with subgroups $\boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\boldsymbol{Z}$ in their fundamental groups required to determine all involutions with some isolated fixed points on orientable 3-dimensional flat space forms. This was accomplished in (4). Theorems 3.2 and 3.6 were applied to perform equivariant surgery on essential annuli.

There is a certain similarity between the methods of proof of this paper and those in [3] to perform equivariant surgery on incompressible tori and Klein bottles.

If the involution has 1-dimensional or 2-dimensional fixed point sets similar but more delicate results hold [2].

I would like to thank the referee for his comments.

§ 2. Notation and preliminaries. We will work throughout in the PL category. Our reference is [1]. A PL homeomorphism we simply call an *isomorphism*.

All 3-manifolds are assumed to be connected.

All involutions are assumed to have at most isolated fixed points.

An $(n-1)$-manifold F contained in an n-manifold M is said to be *proper* if $F\cap\partial M = \partial F$.

A *surface* is a compact, connected 2-manifold.

Let M be a 3-manifold and let $F\subset M$ be a surface that is proper or $F\subset\partial M$. Suppose F is not a 2-sphere that bounds a 3-cell in M, or a 2-cell with $F\subset\partial C$, $C\subset M$ a 3-cell, and with $\overline{C-F} = C\cap\partial M$. Then F is said to be *incompressible* in M, if for each 2-cell $B\subset M$, with $B\cap F = \partial B$, there is a 2-cell $D\subset F$, with $\partial D = \partial B$. The surface F is said to be *∂-incompressible* (*boundary-incompressible*) if for each 2-cell $B\subset M$ with $B\cap F = \partial B\cap F = I$ a 1-cell and $B\cap\partial M = \overline{\partial B-I}$, there is a 2-cell $D\subset F$ with $I\subset\partial D$ and $D\cap\partial F = \overline{\partial D-I}$. The surface F is said to be *boundary parallel*, if there is an embedding $h\colon F\times[0,1]\to M$ with $h(F\times 0) = F$, $h(F\times 1\cup\partial F\times[0,1])\subset\partial M$. Finally the surface F is called *essential* if it is incompressible and not boundary parallel.

Note that if an incompressible proper surface $F\subset M$ has components of ∂F in at least two different components of ∂M, or if $\partial F\neq\emptyset$ and F is ∂-incompressible in M, then F is essential in M.

If S^1 is a 1-sphere, then $A = S^1\times[0,1]$ is an annulus.

EXAMPLE. In Figure 1 an essential annulus A is exhibited in a handlebody of genus 2 that is not ∂-incompressible. (Note: Incompressible proper surfaces in handlebodies must have boundaries and are not ∂-incompressible.)

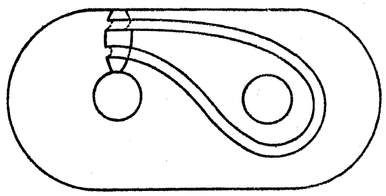

Fig. 1

LEMMA 2.1. *Let M be an n-manifold, let $F\subset M$ be a compact, connected, locally lat, proper $(n-1)$-manifold, let W_0 be a compact, connected n-manifold with $\partial W_0 = F_0\cup F_1$, F_0, F_1 connected $(n-1)$-manifolds such that $F_0\cap F_1 = \partial F_0 = \partial F_1$, and let $f\colon (W_0, F_1, F_0)\to(M,\partial M, F)$ be a map with*

$$(f|\ )_*\colon H_{n-1}(F_0,\partial F_0)\to H_{n-1}(F,\partial F)$$

is an isomorphism, where the homology groups have coefficients in Z_2.

Then F decomposes M into two connected n-manifolds M_0 and W: $M = M_0\cup W$ and $M_0\cap W = \partial M_0\cap\partial W = F$ such that W is compact and $\overline{\partial W-F}$ is connected. (Note: It is possible that $\partial F = \emptyset$.)

Proof. Consider the map $f\colon (W_0,\partial W_0, F_1)\to(M,\partial M\cup F,\partial M)$ and the commutative diagram

$$\begin{array}{ccccccc}
H_n(W_0,F_1) = 0 \to & H_n(W_0,\partial W_0) = Z_2 & \xrightarrow{\partial} & H_{n-1}(\partial W_0, F_1) & \underset{\cong}{\xleftarrow{e_*}} & H_{n-1}(F_0,\partial F_0) = Z_2 \\
\downarrow & \downarrow f_* & & \cong\downarrow (f|\ )_* & & \cong\downarrow (f|\ )_* \\
0\to H_n(M,\partial M)\to & H_n(M,\partial M\cup F) & \xrightarrow{\partial} & H_{n-1}(\partial M\cup F,\partial M) & \underset{\cong}{\xleftarrow{e_*}} & H_{n-1}(F,\partial F) = Z_2
\end{array}$$

where $e\colon (F_0,\partial F_0)\to(\partial W_0, F_0)$, $e\colon (F,\partial F)\to(\partial M\cup F,\partial M)$ are excision maps and the horizontal rows preceding the excision isomorphism are portions of the exact homology sequences of the triples $(W_0,\partial W_0, F_1)$ and $(M,\partial M\cup F,\partial M)$ respectively.

Let $[W_0]\in H_n(W_0,\partial W_0)$. We conclude that $\partial(f_*[W_0])\neq 0$ and that F decomposes M into two connected n-manifolds M_0 and W: $M = M_0\cup W$ and $M_0\cap W = \partial M_0\cap\partial W = F$. Then

$$H_n(M,\partial M\cup F) = H_n(M_0,\partial M_0)\oplus H_n(W,\partial W).$$

We may assume that $f_*[W_0] = (0,[W])$, $[W]\in H_n(W,\partial W)$. Hence W is compact.

Let $F' = \overline{\partial W - F}$.

Consider the map $f\colon (W_0, \partial W_0, F_0) \to (M, \partial M, F)$ and the commutative diagram

$$\begin{array}{ccccccc}
H_n(W_0, F_0) = 0 \to & H_n(W_0, \partial W_0) = Z_2 & \xrightarrow[\cong]{\partial} & H_{n-1}(\partial W_0, F_0) = Z_2 & & \\
& \cong \downarrow f_* & & \downarrow (f|)_* & & \\
H_n(M, F) = 0 \to & H_n(W, \partial W) = Z_2 & \xrightarrow{\partial} & H_{n-1}(\partial W, F) & \xleftarrow[\cong]{e_*} & H_{n-1}(F', \partial F')
\end{array}$$

where $e\colon (F', \partial F') \to (\partial W, F)$ is an excision.

It follows from the definition of ∂ that $H_{n-1}(\partial W, F) = Z_2$. Hence

$$H_{n-1}(F', \partial F') \cong H_{n-1}(\partial W, F) = Z_2$$

and therefore F' is connected. ■

Let W be a compact 3-manifold and let $F \subset \partial W$ be a surface. The pair (W, F) is called an *h-cobordism*, if $\overline{\partial W - F}$ is connected and if the inclusion $\iota\colon F \to W$ induces an isomorphism $\iota_*\colon \pi_1(F) \to \pi_1(W)$. (Equivalently, (W, F) is an h-cobordism if $\iota\colon F \to W$ is a homotopy equivalence.)

A fake 3-cell is a contractible compact 3-manifold that is not a 3-cell. It is not known if fake 3-cells exist. The closed 3-manifold obtained by capping off the boundary of a fake 3-cell by a 3-cell is a fake 3-sphere. If M is a compact 3-manifold, then there is a unique 3-manifold $\mathfrak{P}(M)$ such that M is the connected sum of $\mathfrak{P}(M)$ and a fake 3-sphere X, $M = \mathfrak{P}(M) \# X$, and $\mathfrak{P}(M)$ does not contain fake 3-cells.

If (W, F) is an h-cobordism and F is not a projective plane then there is an isomorphism $h\colon (F \times [0, 1], F \times 0) \to (\mathfrak{P}(W), F)$ (e.g. [1], Theorem 10.2).

