Equivariant surgery on essential annuli and
Moebius bands in 3-manifolds with respect to involutions

by

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Abstract. A complete classification is given for equivariant surgery on essential annuli and Moebius bands in 3-manifolds with respect to involutions with at most isolated fixed points.

§ 1. Introduction. Suppose $M$ is a 3-manifold that contains an essential annulus. Equivariant surgery on it is possible with respect to an involution $\iota : M \to M$ with at most isolated fixed points if there exists an essential annulus $A$ in $M$ with either $A \cap \iota A = \emptyset$ or $\iota A = A$ and with no fixed points of $\iota$ on $A$. We prove that for $\partial$-incompressible essential annuli equivariant surgery is always possible (Theorem 3.6). In contrast, equivariant surgery on incompressible tori in orientable, closed, irreducible 3-manifolds with respect to orientation preserving involutions without fixed points is not possible in general [3], [6]. An essential annulus $A$ in a 3-manifold $M$ is $\partial$-incompressible if the components of $\partial A$ are in different components of $\partial M$, or if $\partial A$ is contained in an incompressible component of $\partial M$ and $M$ is irreducible (Proposition 3.1).

We also prove a relative version of equivariant surgery on an essential annulus $A_0 \subset M$ with $\partial A_0 \cap \partial A_0 = \emptyset$ (Theorem 3.2).

If $M$ is irreducible and if the components of $\partial M$ that contain $\partial A$ are incompressible, we may assume that the annulus $A$ obtained by equivariant surgery in Theorem 3.6 is 2-sided in $M$ (Propositions 3.4 and 3.5).

Our method of proof applies also to essential Moebius bands. The corresponding theorems are given in § 4.

The usual definition of an incompressible proper surface $F$ in a 3-manifold $M$ to be essential requires that there does not exist a homotopy $f_t : (F, \partial F) \to (M, \partial M)$, $0 \leq t \leq 1$, $f_0 : F \to M$ the inclusion, and $f_1(F) \subset \partial M$. This definition is not suitable for our surgery arguments. It follows from Corollary 2.3 that if $M$ does not contain fake 3-cells, then it can be replaced by the requirement that $F$ is not boundary parallel in $M$.

The method of proof of the above Theorems 3.2 and 3.6 is to move a given essential annulus by an $e$-isotopy such that the new annulus and its image under
the involution intersect transversally in 1-spheres and proper 1-cells. We may assume
that the 1-spheres do not bound 2-cells and that the 1-cells do not separate. Then
either the intersection consists only of 1-spheres decomposing the given annulus
into annuli, or it consists only of 1-cells decomposing the given annulus into 2-cells.
The annuli or 2-cells are modified and rearranged to give an essential annulus with
the desired properties. If the given annulus is decomposed into 2-cells, extra care
has to be taken to avoid ending up with a Moebius band.

If we assume that the original annulus $A_0$ is 2-sided in $M$ and further that
its boundary components are not parallel in $\partial M$, then Theorem 3.2, the relative
version of equivariant surgery on annuli, is also a consequence of the generalized
loop theorem [7].

In [5], the characterization of those 3-manifolds with subgroups $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ in
their fundamental groups required to determine all involutions with some isolated
fixed points on orientable 3-dimensional flat space forms. This was accomplished
in (4). Theorems 3.2 and 3.6 were applied to perform equivariant surgery on essential
annuli.

There is a certain similarity between the methods of proof of this paper and those in [3]
to perform equivariant surgery on incompressible tori and Klein bottles.

If the involution has 1-dimensional or 2-dimensional fixed point sets similar
but more delicate results hold [2].

I would like to thank the referee for his comments.

§ 2. Notation and preliminaries. We will work throughout in the PL category.
Our reference is [1]. A PL homeomorphism we simply call an isomorphism.
All 3-manifolds are assumed to be connected.

All involutions are assumed to have at most isolated fixed points.

An $(n-1)$-manifold $F$ contained in an $n$-manifold $M$ is said to be proper
if $F \cap \partial M = \emptyset$.