An h-cobordism (W, P^2), P^2 the projective plane, such that there is not an isomorphism $h\colon (P^2 \times [0, 1], P^2 \times 0) \to (W^2, P^2)$ is called a *fake P^2-h-cobordism*. It is not known if fake P^2-h-cobordisms exist.

THEOREM 2.2. *Let M be a 3-manifold and let $F \subset M$ be an incompressible proper surface such that there is a homotopy $f_t\colon (F, \partial F) \to (M, \partial M)$, $0 \leqslant t \leqslant 1$, with $f_0\colon F \to M$ is the inclusion and $f_1(F) \subset \partial M$.*

Then there is a compact manifold $W \subset M$ with $F \subset \partial W$, $W \cap \partial M = \overline{\partial W - F}$, and (W, F) is an h-cobordism.

Proof. Let $x_0 \in F$ and let $p\colon (\tilde{M}, \tilde{x}_0) \to (M, x_0)$ be the covering projection with $p_*\pi_1(\tilde{M}, \tilde{x}_0) = \pi_1(F, x_0) \subset \pi_1(M, x_0)$. The map

$$f\colon (F \times [0, 1], F \times 1, (x_0, 0)) \to (M, \partial M, x_0), f(x, t) = (f_t(x), t),$$

lifts to a map

$$\tilde{f}\colon (F \times [0, 1], F \times 1, (x_0, 0)) \to (\tilde{M}, \partial\tilde{M}, \tilde{x}_0).$$

Let $F_0 = \tilde{f}(F \times 0)$. Then $\tilde{f}\colon F = F \times 0 \to F_0$ is an isomorphism. The inclusion $\iota\colon F_0 \to \tilde{M}$ induces an isomorphism $\iota_*\colon \pi_1(F_0, \tilde{x}_0) \to \pi_1(\tilde{M}_0, \tilde{x}_0)$.

By Lemma 2.1, there is a compact 3-manifold $W_0 \subset \tilde{M}$ with $F_0 \subset \partial W_0$ and with $\overline{\partial W_0 - F_0} = W_0 \cap \partial\tilde{M}$ is connected. Let $F_0, F_1, \ldots, F_m$ be the components of $p^{-1}(F) \cap W_0$. If $m > 0$ let W_1 be the closure of the component of $W_0 - (F_1 \cup \ldots \cup F_m)$ that contains F_0 and let $\iota_0\colon F_0 \to W_1$ be the inclusion.

CLAIM. *$\iota_{0*}\colon \pi_1(F_0, \tilde{x}_0) \to \pi_1(W_1, \tilde{x}_0)$ is an isomorphism.*

Proof of claim. Let $\iota_0'\colon W_1 \to \tilde{M}$ be the inclusion. Then

$$\iota_{0*}'\colon \pi_1(W_1, \tilde{x}_0) \to \pi_1(\tilde{M}, \tilde{x}_0)$$

is injective. Namely, let $g\colon B^2 \to \operatorname{int}\tilde{M}$ be a map of the 2-cell B^2 into $\operatorname{int}\tilde{M}$ with $g(\partial B^2) \subset \operatorname{int} W$. We may assume that g is transversal with respect to $\partial W_1 \cap \operatorname{int}\tilde{M}$ which consists of $\operatorname{int} F_0$ and some of the interiors of the surfaces $F_1, \ldots, F_m$. Since the surfaces $F_0, \ldots, F_m$ are incompressible in $\tilde{M}$, the map g can be modified by a standard construction to a map $g'\colon B^2 \to W_1$ with $g'|_{\partial B^2} = g|_{\partial B^2}$. We have $\iota_* = \iota_{0*}' \cdot \iota_{0*}$. Since ι_* is an isomorphism and ι_{0*}' is injective, we conclude that ι_{0*}' and ι_{0*}' are isomorphisms.

It follows from Lefschetz duality (or the h-Cobordism Theorem [1], Theorem 10.2) that exactly one of the surfaces $F_1, \ldots, F_m$ is in ∂W_1, and that it is isomorphic to F_0. Suppose this surface is F_1. Then $p|\colon F_1 \to F$ is an isomorphism. Let $x_1 \in F_1$ with $p(x_1) = x_0$. We conclude that the inclusion $\iota_1\colon F_1 \to \tilde{M}$ induces an isomorphism $\iota_{1*}\colon \pi_1(F_1, x_1) \to \pi_1(M, x_1)$. Let $W_2 = \overline{W_0 - W_1}$. We repeat the preceding construction with W_2, F_1, x_1 replacing $W_0, F_0, \tilde{x}_0$. After a finite number of steps we will arrive at an h-cobordism $(\tilde{W}, \tilde{F})$ such that $\tilde{W} \cap \partial\tilde{M} = \overline{\partial\tilde{W} - \tilde{F}}$, $p|\colon \tilde{F} \to F$ is an isomorphism, and $p^{-1}(F) \cap \tilde{W} = \tilde{F}$. Define $W = p(\tilde{W})$. Then $p|\colon \tilde{W} \to W$ is an isomorphism and W has the desired properties. ■

COROLLARY 2.3. *Let M be a 3-manifold which does not contain fake 3-cells and let $F \subset M$ be an incompressible proper surface. If $F = P^2$ is a projective plane assume in addition that M does not contain fake P^2-h-cobordisms.*

Then there is a homotopy $f_t\colon (F, \partial F) \to (M, \partial M)$, $0 \leqslant t \leqslant 1$, with $f_0\colon F \to M$ is the inclusion and $f_1(F) \subset \partial M$ if and only if F is boundary parallel in M.

Proof. Theorem 2.2 and the h-Cobordism Theorem. ■

A 3-manifold M is *irreducible* if each 2-sphere in M bounds a 3-cell in M.

Note. If M is irreducible and if it is not a fake 3-sphere, then M does not contain fake 3-cells.

Corollary 2.3 should be known, but we do not know a reference.

Regular neighbourhoods in this paper will always be defined via second barycentric subdivisions of simplicial subdivisions. The following lemma will be frequently applied.

LEMMA (2.4) ([3], Lemma 2.1). *Let P be a polyhedron, let $\iota\colon P \to P$ be an isomorphism with $\iota^m = \mathrm{id}$, and let K be a simplicial subdivision.*

Then there is a subdivision K' of K so that ι is simplicial with respect to K'.

PROPOSITION 2.5. *Let M be a 3-manifold and let $\iota\colon M \to M$ be an involution. Suppose that $F_0 \subset M$ is a 2-sided (1-sided), ∂-incompressible, incompressible proper surface with $\partial F_0 \neq \emptyset$ that is not a 2-cell.*

Then there is a 2-sided (1-sided), ∂-incompressible, incompressible proper surface $F \subset M$ isomorphic to F such that

(0) *There are no fixed points of ι on F.*

(1) *F and ιF intersect transversally.*

(2) *If 0 is an open neighbourhood of $F_0 \cup \iota F_0$, then we may assume that $F \subset 0$.*

(3) *There are no 1-spheres in $F \cap \iota F$ that bound 2-cells in F.*

(4) *There are no 1-cells I in $F \cap \iota F$ such that there is a 2-cell $D \subset F$ with $I \subset \partial D$ and $D \cap \partial F = \overline{\partial D - I}$.*

If M is irreducible and the components of ∂M that contain boundaries of 1-cells of (3) are incompressible in M, then there is an ambient isotopy on M that maps F_0 to ιF.

Proof. It follows from Proposition 2.6 of [3] that there is a 2-sided (1-sided), ∂-incompressible, incompressible surface $F \subset M$ isomorphic to F_0 which satisfies properties (0), (1), (2), and (3). Therefore we may assume that F_0 satisfies properties (0), (1), (2), and (3).

Suppose there is a 1-cell in $F \cap \iota F$ as described in (4) Then there is a 1-cell I in $F \cap \iota F$, a 2-cell $D \subset \iota F$ with $D \cap F_0 = I$ and with $D \cap \partial \iota F_0 = \overline{\partial D - I} = D \cap \partial M$. Since F_0 is ∂-incompressible, then there is also a 2-cell $D_0 \subset F_0$ with $I \subset \partial D_0$ and with $D_0 \cap \partial F_0 = \overline{\partial D_0 - I}$. Define $F_0' = \overline{F_0 - D_0}$.