A surface is a compact, connected 2-manifold.
Let $M$ be a 3-manifold and let $F \subset M$ be a surface that is proper or $F \subset \partial M$.
Suppose $F$ is not a 2-sphere that bounds a 3-cell in $M$, or a 2-cell with $F \subset \partial C$,
$C \subset M$ a 3-cell, and with $C = F = C \cap \partial M$. Then $F$ is said to be incompressible
in $M$, if for each 2-cell $B \subset M$, with $B \cap F = \partial B$, there is a 2-cell $D \subset F$, with
$\partial D = \partial B$. The surface $F$ is said to be $\delta$-incompressible (boundary-incompressible) if
for each 2-cell $B \subset M$ with $B \cap F = \partial B \cap F = I$ a 1-cell and $B \cap \partial M = \partial B - I$,
there is a 2-cell $D \subset F$ with $I \subset D$ and $\partial D \cap F = \partial D - I$. The surface $F$ is said to be
boundary parallel, if there is an embedding $h: F \times [0, 1] \to M$ with $h(F \times 0) = F$,
h($F \times 1 \cup \partial F \times [0, 1]) \subset \partial M$. Finally the surface $F$ is called essential if it is
incompressible and not boundary parallel.

Note that if an incompressible proper surface $F \subset M$ has components of $\partial F$
in at least two different components of $\partial M$, or if $\partial F \neq \emptyset$ and $F$ is $\delta$-incompressible
in $M$, then $F$ is essential in $M$.

If $S^1$ is a 1-sphere, then $A = S^1 \times [0, 1]$ is an annulus.

EXAMPLE. In Figure 1 an essential annulus $A$ is exhibited in a handlebody
of genus 2 that is not $\delta$-incompressible. (Note: Incompressible proper surfaces in
handlebodies must have boundaries and are not $\delta$-incompressible.)

![Figure 1](image)

LEMMA 2.1. Let $M$ be an $n$-manifold, let $F \subset M$ be a compact, connected, locally
lat. proper $(n-1)$-manifold, let $W_0$ be a compact, connected $n$-manifold with
$\partial W = F_0 \cup F_1, F_0, F_1$ connected $(n-1)$-manifolds such that $F_0 \cap F_1 = \emptyset$,
and let $f: (W_0, F_1, F_0) \to (M, \partial M, F)$ be a map with

$$(f|_*) : H_{n-1}(F_0, \partial F_0) \to H_{n-1}(F_1, \partial F_1)$$

is an isomorphism, where the homology groups have coefficients in $\mathbb{Z}_2$.

Then $F$ decomposes $M$ into two connected $n$-manifolds $M_0$ and $W$:
$M = M_0 \cup W$ and $M_0 \cap W = \partial M_0 \cap \partial W = F$ such that $W$ is compact and
$\partial W - F$ is connected.
(Note: It is possible that $\partial F = \emptyset$.)

Proof. Consider the map $f: (W_0, \partial W_0, F) \to (M, \partial M \cup F, \partial M)$ and the
commutative diagram

$$
\begin{array}{c}
H_n(W_0, F) = 0 \to H_n(W_0, \partial W_0) = Z_2 \xrightarrow{\text{in}} H_{n-1}(\partial W_0, F_1) \xrightarrow{\text{in}} H_{n-1}(F_0, \partial F_0) = Z_2 \\
\text{in} \downarrow f \text{in} \downarrow f \\
H_n(M, \partial M) \xrightarrow{\text{in}} H_n(M, \partial M \cup F) \xrightarrow{\text{in}} H_{n-1}(\partial M \cup F, \partial M) \xrightarrow{\text{in}} H_{n-1}(F, \partial F) = Z_2
\end{array}
$$

where $e: (F_0, \partial F_0) \to (\partial W_0, F_0)$, $e: (F, \partial F) \to (\partial M \cup F, \partial M)$ are excision maps and
the horizontal rows preceding the excision isomorphism are portions of the exact
homology sequences of the triples $(W_0, \partial W_0, F_1)$ and $(M, \partial M \cup F, \partial M)$
respectively.

Let $(W_0) \in H_1(W_0, \partial W_0)$. We conclude that $\partial f_*[W_0] \neq 0$ and that $F$
de-composes $M$ into two connected $n$-manifolds $M_0$ and $W$: $M = M_0 \cup W$ and
$M_0 \cap W = \partial M_0 \cap \partial W = F$. Then

$$H_n(M, \partial M \cup F) = H_n(M_0, \partial M_0) \oplus H_n(W, \partial W).$$

We may assume that $f_*[W] = (0, [W])$, $[W] \in H_n(W, \partial W)$. Hence $W$ is compact.
Let $F' = \partial W - F$.

Consider the map $f: (W_0, \partial W_0, F_0) \to (M, \partial M, F)$ and the commutative diagram

$$
\begin{array}{c}
H_n(W_0, F_0) = 0 \to H_n(W_0, \partial W_0) = Z_2 \to H_{n-1}(\partial W_0, F_0) = Z_2 \\
\downarrow f_* \downarrow \downarrow \downarrow \downarrow \\
H_n(M, F) = 0 \to H_n(W, \partial W) = Z_2 \to H_{n-1}(\partial W, F) \cong H_{n-1}(F', \partial F')
\end{array}
$$

where $e: (F', \partial F') \to (\partial W, F)$ is an excision.

It follows from the definition of $\delta$ that $H_{n-1}(\partial W, F) = Z_2$. Hence

$$H_{n-1}(F', \partial F') \cong H_{n-1}(\partial W, F) = Z_2
$$

and therefore $F'$ is connected. $\blacksquare$

Let $W$ be a compact 3-manifold and let $F = \partial W$ be a surface. The pair $(W, F)$ is called an $h$-cobordism, if $\partial W - F$ is connected and if the inclusion $i: F \to W$ induces an isomorphism $i_*: \pi_1(F) \to \pi_1(W)$. (Equivalently, $(W, F)$ is an $h$-cobordism if $i: F \to W$ is a homotopy equivalence.)

A fake 3-cell is a contractible compact 3-manifold that is not a 3-cell. It is not known if fake 3-cells exist. The closed 3-manifold obtained by capping off the boundary of a fake 3-cell by a 3-cell is a fake 3-sphere. If $M$ is a compact 3-manifold, then there is a unique 3-manifold $\Psi(M)$ such that $M$ is the connected sum of $\Psi(M)$ and a fake 3-sphere $X$, $M = \Psi(M) \# X$, and $\Psi(M)$ does not contain fake 3-cells.

If $(W, F)$ is an $h$-cobordism and $F$ is not a projective plane then there is an isomorphism $h: (F \times [0, 1], F \times 0) \to (\Psi(W), F)$ (cf. [1], Theorem 10.2).

An $h$-cobordism $(W, P^2)$, $P^2$ the projective plane, such that there is not an isomorphism $h: (P^2 \times [0, 1], P^2 \times 0) \to (W^2, P^2)$ is called a fake $P^2$-h-cobordism. It is not known if fake $P^2$-h-cobordisms exist.

**Theorem 2.2.** Let $M$ be a 3-manifold and let $F = \partial M$ be an incompressible proper surface such that there is a homotopy $f_1: (F, \partial F) \to (M, \partial M)$, $0 \leq i \leq 1$, with $f_0: F \to M$ is the inclusion and $f_1(F) \subset \partial M$. Then there is a compact manifold $W \subset M$ with $\partial W \cdot \partial W = \partial W - F$, $(W, F)$ is an $h$-cobordism.