Let $D \times [0, \varepsilon]$ be a sufficiently thin collar of $D = D \times 0$ such that

$$D \times [0, \varepsilon] \cap F_0 = D \times [0, \varepsilon] \cap F_0' = I \times [0, \varepsilon], \quad \text{and}$$
$$D \times [0, \varepsilon] \cap \partial M = \overline{\partial D - I} \times [0, \varepsilon] .$$

We must have $D \times \varepsilon \cap \iota(D \times \varepsilon) = \emptyset$. Otherwise $\iota D = D_0$ and hence $\iota I = I$. Consequently, there must be a fixed point of ι on I, a contradiction: there are no fixed points of ι on $F_0 \cup \iota F_0$.

Define the proper surface $F_1 = \overline{F_0' - I \times [0, \varepsilon]} \cup D \times \varepsilon$.

Then $\iota(D \times \varepsilon) \cap F_1 = \emptyset$, and hence

$$F_1 \cap \iota F_1 \subset F_0 \cap \iota F_0 - (I \cup \iota I).$$

The surface F_1 is isomorphic to F_0 and is 2-sided (1-sided), ∂-incompressible, and incompressible in M. At least two 1-cells have been eliminated from $F_0 \cap \iota F_0$. In a finite number of steps we arrive at F.

Suppose now that the component of ∂M that contains ∂I is incompressible in M. Then the proper 2-cell $D_0 \cup D$ determines a 2-cell $B \subset \partial M$ with $\partial B = \partial(D_0 \cup D)$. If M is irreducible, the 2-sphere $D_0 \cup D \cup B$ bounds a 3-cell in M. Hence there is an ambient isotopy on M that maps F_0 to F_1. ∎

LEMMA 2.6. *Let M be a 3-manifold and let $F_1, F_2 \subset M$ be two proper surfaces that intersect transversally. Let $S \subset F_1 \cap F_2$ be a 1-sphere.*

(1) *If both F_1, F_2 are 2-sided in M, then S is 2-sided in both F_1, F_2.*

(2) *If S is 2-sided in both F_1, F_2, then a regular neighbourhood V of S in M is a solid torus.*

§ 3. Equivariant surgery on essential annuli with respect to involutions. Note that a ∂-incompressible, incompressible proper annulus in a 3-manifold is essential. We have the following converse.

PROPOSITION 3.1. *Let M be an irreducible 3-manifold and let $A \subset M$ be an essential annulus. Suppose that $\partial A \subset R$, $R \subset \partial M$ an incompressible component.*

Then A is ∂-incompressible in M.

Proof. If A is not ∂-incompressible in M, then there is a 2-cell $B \subset M$ with $B \cap A = \partial B \cap A$ is a nonseparating proper 1-cell in A and $B \cap \partial M = \overline{\partial B - I}$.

Let $B \times [-\varepsilon, \varepsilon]$ be a regular neighbourhood of $B = B \times 0$ in M with $B \times [-\varepsilon, \varepsilon] \cap A = I \times [-\varepsilon, \varepsilon]$ and $B \times [-\varepsilon, \varepsilon] \cap \partial M = \overline{\partial B - I} \times [-\varepsilon, \varepsilon]$. Then

$$D = \overline{A - I \times [-\varepsilon, \varepsilon]} \cup B \times -\varepsilon \cup B \times \varepsilon$$

is a proper 2-cell in M with $\partial D \subset R$. Since R is incompressible there is a 2-cell $D_0 \subset R$ with $\partial D_0 = \partial D$. Since M is irreducible, the 2-sphere $D \cup D_0$ bounds a 3-cell C in M. If $B \times [-\varepsilon, v] \subset C$, then A is not incompressible, and if $B \times [-\varepsilon, \varepsilon] \subset \overline{M - C}$, then A is boundary parallel in M. A contradiction. ∎

If M is not assumed to be irreducible or if R is not assumed to be incompressible, it cannot be concluded that A is ∂-incompressible. The essential anulus of the example in § 2 is contained in an irreducible 3-manifold M, but it is not ∂-incompressible: ∂M is not incompressible. An example of a 3-manifold M with ∂M incompressible that contains an essential annulus that is not ∂-incompressible can be constructed as follows: Let F be a closed surface not a 2-sphere or projective plane, and let Y be a 3-manifold not a 3-cell with ∂Y a 2-sphere. Choose an annulus $A_0 \subset F$ that is not nullhomotopic in F and a 3-cell $C \subset \operatorname{int}(A_0 \times [0, \frac{1}{2}])$. Define

$$M = \overline{F \times [0, 1] - C} / \partial C = \partial Y.$$

Then ∂M is incompressible. The annulus $A = \partial A_0 \times [0, \frac{1}{2}] \cup A_0 \times \frac{1}{2}$ is essential in M but not ∂-incompressible.

Note that an incompressible proper annulus in a 3-manifold M that has its boundary components in two different components of ∂M is essential and ∂-incompressible.

THEOREM 3.2. *Let M be a 3-manifold and let $\iota\colon M \to M$ be an involution. Suppose that $A_0 \subset M$ is an essential annulus with $\partial A_0 \cap \iota \partial A_0 = \emptyset$.*

Then either there is an essential annulus $A \subset M$ *with*

$$A \cap \iota A = \emptyset \quad \text{and} \quad \partial A \cup \iota\partial A = \partial A_0 \cup \iota\partial A_0\,,$$

or there are two disjoint annuli $A_1, A_2 \subset M$ *with*

$$\iota A_i = A_i, \quad i = 1, 2, \quad \partial(A_1 \cup A_2) = \partial A_0 \cup \iota\partial A_0\,,$$

and with no fixed points on $A_1 \cup A_2$.

Furthermore, if O *is a given neighbourhood of* $A_0 \cup \iota A_0$, *we may assume that* $A, A_1, A_2 \subset O$ *respectively.*

Note also that if A_0 *is a 2-sided (1-sided) then* A *must be 2-sided (1-sided) respectively. If* A_1, A_2 *occurs then* A_0, A_1, A_2 *must be 2-sided.*

Proof. We apply Proposition 2.5. We may assume that there are no fixed points of ι on A_0, that A_0 and ιA_0 intersect transversally, and that there are no 1-spheres in $A_0 \cap \iota A_0$ that bound 2-cells in A_0.

Therefore $A_0 \cap \iota A_0$ consists of disjoint 1-spheres that decompose A_0 and ιA_0 into annuli.

Our goal is to successively eliminate the 1-spheres from $A_0 \cap \iota A_0$.

We call a *regular neighbourhood* $V \subset \operatorname{int} M$ of a 1-sphere $S \subset A_0 \cap \iota A_0$ a standard regular neighborhood of S in M if the following properties hold:

(1) $\iota V = V$ if $\iota S = S$, and $V \cap \iota V = \emptyset$ if $\iota S \neq S$.

(2) there are no fixed points of i on V, and

(3) $A_0 \cap V$, $\iota A_0 \cap V$ are proper annuli in V.

By Lemma 2.4, standard regular neighborhoods exist.

Note V is a solid torus by Lemma 2.6.

Step 1. Suppose there are annuli $\tilde{A} \subset \iota A_0$ and $A' \subset A_0$ with $\tilde{A} \cap A_0 = \partial\tilde{A} = \partial A'$ and suppose further that there is a solid torus V_0 with $\partial V_0 = A' \cup \tilde{A}$ and such that there is an isomorphism $h\colon (A' \times [0,1], A' \times 0) \to (V_0, A')$.

We show that at least two 1-spheres of $A_0 \cap \iota A_0$ can be eliminated.

Let $\partial\tilde{A} = S_1 \cup S_2$.

Case 1. $\iota\partial\tilde{A} = \partial\tilde{A}$.

Then necessarily $\iota\tilde{A} = A'$.

Let V_1, V_2 be disjoint standard regular neighborhoods of S_1, S_2 in M respectively with either $\iota V_i = V_i$, $i = 1, 2$, or $\iota V_1 = V_2$. See Figure 2.

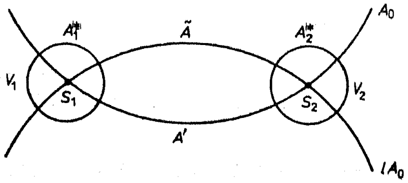

Fig. 2

$\partial(A_0 \cap V_i)$, $\partial(\iota A_0 \cap V_i)$ subdivide ∂V_i into four annuli, $i = 1, 2$. Let $A_i^* \subset \partial V_i$ be the one of the four annuli with one boundary component in $\tilde{A}$ and the other component in $\overline{A - A'}$. Define the proper annulus

$$A_0' = \overline{A_0 - (A' \cup (V_1 \cap A_0) \cup (V_2 \cap A_0))} \cup A_1^* \cup A_2^* \cup \tilde{A} \cup \overline{\tilde{A} - (\tilde{A} \cap (V_1 \cup V_2))}\,.$$

Then $A_1' \cap \iota A_1 = A_0 \cap \iota A_0 - \partial\tilde{A}$.

Case 2. $\partial\tilde{A} \cap \iota\partial\tilde{A}$ is single 1-sphere.

Suppose that $\partial\tilde{A} \cap \iota\partial\tilde{A} = S_1$. Necessarily $\iota S_1 = S_1$. Let V be a standard regular neighborhood of S_1 in M. The proper annuli $A_0 \cap V$, $\iota A_0 \cap V$ subdivide V into four solid tori. Let V_1^* be the solid torus with $V_1^* \cap \partial V$ is the annulus with one component in $\tilde{A}$ and the other component in $\overline{A_0 - A'}$ and let V_2^* be the solid torus with $V_2^* \cap \partial V$ is the annulus with one component in $\tilde{A}$ and the other component in A_0. See Figure 3.

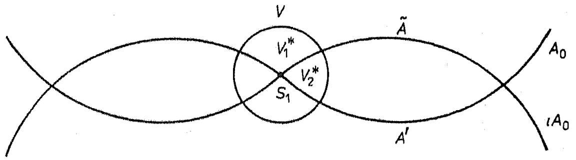

Fig. 3

Then either $\iota V_1^* = V_1^*$, $V_2^* \cap \iota V_2^* = S_1$, or $V_1^* \cap \iota V_1^* = S_1$, $\iota V_2^* = V_2^*$.

Let $\tilde{A} \times [0, \varepsilon]$ be a sufficiently thin collar of $\tilde{A} = \tilde{A} \times 0$ in M such that $\partial\tilde{A} \times [0, \varepsilon] \subset \overline{A_0 - A'}$ and $\tilde{A} \times [0, \varepsilon] \cap \partial V = \tilde{A} \times [0, \varepsilon] \cap \partial V_1^* = (\tilde{A} \cap \partial V) \times [0, \varepsilon]$.

If $\iota V_1^* = V_1^*$, we may assume that $\tilde{A} \times \varepsilon$ and $\iota(\tilde{A} \times \varepsilon)$ intersect transversally in a single 1-sphere S_* in $\operatorname{int} V_1^*$. (First construct the collar in V by means of the solid Klein bottle $V/\iota|$.)

Define the proper annulus

$$A_0' = \overline{A_0 - (A' \cup S_1 \times [0, \varepsilon] \cup S_2 \times [0, \varepsilon])} \cup \tilde{A} \times \varepsilon\,.$$

Then

$$A_0' \cap \iota A_0' = \begin{cases} (A_0 \cap \iota A - (\partial\tilde{A} \cup \partial\iota\tilde{A})) \cup S_*, & \text{if } \iota V_1^* = V_1^*, \text{ and} \\ A_0 \cap \iota A_0 - (\partial\tilde{A} \cup \partial\iota\tilde{A}), & \text{if } V_1^* \cap \iota V_1^* = S_1 \end{cases}$$

Case 3. $\tilde{A} \cap \iota\partial\tilde{A} = \emptyset$.

Let $\tilde{A} \times [0, \varepsilon]$ be a sufficiently thin collar of $\tilde{A} = \tilde{A} \times 0$ in M such that $\partial\tilde{A} \times [0, \varepsilon] \subset \overline{A_0 - A'}$. Then necessarily $\tilde{A} \times \varepsilon \cap \iota(\tilde{A} \times \varepsilon) = \emptyset$.

Define the proper annulus

$$A_0' = \overline{A_0 - (A' \cup S_1 \times [0, \varepsilon] \cup S_2 \times [0, \varepsilon])} \cup \tilde{A} \times \varepsilon\,.$$

Then $A_0' \cap \iota A_0' = A_0 \cap \iota A_0 - (\partial\tilde{A} \cup \iota\partial\tilde{A})$.

In all three cases $\partial A_0' = \partial A_0$. Using the solid torus specified in the hypothesis an ambient isotopy on M can be constructed that maps A_0 onto A_0'. Therefore A_0' is an essential annulus with $\partial A_0' = \partial A_0$ and with $A_0' \cap \iota A_0'$ containing at least two 1-spheres less than $A_0 \cap \iota A_0$.

Now let $\tilde{A} \subset \iota A_0$ be one of the two annuli defined by the 1-spheres of $A_0 \cap \iota A_0$ in ιA_0 such that one component of $\partial\tilde{A}$ is a component of $\partial\iota A_0$ and the other component is $S = \tilde{A} \cap A_0 = \partial\tilde{A} \cap A_0$. Then S decomposes A_0 into two annuli A' and A'':

$$A_0 = A' \cup A'' \quad \text{and} \quad A' \cap A'' = \partial A' \cap \partial A'' = S.$$

See Figure 4.

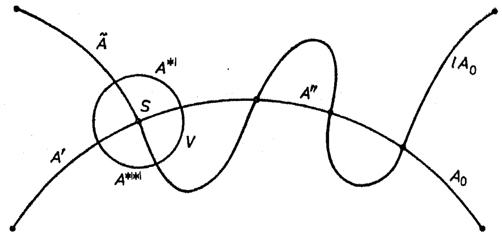

Fig. 4

We consider the proper annuli $A' \cup \tilde{A}$ and $A'' \cup \tilde{A}$.

Step 2. Suppose $A' \cup \tilde{A}$ is boundary parallel.

Applying Step 1 we may assume that $A' \cap \iota A_0 = S$.

First assume that $\iota S = S$.

Let V be a standard regular neighborhood of S in M. Then $\partial(A_0 \cap V)$, $\partial(\iota A_0 \cap V)$ subdivide ∂V into four annuli. Let A^* be the one of these four annuli with one component of ∂A^* in $\tilde{A}$ and the other component of ∂A^* in A''. See Figure 4.

Case 1. $\iota\tilde{A} = A'$.

Define the proper annulus

$$A_0' = \overline{A'' - (V \cap A'')} \cup A^* \cap \overline{\tilde{A} - (V \cap \tilde{A})}.$$

There is an ambient isotopy on M that maps A_0 to A_0'. Therefore A_0' is an essential annulus with

$$\partial A_0' \cap \iota\partial A_0' = \partial A_0 \cap \iota\partial A_0$$

and with $A_0' \cap \iota A_0' = A_0 \cap \iota A_0 - S$.

Thus we have eliminated a 1-sphere from $A_0 \cap \iota A_0$.

Case 2. $\iota\tilde{A} = A''$.

Then $A_0 \cap \iota A_0 = S$ and $\iota\overline{(\iota A_0 - \tilde{A})} = A'$. Let A^{**} be the one of the preceding four annuli with $A^{**} \cap A^* = \emptyset$. Then $\iota A^* = A^*$ and $\iota A^{**} = A^{**}$.

Define the proper annuli

$$A_1 = \overline{A'' - (V \cap A'')} \cup A^* \cup \overline{\tilde{A} - (V \cap \tilde{A})} \quad \text{and}$$
$$A_2 = \overline{A' - (V \cap A')} \cup A^{**} \cup \overline{\iota A_0 - (\tilde{A} \cup (\iota A_0 \cap V))}.$$

There are ambient isotopies on M that map A_0 to A_i, $i = 1, 2$. Therefore A_1, A_2 are disjoint essential annuli with $\iota A_i = A_i$, $i = 1, 2$, with

$$\partial(A_1 \cup A_2) = \partial A_0 \cup \iota\partial A_0,$$

and with no fixed points of ι on $A_1 \cup A_2$.