**Proof.** Let $x_0 \in F$ and let $p: (\tilde{M}, \tilde{x}_0) \to (M, x_0)$ be the covering projection with $p \circ \pi = \pi_1(F, x_0) \to \pi_1(M, x_0)$. The map

$$f: (F \times [0, 1], F \times 0, x_0) \to (M, \partial M, x_0), f(x, t) = (f_0(x), t),$$

lifts to a map

$$\tilde{f}: (F \times [0, 1], F \times 0, 0) \to (\tilde{M}, \partial \tilde{M}, \tilde{x}_0).$$

Let $F_0 = \tilde{f}(F \times 0)$. Then $\tilde{f}: F = F \times 0 \to F_0$ is an inclusion. The inclusion $i: F_0 \to \tilde{M}$ induces an isomorphism $i_*: \pi_1(F_0, \tilde{x}_0) \to \pi_1(\tilde{M}, \tilde{x}_0).$

By Lemma 2.1, there is a compact 3-manifold $W_0 \subset \tilde{M}$ with $F_0 \subset \partial W_0$ and with $\partial W_0 - F_0 \equiv W_0 \cap \partial \tilde{M}$ is connected. Let $F_0, F_1, \ldots, F_n$ be the components of $\partial W_0 - F_0$. If $m > 0$ let $W_1$ be the closure of the component of $W_0 - (F_1 \cup \ldots \cup F_m)$ that contains $F_0$ and let $\iota: F_0 \to W_1$ be the inclusion.

**Claim.** $i_*: \pi_1(F_0, \tilde{x}_0) \to \pi_1(W_1, \tilde{x}_0)$ is an isomorphism.

**Proof of claim.** Let $\iota: W_1 \to \tilde{M}$ be the inclusion. Then

$$\iota_*: \pi_1(W_1, \tilde{x}_0) \to \pi_1(\tilde{M}, \tilde{x}_0)$$

is injective. Namely, let $g: B^3 \to \partial \tilde{M}$ be a map of the 2-cell $B^2$ into $\partial \tilde{M}$ with $g(\partial B^2) \subset \partial W_1$. We may assume that $g$ is transversal with respect to $\partial W_1 \cap \partial \tilde{M}$ which consists of $\partial F_0$ and some of the interiors of the surfaces $F_1, \ldots, F_n$. Since the surfaces $F_0, \ldots, F_n$ are incompressible in $\tilde{M}$, the map $g$ can be modified by a standard construction to a map $g': B^2 \to W_1$ with $g'|_{\partial B^2} = g|_{\partial B^2}$. We have

$$i_* = \iota_* \circ g_*.$$ Since $i_*$ is an isomorphism and $\iota_*$ is injective, we conclude that $i_*$ and $\iota_*$ are isomorphisms.

It follows from Lefschetz duality (or the h-Cobordism Theorem [1], Theorem 10.2) that exactly one of the surfaces $F_i, \ldots, F_n$ is in $\partial W_0$, and that it is isomorphic to $F_0$. Suppose this surface is $F_i$. Then $p_i: F_i \to F$ is an isomorphism. Let $x_i \in F_i$ with $p_i(x_i) = x_0$. We conclude that the inclusion $i_1: F_1 \to \tilde{M}$ induces an isomorphism

$$i_{1*}: \pi_1(F_1, x_1) \to \pi_1(M, x_i).$$

Let $W_2 = W_0 - W_1$. We repeat the preceding construction with $W_2, F_1, x_1$ replacing $W_0, F_0, x_0$. After a finite number of steps we will arrive at an $h$-cobordism $(\tilde{W}, F)$ such that $\tilde{W} \cap \partial \tilde{M} = \partial \tilde{W} - F$, $p: F \to F$ is an isomorphism, and $p^{-1}(F) \cap \tilde{W} = F$. Define $W = p(\tilde{W})$. Then $p: \tilde{W} \to W$ is an isomorphism and $W$ has the desired properties. $\blacksquare$

**Corollary 2.3.** Let $M$ be a 3-manifold which does not contain fake 3-cells and let $F = \partial M$ be an incompressible proper surface. If $F = P^2$ is a projective plane assume in addition that $M$ does not contain fake $P^2$-h-cobordisms.

Then there is a homotopy $f_1: (F, \partial F) \to (M, \partial M)$, $0 \leq i \leq 1$, with $f_0: F \to M$ is the inclusion and $f_1(F) \subset \partial M$ if and only if $F$ is boundary parallel in $M$.

**Proof.** Theorem 2.2 and the h-Cobordism Theorem. $\blacksquare$

A 3-manifold $M$ is irreducible if each 2-sphere in $M$ bounds a 3-cell in $M$. Note. If $M$ is irreducible and if it is not a fake 3-sphere, then $M$ does not contain fake 3-cell.

Corollary 2.3 should be known, but we do not know a reference. Regular neighbourhoods in this paper will always be defined via second barycentric subdivisions of simplicial subdivisions. The following lemma will be frequently applied.

**Lemma (2.4) (3), Lemma 2.1.** Let $P$ be a polyhedron, let $s: P \to P$ be an isomorphism with $s = \id$, and let $K$ be a simplicial subdivision. Then there is a subdivision $K'$ of $K$ so that $s$ is simplicial with respect to $K'$. 
Proposition 2.5. Let $M$ be a 3-manifold and let $\iota: M \to M$ be an involution. Suppose that $F_0 \subset M$ is a 2-sided (1-sided), $\delta$-incompressible, incompressible proper surface with $\partial F_0 \neq \emptyset$ that is not a 2-cell.

Then there is a 2-sided (1-sided), $\delta$-incompressible, incompressible proper surface $F \subset M$ isomorphic to $F$ such that

(0) There are no fixed points of $\iota$ on $F$. 
(1) $F$ and $\iota F$ intersect transversely. 
(2) If $0$ is an open neighbourhood of $F_0 \cup F_0$, then we may assume that $F \subset 0$. 
(3) There are no 1-spheres in $F \cap \iota F$ that bound 2-cells in $F$. 
(4) There are no 1-cells $I$ in $F \cap \iota F$ such that there is a 2-cell $D \subset F$ with $I \subset \partial D$ and $D \cap \iota F = \partial D \cap I$.

If $M$ is irreducible and the components of $\partial M$ that contain boundaries of 1-cells of $\delta$ are not incompressible in $M$, then there is an ambient isotopy on $M$ that maps $F_0$ to $\iota F$.

Proof. It follows from Proposition 2.6 of [3] that there is a 2-sided (1-sided), $\delta$-incompressible, incompressible proper surface $F \subset M$ isomorphic to $F_0$ which satisfies properties (0), (1), (2), and (3). Therefore, we may assume that $F_0$ satisfies properties (0), (1), (2), and (3).

Suppose there is a 1-cell in $F \cap \iota F$ as described in (4). Then there is a 1-cell $I$ in $F \cap \iota F$, a 2-cell $D \subset \iota F$ with $D \cap F_0 = I$ and with $D \cap \iota D = \partial D \cap I = D \cap \iota M$. Since $F_0$ is $\delta$-incompressible, then there is also a 2-cell $D_0 \subset F_0$ with $I \subset D_0$ and $D_0 \cap \iota D_0 = \partial D_0 \cap I$. Define $F' = F_0 - D_0$.

Let $D \times [0, e]$ be a sufficiently thin collar of $D = D \times 0$ such that

$D \times [0, e] \cap F_0 = D \times [0, e] \cap F' = I \times [0, e]$, and

$D \times [0, e] \cap \partial M = \partial D \times I \times [0, e]$.

We must have $D \times e \cap (D \times e) = \emptyset$. Otherwise $U = D_0$ and hence $U = I$. Consequently, there must be a fixed point of $\iota$ on $I$, a contradiction: there are no fixed points of $\iota$ on $F_0 \cup F_0$.

Define the proper surface $F_1 = F_0 - I \times [0, e] \cup D \times e$. Then $(D \times e) \cap F_1 = \emptyset$, and hence $F_1 \cap F_0 = F_0 \cap F_0 = (I \cup U)$.

The surface $F_1$ is isomorphic to $F_0$ and is 2-sided (1-sided), $\delta$-incompressible, and incompressible in $M$. At least two 1-cells have been eliminated from $F_0 \cap F_0$.