Next assume that $\iota S \neq S$.

Let V be a standard regular neighborhood of S in M with $V \cap \iota V = \emptyset$. Then $\partial(A_0 \cap V)$, $\partial(\iota A_0 \cap V)$ subdivide ∂V into four annuli. Again let A^* be the annulus with one component of ∂A^* in $\tilde{A}$ and the other component in $A_0 - A'$. Let A^{**} be the annulus with $A^{**} \cap A^* = \emptyset$.

Define the proper annulus

$$A_1' = \overline{A_0 - (A' \cup \iota\tilde{A} \cup (V \cap A_0) \cup (\iota V \cap A_0))} \cup A^* \cup \overline{\tilde{A} - (\tilde{A} \cap V)}$$
$$\cup\, \iota(A^{**} \cup \overline{A' - (A' \cap V)}).$$

There is an ambient isotopy on M that maps A_0 to A_0'. Therefore A_0' is an essential annulus with $\partial A_0' = \iota\partial A_0$ and with $A_0' \cap \iota A_0' = A_0 \cap \iota A_0 - (S \cup \iota S)$.

We have eliminated two 1-spheres from $A_0 \cap \iota A_0$.

Since one boundary component of each is a boundary component of A_0, they are also incompressible, and therefore essential.

Step 3. Both proper annuli $A' \cup \tilde{A}$ and $A'' \cup \tilde{A}$ are not boundary parallel.

We may assume that $\iota\tilde{A} \subset A'$.

If $\iota S \neq S$, define $A_0' = A'' \cup \tilde{A}$.

If $\iota S = S$, define as in Step 2, Case 1, the essential annulus

$$A_0' = \overline{A'' - (V \cap A'')} \cup A^* \cup \overline{\tilde{A} - (V \cap \tilde{A})}.$$

Then A_0' satisfies

$$\partial A_0' \cup \iota\partial A_0' = \partial A_0 \cup \iota\partial A_0$$

and $A_0' \cap \iota A_0' \subset A_0 \cap \iota A_0 - (S \cup \iota S)$.

After a finite number of steps we arrive at the annulus A or the annuli A_1, A_2 with the desired properties.

Note that a proper annulus in M is 2-sided (1-sided) if and only if its boundary is 2-sided (1-sided) in ∂M.

If $\iota A_i = A_i$, $i = 1, 2$, applying Lemma 2.4 there is a regular neighborhood W_i of A_i, $1 = 1, 2$, with $\iota W_i = W_i$ and with no fixed points if ι on W_i. If A_i is 1-sided then W_i is a solid Klein bottle (see proof of Proposition 3.4). But a solid Klein bottle does not admit a fixed point free involution. ■

If A_0 is 2-sided in M and if the components of ∂A_0 are not parallel in ∂M, the generalized loop theorem of [7] proves also Theorem 3.2 as follows: Pushing off A_0 from the fixed points of ι and cutting out an ι-invariant regular neighbourhood of the fixed point set, we may assume that ι has no fixed points. Let $p\colon M \to M/\iota$ be the natural projection. Then $p(A_0) \subset M/\iota$ is a singular annulus. It can be replaced by an essential annulus. This lifts to an essential annulus A in M with $A \cap \iota A = \emptyset$ (this argument was pointed out by the referee).

Corollary 3.3 *If in addition A_0 is ∂-incompressible in M we may assume that A, A_1, A_2 respectively are ∂-incompressible in M.*

Proof. The same four steps of the proof of Theorem 3.2 may be applied with the following modifications.

Step 1. Suppose there are annuli $\tilde{A} \subset \iota A_0$ and $A' \subset A_0$ with $\tilde{A} \cap A_0 = \partial \tilde{A} = \partial A'$ and suppose further that there is a 2-cell $D \subset M$ with $D \cap A_0 = \partial D \cap A' = I$ is a nonseparating proper 1-cell in A' and $D \cap \tilde{A} = \overline{\partial D - I}$.

Step 2. Suppose $A' \cup \tilde{A}$ is not ∂-incompressible.

Then the annuli A_0', A_1, A_2 constructed in the four steps will be incompressible and ∂-incompressible.

Proposition 3.4. *Let M be an irreducible 3-manifold that is not $P^2 \times [0, 1]$, P^2 the projective plane, and let $A \subset M$ be a 1-sided incompressible annulus such that the components of ∂M that contain ∂A are incompressible. Let W be a regular neighborhood of A in M. Then $\tilde{A} = \overline{\partial W - (W \cap \partial M)}$ is a 2-sided essential annulus in M.*

Proof. Let $MO = [0, 1] \times [-1, 1]/(0, x) \sim (1, -x)$ be a Moebius band and let

$$S^1 = [0, 1] \times 0/(0, 0) \sim (1, 0) \subset MO\,.$$

Then $(W, \tilde{A}, A) = (MO \times [0, 1],\ \partial MO \times [0, 1],\ S^1 \times [0, 1])$ and $W \cap \partial M = MO \times 0 \cup MO \times 1$.

Suppose that $\tilde{A}$ is not incompressible. Since M is irreducible and since the components of ∂M that contain ∂A are incompressible these components must be two distinct projective planes and we must have $M = P^2 \times [0, 1]$, which is excluded.

Suppose $\tilde{A}$ is boundary parallel. Then M is a solid Klein bottle and ∂M is not incompressible, a contradiction. ■

Proposition 3.5. *$P^2 \times [0, 1]$ does not admit an involution.*

Proof. Suppose $\iota\colon P^2 \times [0, 1] \to P^2 \times [0, 1]$ is an involution. Then $\iota|\colon \partial(P^2 \times [0, 1]) \to \partial(P^2 \times [0, 1])$ is fixed point free. Since a projective plane does not admit a fixed point free involution, we must have $\iota(P^2 \times 0) = P^2 \times 1$. If ι has isolated fixed points $x_1, \ldots, x_k$, let $C_1, \ldots, C_k \subset \operatorname{int}(P^2 \times [0, 1])$ be disjoint 3-cells with $x_i \in \operatorname{int} C_i$ and $\iota C_i = C_i$, $i = 1, \ldots, k$ (Lemma 2.4). Let

$$M = \overline{P^2 \times [0, 1] - (C_1 \cup \ldots \cup C_k)}\,.$$

Then $M \to M/\iota|$ is a 2-sheeted covering. Since $\pi_1(M) = Z_2$, the order of $\pi_1(M/\iota|)$ must be 4. It follows from the Lefschetz Fixed Point Theorem that ι must have fixed points. Since $\partial(M/\iota|)$ contains the projective planes $P^2 \times 0$ and $\partial C_1/\iota|$, we conclude that $\pi_1(M/\iota|) = Z_2 \oplus Z_2$. A contradiction: $Z_2 \oplus Z_2$ cannot be the fundamental group of a 3-manifold ([1], Theorem 9.13). ■

Theorem 3.6. *Let M be a 3-manifold and let $\iota\colon M \to M$ be an involution. Suppose that $A_0 \subset M$ is a ∂-incompressible essential annulus.*

Then there is a ∂-incompressible essential annulus $A \subset M$ with either $A \cap \iota A = \emptyset$ or $\iota A = A$ and with no fixed points of ι on A.

If $\iota A = A$, then A is 2-sided in M.

If M is irreducible and if the components of ∂M that contain ∂A are incompressible, we may also assume that A is 2-sided in M.

Furthermore, if O is a given neighborhood of $A_0 \cup \iota A_0$ in M, we may assume that $A \subset O$.

Proof. By Proposition 2.5 we may assume that there are no fixed points of ι on A_0, that A_0 and ιA_0 intersect transversally, that there are no 1-spheres in $A_0 \cap \iota A_0$ bounding 2-cells in A_0, and that there are no 1-cells in $A_0 \cap \iota A_0$ separating A_0. Consequently, $A_0 \cap \iota A_0$ consists either of 1-spheres that decompose A_0 into annuli, or of nonseparating 1-cells that decompose A_0 into 2-cells. In the first case we must have $\partial A_0 \cap \iota\partial A_0 = \emptyset$. Corollary 3.3 proves the theorem in this case. Hence we may assume that $A_0 \cap \iota A_0$ consists of nonseparating proper 1-cells. Our goal is to successively eliminate the 1-cells from $A_0 \cap \iota A_0$.