In a finite number of steps we arrive at $F$.

Suppose now that the component of $\partial M$ that contains $\partial I$ is incompressible in $M$.

Then the proper 2-cell $D_0 \cup D$ determines a 2-cell $B \subset M$ with $\partial B = \partial (D_0 \cup D)$. If $M$ is irreducible, the 2-sphere $D_0 \cup D \cup B$ bounds a 3-cell in $M$. Hence there is an ambient isotopy on $M$ that maps $F_0$ to $F$.

Lemma 2.6. Let $M$ be a 3-manifold and let $F_1, F_2 \subset M$ be two proper surfaces that intersect transversally. Let $S \subset F_1 \cap F_2$ be a 1-sphere.

(1) If both $F_1, F_2$ are 2-sided in $M$, then $S$ is 2-sided in both $F_1, F_2$.

(2) If $S$ is 2-sided in both $F_1, F_2$, then a regular neighbourhood $V$ of $S$ in $M$ is a solid torus.

§ 3. Equivariant surgery on essential annuli with respect to involutions. Note that a $\delta$-incompressible, incompressible proper annulus in a 3-manifold is essential. We have the following converse.

Proposition 3.1. Let $M$ be an irreducible 3-manifold and let $A \subset M$ be an essential annulus. Suppose that $\partial A \subset R, R \subset \partial M$ is an incompressible component.

Then $A$ is $\delta$-incompressible in $M$.

Proof. If $A$ is not $\delta$-incompressible in $M$, then there is a 2-cell $B \subset M$ with $B \cap A = \partial B \cap A$ of a non-separating proper 1-cell $A$ in $A$ and $B \cap \partial M = \partial B \cap I$.

Let $B \times [-e, e]$ be a regular neighbourhood of $B = B \times 0 \subset M$ with $B \times [-e, e] \cap A = I \times [-e, e]$ and $B \times [-e, e] \cap \partial M = \partial B \times I \times [-e, e]$. Then $D = A - I \times [-e, e] \cup (B - [-e, e] \times B \times e)$ is a proper 2-cell in $M$ with $\partial D \subset R$. Since $R$ is incompressible there is a 2-cell $D_0 \subset R$ with $\partial D_0 = \partial D$. Since $M$ is irreducible, the 2-sphere $D_0 \cup D$ bounds a 2-cell $C$ in $M$. If $B \times [-e, e] \subset C$, then $A$ is not incompressible, and if $B \times [-e, e] \subset M - C$, then $A$ is boundary parallel in $M$. A contradiction.

If $M$ is not assumed to be irreducible or if $R$ is not assumed to be incompressible, it cannot be concluded that $A$ is $\delta$-incompressible. The essential annulus of the example in §2 is contained in an irreducible 3-manifold $M$, but it is not $\delta$-incompressible: $\partial M$ is not incompressible. An example of a 3-manifold $M$ with $\partial M$ incompressible that contains an essential annulus that is not $\delta$-incompressible can be constructed as follows: Let $F$ be a closed surface not a 2-sphere or projective plane, and let $Y$ be a 3-manifold not a 3-cell with $\partial Y = 2$-sphere. Choose an annulus $A_0 \subset F$ that is not nullhomotopic in $F$ and a 3-cell $C = \text{int}(A_0 \times [0, e])$. Define $M = F \times [0, 1] - C \cup \partial C = \partial Y$.

Then $\partial M$ is incompressible. The annulus $A = \partial A_0 \times [0, e] \cup A_0 \times 1/2$ is essential in $M$ but not $\delta$-incompressible.

Note that an incompressible proper annulus in a 3-manifold $M$ that has its boundary components in two different components of $\partial M$ is essential and $\delta$-incompressible.

Theorem 3.2. Let $M$ be a 3-manifold and let $\iota: M \to M$ be an involution. Suppose that $A_0 \subset M$ is an essential annulus with $\partial A_0 \cap \iota A_0 = \emptyset$. 


Then either there is an essential annulus $A \subset M$ with

$$A \cap \partial A = \emptyset \quad \text{and} \quad \partial A \cup \partial A = \partial A_0 \cup \partial A_1,$$

or there are two disjoint annuli $A_1, A_2 \subset M$ with

$$\forall i = 1, 2, \quad \partial(A_i \cup A_j) = \partial A_0 \cup \partial A_0,$$

and with no fixed points on $A_1 \cup A_2$.

Furthermore, if $O$ is a given neighbourhood of $A_0 \cup A_0$, we may assume that $A, A_1, A_2 \subset O$ respectively.

Note also that if $A_0$ is a 2-sided (1-sided) then $A$ must be 2-sided (1-sided) respectively. If $A_1, A_2$ occurs then $A_0, A_1, A_2$ must be 2-sided.

Proof. We apply Proposition 2.5. We may assume that there are no fixed points of $i$ on $A_0$, that $A_0$ and $iA_0$ intersect transversally, and that there are no 1-spheres in $A_2 \cap iA_0$ that bound 2-cells in $A_0$.

Therefore $A_0 \cap iA_0$ consists of disjoint 1-spheres that decompose $A_0$ and $iA_0$ into annuli.

Our goal is to successively eliminate the 1-spheres from $A_0 \cap iA_0$.

We call a regular neighbourhood $V = \text{int} M$ of a 1-sphere $S \subset A_0 \cap iA_0$ a standard regular neighborhood of $S$ in $M$ if the following properties hold:

1. $V \cap V = S$ if $iS = S$.
2. $V \cap V = \emptyset$ if $iS \neq S$.
3. There are no fixed points of $i$ on $V$, and
4. $A_0 \cap V, iA_0 \cap V$ are proper annuli in $V$.

By Lemma 2.3, standard regular neighborhoods exist.

Note $V$ is a solid torus by Lemma 2.4.

Step 1. Suppose there are annuli $A = iA_0$ and $A' = A_0$ with $A \cap A_0 = \partial A = \partial A'$ and suppose further that there is a solid torus $V_0$ with $V_0 = A' \cup A$ and such that there is an isomorphism $h : (A' \times [0, 1], A' \times 0) \to (V_0, A_0)$.

We show that at least two 1-spheres of $A_0 \cap iA_0$ can be eliminated.

Let $\partial A = S_1 \cup S_2$.

Case 1. $iA_0 = \partial A$.

Then necessarily $A' = A$.

Let $V_1, V_2$ be disjoint standard regular neighborhoods of $S_1, S_2$ in $M$ respectively with either $iV_i = V_0, i = 1, 2$, or $iV_i = V_2$. See Figure 2.

Fig. 2

Then either $iV_1 = V_1, V_1 \cap iV_2 = S_1$, or $iV_1 \cap V_2 = S_1, iV_2 = V_2$.

Let $\tilde{A} \times [0, e]$ be a sufficiently thin collar of $\tilde{A} = \tilde{A} \cap \partial A = \partial A \times [0, e]$ in $A_0 \cap A_i$ such that $\partial \tilde{A} \times [0, e] \subset A_0 - A'$. Then necessarily $\tilde{A} \times \partial (A \times e)$ is a 1-sphere $S_0$ in $M$.

Define the proper annulus

$$A_0 = A_0 - (A' \cup S_1 \times [0, e] \cup S_2 \times [0, e] \cup \tilde{A} \times e).$$

Then $A_0 \cap \partial A_0 = (A_0 \cap \partial A) = \partial A_0 \cup \partial A_0$.

Furthermore, if $O$ is a given neighbourhood of $A_0 \cap A_0$, we may assume that $A, A_1, A_2 \subset O$ respectively.

Then $A_0 \cap \partial A_0 = (A_0 \cap \partial A_0)$. 

Case 2. $A \cap \partial A_0$ is single 1-sphere.

Then $\partial A \cap \partial A_0 = S_0$. Let $S_1$ be a standard regular neighborhood of $S_0$ in $A_0 - A'$.