Note that if $I \subset A_0 \cap \iota A_0$ is a 1-cell, we must have $I \cap \iota I = \emptyset$, since ι has no fixed points on A_0. Therefore if $A_0 \cap \iota A_0 \neq \emptyset$, it must consist of more than one 1-cell.

Let $D \subset \iota A_0$ be a 2-cell with $D \cap A_0 = I_1 \cup I_2$, I_1, I_2 proper 1-cells in A_0 and ιA_0, and with $D \cap \partial M = \overline{\partial D - (I_1 \cup I_2)}$. The cells I_1, I_2 decompose A_0 into two 2-cells D' and D'': $A_0 = D' \cup D''$ with $D' \cap D'' = \partial D' = \partial D' = I_1 \cup I_2$.

Then $A' = D' \cup D$ and $A'' = D'' \cup D$ are either both proper annuli or both proper Moebius bands. Since A_0 is ∂-incompressible, both A' and A'' must be ∂-incompressible. Since A_0 is incompressible, at least one of A', A'' must be incompressible.

Step 1. Suppose both A' and A'' are proper annuli. We may assume that A' is incompressible. Therefore A' is a ∂-incompressible essential annulus.

Case 1. $\iota(I_1 \cup I_2) = I_1 \cup I_2$ and $\iota D = D'$.

Define $A = A'$. Then A is a ∂-incompressible essential annulus with $\iota A = A$.

Case 2. $\iota(I_1 \cup I_2) = I_1 \cup I_2$ and $\iota D = D''$.

Let C_1, C_2 be disjoint regular neighborhoods of I_1, I_2 in M respectively with $\iota C_1 = C_2$ and with $A_0 \cap C_i$, $\iota A_0 \cap C_i$ are proper 2-cells in the 3-cell C_i, $i = 1, 2$. The annulus $A_i^* = \overline{\partial C_i - (C_i \cap \partial M)}$ is subdivided by the four proper 1-cells $(A_0 \cup \iota A_0) \cap A_i^*$ into four 2-cells. Let $B_i^* \subset \partial C_i$ be the one of these four 2-cells that meets D' and D. See Figure 5.

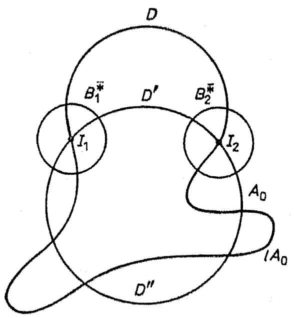

Fig. 5

Define the annulus

$$A_1 = \overline{D' - (D' \cap (C_1 \cup C_2))} \cup B_1^* \cup B_2^* \cup \overline{D - (D \cap (C_1 \cup C_2))}.$$

Then

$$A_1 \cap \iota A_1 = A_0 \cap \iota A_0 - (I_1 \cup I_2).$$

There is an ambient isotopy on M that maps A' onto A_1. We have removed two 1-cells from $A_0 \cap \iota A_0$.

Note it is not possible that $(I_1 \cup I_2) \cap \iota(I_1 \cup I_2)$ is a single 1-cell, say I_1. Namely, then necessarily $\iota I_1 = I_1$, and consequently there must be a fixed point of ι on $I_1 \subset A_0$, a contradiction.

Case 3. $(I_1 \cup I_2) \cap \iota(I_1 \cup I_2) = \emptyset$.

If $\iota D \subset \operatorname{int} D''$, define $A_1 = A'$.

Next suppose $\iota D \subset \operatorname{int} D'$. Let $D \times [0, \varepsilon]$ be a sufficiently thin collar of $D = D \times 0$ in M with $D \times [0, \varepsilon] \cap A_0 = \partial D \times [0, \varepsilon]$, $I_1 \times [0, \varepsilon] \subset D'$, and $D \times [0, \varepsilon] \cap \partial M = \overline{\partial D - (I_1 \cup I_2)} \times [0, \varepsilon]$.

Then $D \times \varepsilon \cap \iota(D \times \varepsilon) = \emptyset$. Define the proper annulus

$$A_1 = \begin{cases} \overline{D' - \partial D \times [0, \varepsilon]} \cup D \times \varepsilon, & \text{if } D \times \varepsilon \cap D'' = \emptyset, \quad \text{and} \\ \overline{D' - I_1 \times [0, \varepsilon]} \cup D \times \varepsilon \cup I_2 \times \varepsilon, & \text{if } D \times \varepsilon \cap D'' = I_2 \times \varepsilon. \end{cases}$$

Then $A_1 \cap \iota A_1 \subset A_0 \cap \iota A_0 - (I_1 \cup \iota I_1)$.

There is an ambient isotopy on M that maps A' onto A_1. At least two 1-cells have been removed from $A_0 \cap \iota A_0$.

Step 2. We may now assume that for all 2-cells $D \subset \iota A_0$, both A' and A'' are Moebius bands.

$A_0 \cap \iota A_0$ must contain more than two 1-cells. Namely otherwise, there is a $D \subset \iota A_0$ with $\iota \partial D = \partial D$ and $\iota|_{D \cup \iota D}$ is a fixed point free involution on $D \cup \iota D$. Since a Moebius band does not admit a fixed point free involution, $D \cup \iota D$ must be an annulus, and A' and A'' determined by D are annuli, a contradiction.

Let $D_1, D_2 \subset \iota A_0$ be two consecutive 2-cells with $D_1 \cap A_0 = I_1 \cup I_2$ and $D_2 \cap A_0 = I_2 \cup I_3$, I_1, I_2, I_3 1-cells. The three 1-cells I_1, I_2, I_3 decompose A_0 into three 2-cells. Let $D' \subset A_0$ be the 2-cell with $I_1, I_2 \subset \partial D'$ and $D' \cap I_2 = \emptyset$ See Figure 6.

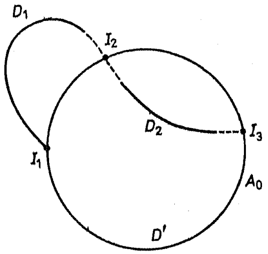

Fig. 6

Define $A' = D' \cup D_1 \cup D_2$. Since A_0 is ∂-incompressible, A' is ∂-incompressible. Since both D_1 and D_2 define Moebius bands, A' must be an annulus.

Claim. *A' is incompressible.*

Proof of Claim. Suppose A' is not incompressible. Then A' has a compressing 2-cell B such that $B \cap A' = \partial B$ is a nonseparating 1-sphere on A'. We may assume that A' intersects ιA_0 transversely. There is a 2-cell $B_0 \subset B$ such that

$$B_0 \cap A_0 = \partial B_0 \cap A_0 = I$$

is a 1-cell and the 1-cell $B_0 \cap \iota A_0 = \overline{\partial B_0 - I}$ is contained either in D_1 or in D_2. We conclude that either D_1 or D_2 respectively determine annuli instead of Moebius bands, a contradiction.

Thus A' is a ∂ incompressible essential annulus.

Define $D = D_1 \cup D_2$. The construction of Step 1 can be applied again.

Case 1. $\iota(I_1 \cup I_3) = I_1 \cup I_3$ and $\iota I_2 \subset D'$.

Define $A = A'$. Then A is a ∂ incompressible essential annulus with $\iota A = A$.

Case 2. $\iota(I_1 \cup I_3) = I_1 \cup I_3$ and $\iota I_2 \subset \overline{A_0 - D'}$.

Let C_1, C_3 be disjoint regular neighborhoods of I_1, I_3 in M respectively with $\iota C_1 = C_3$ and with $A_0 \cap C_i$, $\iota A_0 \cap C_i$ are proper 2-cells in the 3-cell C_i, $i = 1, 3$. The annulus $A_i^* = \overline{\partial C_i - (C_i \cap \partial M)}$ is subdivided by the four proper 1-cells $(A_0 \cup \iota A_0) \cap A_i^*$ into four 2-cells. Let $B_i^* \subset \partial C_i$ be the one of these four 2-cells that meets D' and D. Define the annulus

$$A_1 = \overline{D' - (D' \cap (C_1 \cup C_3))} \cup B_1^* \cup B_3^* \cup \overline{D - (D \cap (C_1 \cup C_3))}.$$

Then $A_1 \cap \iota A_2 = A_0 \cap \iota A_0 - (I_1 \cup I_2 \cup I_3)$. There is an ambient isotopy on M that maps A' onto A_1. We have removed three 1-cells from $A_0 \cap \iota A_0$.

Again it is not possible that $(I_1 \cup I_3) \cap \iota(I_1 \cup I_3)$ is a single 1-cell.

Case 3. $(I_1 \cup I_3) \cap \iota(I_1 \cup I_3) = \emptyset$.

It is not possible that $\iota I_2 = I_1$. Otherwise $D_1 \cap \iota D_1 = \partial D_1 = \partial \iota D_1$ and $\iota|_{D_1 \cup \iota D_1}$ defines a fixed point free involution on $D_1 \cup \iota D_1$. Since a Moebius band does not admit a fixed point free involution, $D_1 \cup \iota D_1$ must be an annulus. Therefore D_1 does not determine Moebius bands, a contradiction. Similarly, $\iota I_2 = I_3$ is not possible. Consequently, $\iota D \subset \operatorname{int} D'$ or $\iota D \subset A_0 - D'$.

If $\iota D \subset A_0 - D'$, define $A_1 = A'$.

Next suppose $\iota D \subset \operatorname{int} D'$. Let $D \times [0, \varepsilon]$ be a sufficiently thin collar of $D = D \times 0$ in M with $D \times [0, \varepsilon] \cap A_0 = \partial D \times [0, \varepsilon]$, $I_1 \times [0, \varepsilon] \subset D'$, and $D \times [0, \varepsilon] \cap \partial M = \overline{\partial D - (I_1 \cup I_3)} \times [0, \varepsilon]$.

Then $D \times \varepsilon \cap \iota(D \times \varepsilon) = \emptyset$. Define the proper annulus

$$A_1 = \begin{cases} \overline{D' - \partial D \times [0, \varepsilon]} \cup D \times \varepsilon, & \text{if } D \times \varepsilon \cap D'' = \emptyset, \quad \text{and} \\ D' - I_1 \times [0, \varepsilon] \cup D \times \varepsilon \cup I_2 \times \varepsilon, & \text{if } D \times \varepsilon \cap D'' = I_2 \times \varepsilon. \end{cases}$$

Then $A_1 \cap \iota A_1 \subset A_0 \cap \iota A_0 - (I_1 \cup I_2)$.

There is an ambient isotopy on M that maps A' onto A_1. At least two 1-cells have been removed from $A_0 \cap \iota A_0$.

Therefore, in a finite number of steps we arrive at the ∂-incompressible essential annulus A with either $A \cap \iota A = \emptyset$ or $\iota A = A$ and with no fixed points of ι on A.

If $\iota A = A$, A must be 2-sided in M. Namely otherwise, by Lemma 2.4, there is a regular neighborhood W of A in M with $\iota W = W$ and with no fixed points of ι on W. But W is a solid Klein bottle. It does not admit a fixed point free involution, a contradiction.

Suppose $A \cap \iota A = \emptyset$ and A is 1-sided in M. If M is irreducible and if the components of ∂M that contain ∂A are incompressible, let W be a regular neighborhood of A in M with $W \cap \iota W = \emptyset$. By Proposition 3.4, $\tilde{A} = \overline{\partial W - (W \cap \partial M)}$ is a 2-sided essential annulus in M with $\tilde{A} \cap \iota \tilde{A} = \emptyset$. Since A is ∂-incompressible, so is $\tilde{A}$. Replace A by $\tilde{A}$. ■

COROLLARY 3.7. *Let M be an irreducible 3-manifold and let ι: $M \to M$ be an involution. Suppose that $A_0 \subset M$ is an essential annulus such that the components of ∂M that contain ∂A are incompressible.*

Then there is a 2-sided essential annulus $A \subset M$ with either $A \cap \iota A = \emptyset$ or $\iota A = A$ and with no fixed points of ι on A.

Furthermore, if O is a given neighborhood of $A_0 \cup \iota A_0$ in M, we may assume that $A \subset O$.

Proof. Proposition 3.1 and Theorem 3.6. ■

Note also that if the 3-manifold M is orientable each proper annulus in M is 2-sided.

§ 4. Equivariant surgery on essential Moebius bands with respect to involutions.

Again note that a ∂-incompressible, incompressible proper Moebius band in a 3-manifold is essential. Also again we have the following converse.

PROPOSITION 4.1. *Let M be an irreducible 3-manifold and let $MO \subset M$ be an essential Moebius band. Suppose that $\partial MO \subset R$, $R \subset \partial M$ an incompressible component.*

Then MO is ∂-incompressible in M.

Proof. If MO is not ∂-incompressible, then as in the proof of Proposition 3.1 it follows that MO is boundary parallel, a contradiction. ■

Again if M is not assumed to be irreducible or if R is not assumed to be incompressible, it cannot be concluded that MO is ∂-incompressible. If for example $V = [0, 1] \times [-1, 1] \times [-1, 1]/(0, x, y) \sim (1, -x, -y)$ is the solid torus, then $[0, 1] \times [-1, 1] \times 0/(0, x, 0) \sim (1, -x, 0)$ is an essential Moebius band in V that is not ∂-incompressible. Similarly as in case of the annulus examples can be constructed of essential Moebius bands MO in 3-manifolds M that are not irreducible and with R incompressible such that MO is not ∂-incompressible.

THEOREM 4.2. *Let M be a 3-manifold and let ι: $M \to M$ be an involution. Suppose that $MO_0 \subset M$ is an essential Moebius band with $\partial MO_0 \cap \iota \partial MO_0 = \emptyset$.*

Then one of the following three properties holds.

(I) *Either there is an essential Moebius band $MO \subset M$ with $MO \cap \iota MO = \emptyset$ $\partial MO = \partial MO$, or there is a 2-sided essential annulus $A \subset M$ with $\iota A = A$ and with no fixed points of ι on A. If MO_0 is 2-sided (1-sided) so is MO.*

(II) *MO_0 is 1-sided in M and there is a 2-sided essential annulus A in M with $\partial A = \partial MO_0 \cup \iota \partial MO_0$ and $A \cap \iota A = \partial A = \partial \iota A$.*

(III) *M is a solid torus. There is an essential Moebius band $MO \subset M$ with $\partial MO = \partial MO_0$ and $MO \cap \iota MO = S$ is a single nonseparating 1-sphere in MO.*

Proof. We apply Proposition 2.5. We may assume that there are no fixed points of ι on MO_0, that MO_0 and ιMO_0 intersect transversally, and that there are no 1-spheres in $MO_0 \cap \iota MO_0$ that bound 2-cells in MO_0.

Therefore $MO_0 \cap \iota MO_0$ consists of disjoint 1-spheres. If MO_0 is 2-sided in M then each 1-sphere separates MO_0 into an annulus and a Moebius band (Lemma 2.6). If MO_0 is 1-sided in M then it is not possible that a 1-sphere separates one and not the other of MO_0 and ιMO_0 (otherwise MO_0 must be 2-sided in M, a contradiction).

Our goal is to successively eliminate the 1-spheres from $MO_0 \cap \iota MO_0$.

Step 1. Suppose there is a 1-sphere in $MO_0 \cap \iota MO_0$ that separates MO_0.

Then there is a unique annulus $\tilde{A} \subset \iota MO_0$ with $\tilde{A} \cap MO_0 = \partial\tilde{A} \cap MO_0 = S$ a single 1-sphere and $\partial A = S \cup \partial MO_0$. Necessarily $\iota S = S$. Consider the annulus $A = \tilde{A} \cup \iota\tilde{A}$. Note A is incompressible.

Case 1. The annulus A is not boundary parallel. Then A is an essential annulus with $\iota A = A$ and with no fixed points of ι on A. Further A must be 2-sided in M (see remark in proof of Theorem 3.6).

Case 2. The annulus A is boundary parallel. Then the incompressible Moebius band $MO = \overline{MO - \iota\tilde{A}} \cup \tilde{A}$ is not boundary parallel and therefore essential. Applying Lemma 2.4, let V be a regular neighborhood of S in $\operatorname{int} M$ such that

$$\iota V = V, \quad MO_0 \cap V, \quad \iota MO_0 \cap V$$

are proper annuli in V, and such that there are no fixed points of ι on V. Then V is a solid torus and $\partial(MO_0 \cap V)$, $\partial(\iota MO_0 \cap V)$ subdivide ∂V into four annuli. Let $A^* \subset \partial V$ be the one of the four annuli with one boundary component in $\tilde{A}$ and the other component in $\overline{MO_0 - \iota\tilde{A}}$. Define the essential Moebius band

$$MO_1 = \overline{MO_0 - (\iota\tilde{A} \cup (MO_0 \cap V)} \cup A^* \cup \tilde{A}.$$

Then $MO_1 \cap \iota MO_1 = MO_0 \cap \iota MO_0 - S$. One 1-sphere has been removed from $MO_0 \cap \iota MO_0$. Note $\partial MO_1 = \partial MO_0$.

Applying Step 1 a finite number of times we either arrive at property I of the theorem or at the following.

Step 2. $MO_0 \cap \iota MO_0 = S$ a single 1-sphere and S does not separate MO_0l and not ιMO_0.

Applying Lemma 2.4, let V be a regular neighborhood of S in $\operatorname{int} V$ such that $\iota V = V$, $MO \cap V$, $\iota MO_0 \cap V$ are proper Moebius bands, and there are no fixed points of ι on V. Then V is a solid torus and the two 1-spheres $\partial(MO_0 \cap V)$, $\partial(\iota MO_0 \cap V)$ subdivide ∂V into two annuli A^*, ιA^*. Consider the incompressible 2-sided annulus $A' = \overline{MO_0 - (V \cap MO_0)} \cup A^* \cup \iota(\overline{MO_0 - (V \cap MO_0)})$. If A' is not boundary parallel it is easy to modify A' to obtain the 2-sided essential annulus A of property II of the theorem. If A' is boundary parallel, it follows that M is a solid torus and we have arrived at property III of the theorem. ■

Example. Let $V = [0, 1] \times [-1, 1] \times [-1, 1]/(0, x, y) \sim (1, -x, -y)$ be the solid torus and let $MO = [0, 1] \times [-1, 1] \times 0/(0, x, 0) \sim (1, -x, 0)$ be an essential Moebius band in V. Define the fixed point free involution $\iota\colon V \to V$ by

$$\iota(t, x, y) = (1-t, y, x).$$

Then $MO \cap \iota MO = S$ is a nonseparating 1-sphere in MO.

Corollary 4.3. *If in addition MO_0 is ∂-incompressible in M only properties* (I) *or* (II) *hold and we may assume that the annulus A and the Moebius band MO are ∂-incompressible.*

Proof. In Step 1 of the proof of Theorem 4.2 either the annulus A or the Moebius band MO' must be ∂-incompressible. In Step 2, the annulus A must be ∂-incompressible. Property III cannot hold since a Moebius band in a solid torus is not ∂-incompressible. ■

Theorem 4.4. *Let M be a 3-manifold and let $\iota\colon M \to M$ be an involution. Suppose that $MO_0 \subset M$ is a ∂-incompressible essential Moebius band.*

Then there is either a ∂-incompressible essential annulus or Moebius band $F \subset M$ with $F \cap \iota F = \emptyset$ or there is a 2-sided ∂-incompressible essential annulus $A \subset M$ with $\iota A = A$ and with no fixed points of ι on A.

Furthermore, if O is a given neighborhood of $A_0 \cup \iota A_0$ in M, we may assume that $A \subset O$.

Proof. By Proposition 2.5 we may assume that there are no fixed points of ι on M_0, that MO_0 and ιMO_0 intersect transversally, that there are no 1-spheres in $MO_0 \cap \iota MO_0$ bounding 2-cells in MO_0, and that there are no 1-cells in $MO_0 \cap \iota MO_0$ separating MO_0. Consequently either $\partial MO_0 \cap \partial\iota MO_0 = \emptyset$ and $MO_0 \cap \iota MO_0$ consists of 1-spheres or $MO_0 \cap \iota MO_0$ consists of nonseparating 1-cells that decompose MO_0 into 2-cells.

In the first case Corollary 4.3 proves the theorem. If property II holds, let W be a regular neighborhood of $A \cup \iota A$, A the 2-sided ∂-incompressible essential annulus with $A \cap \iota A = \partial A$, such that $\iota W = W$. Let $A' \subset \partial W$ be an annulus that is proper in M. Then A' is a 2-sided ∂-incompressible essential annulus with $A' \cap \iota A' = \emptyset$.

In the second case the proof of Theorem 3.6 applies. If $MO_0 \cap \iota MO_0 \neq \emptyset$ we actually can obtain an annulus with the required properties. Step 1 of the proof alone will do. ■

Corollary 4.5. *Let M be an irreducible 3-manifold and let $\iota\colon M \to M$ be an involution. Suppose that $MO_0 \subset M$ is an essential Moebius band such that the component of ∂M that contains ∂MO_0 is incompressible.*

Then there is either a 2-sided essential annulus or Moebius band $F \subset M$ with $F \cap \iota F = \emptyset$ or there is a 2-sided essential annulus $A \subset M$ with $\iota A = A$ and with no fixed points of ι on A.

Furthermore, if O is a given neighborhood of $A_0 \cup \iota A_0$ in M, we may assume that $A \subset O$.

Proof. Proposition 4.1 and Theorem 4.4. By Corollary 3.7 we may assume that if F is an annulus it is 2-sided. If F is a 1-sided Moebius band, let W be a regular neighborhood of F in M with $W \cap \iota W = \emptyset$. Then $A = \partial W - (W \cap \partial M)$ must be an essential annulus (since $\partial(W \cap \partial M)$ consists of two 1-spheres it is incompressible. If it is boundary parallel, M must be a solid torus and ∂M is not incompressible). Thus A is 2-sided and $A \cap \iota A = \emptyset$. ■

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On the Cauchy equation modulo Z

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Abstract. Assume that X is a real linear topological space (which always is assumed to be Hausdorff) and let $f\colon X \to \mathbf{R}$ be a function such that

$$f(x+y)-f(x)-f(y) \in \mathbf{Z}$$

for all $x, y \in X$. Some conditions are established under which f has the form $g+k$, where g is a continuous linear functional on the space X and the function k takes integer values only. An application to the Cauchy equation

$$f(x+y) = f(x)+f(y)$$

for functions acting between linear topological spaces is also given.

Let a function $f\colon \mathbf{R} \to \mathbf{R}$ be given and assume that

$$f(x+y)-f(x)-f(y) \in \mathbf{Z} \tag{1}$$

for all $x, y \in \mathbf{R}$, where $\mathbf{Z}$ denotes the set of all integers. As follows from an example of G. Godini [6, Example 2], it is not generally true that such a function f must be of the form $g+k$ where g is an additive function and k takes integers values only. However, the following theorem has been proved in paper [1]:

THEOREM 1. *If the Cauchy difference $f(x+y)-f(x)-f(y)$, as a function of two real variables, is Lebesgue measurable and takes integer values only, then there exists an additive function $g\colon \mathbf{R} \to \mathbf{R}$ and a Lebesgue measurable function $k\colon \mathbf{R} \to \mathbf{Z}$ such that*

$$f = g+k. \tag{2}$$

In the present paper, the following theorem will be shown:

THEOREM 2. *Assume that X is a real linear topological space.*
If a function $f\colon X \to \mathbf{R}$ satisfies condition (1) *for all $x, y \in X$ and there exists a set $E \subset X$ such that*

$$0 \in \operatorname{Int}(E-E) \tag{3}$$