Let $V_0 \cap \partial V = V_0 \cap \partial V = \emptyset$.

Thus $V_0 \cup \partial V$ is the annulus with one component $\tilde{A}$ and the other component in $A_0 - A'$ and let $V_0$ be the solid torus $V_0 \cap \partial V$ is the annulus with one component in $\tilde{A}$ and the other component in $A_0$. See Figure 3.

Fig. 3

Then $iV_1 = V_1, V_1 \cap iV_2 = S_1$, or $iV_1 \cap V_2 = S_1, iV_2 = V_2$.

Let $\tilde{A} \times [0, e]$ be a sufficiently thin collar of $\tilde{A} = \tilde{A} \cap \partial A = \partial A \times [0, e]$ in $A_0 - A'$ such that $\tilde{A} \times [0, e] \subset A_0 - A'$. Then necessarily $\tilde{A} \times \partial (A \times e)$ is a 1-sphere $S_0$ in $M$.

Define the proper annulus

$$A_0 = A_0 - (A' \cup S_1 \times [0, e] \cup S_2 \times [0, e] \cup \tilde{A} \times e).$$

Then $A_0 \cap \partial A_0 = (A_0 \cap \partial A_0)$. 

Case 3. $A \cap \partial A_0$ is null.

Then $\partial A \cap \partial A_0 = \emptyset$. 

Let $\tilde{A} \times [0, e]$ be a sufficiently thin collar $\tilde{A} = \tilde{A} \cap \partial A = \partial A \times [0, e]$ in $A_0 - A'$ such that $\tilde{A} \times [0, e] \subset A_0 - A'$. Then necessarily $\tilde{A} \times \partial (A \times e)$ is a 1-sphere $S_0$ in $M$.

Define the proper annulus

$$A_0 = A_0 - (A' \cup S_1 \times [0, e] \cup S_2 \times [0, e] \cup \tilde{A} \times e).$$

Then $A_0 \cap \partial A_0 = (A_0 \cap \partial A_0)$. 

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In all three cases $\partial A_0 = \partial A_0$. Using the solid torus specified in the hypothesis an ambient isotopy on $M$ can be constructed that maps $A_0$ onto $A_0'$. Therefore $A_0'$ is an essential annulus with $\partial A_0' = \partial A_0$ and with $A_0' \cap \partial A_0$ containing at least two 1-spheres less than $A_0 \cap \partial A_0$.

Now let $\bar{A} \subset \partial A_0$ be one of the two annuli defined by the 1-spheres of $A_0 \cap \partial A_0$ in $\partial A_0$ such that one component of $\partial \bar{A}$ is a component of $\partial A_0$ and the other component is $S = \bar{A} \cap \partial A_0 = \partial A_0 \cap \bar{A}$. Then $S$ decomposes $A_0$ into two annuli $A'$ and $A''$: 

$$A_0 = A' \cup A'' \quad \text{and} \quad A' \cap A'' = \partial A' \cap \partial A'' = S.$$ 

See Figure 4.

![Figure 4](image)

We consider the proper annuli $A' \cup \bar{A}$ and $A'' \cup \bar{A}$.

Step 2. Suppose $A' \cup \bar{A}$ is boundary parallel.

Applying Step 1 we may assume that $\partial A' \cap \partial A_0 = S$.

Let $V$ be a standard regular neighborhood of $S$ in $M$. Then $\partial(A_0 \cap V)$, $\partial(A_0 \cap V)$ subdivide $\partial V$ into four annuli. Let $A^*$ be the one of these four annuli with one component of $\partial A^*$ in $\bar{A}$ and the other component of $\partial A^*$ in $A''$. See Figure 4.

**Case 1.** $\bar{A} = A'$.

Define the proper annulus 

$$A_0' = (V \cap A') \cup A^* \cap \bar{A} = (V \cap \bar{A}).$$

There is an ambient isotopy on $M$ that maps $A_0$ to $A_0'$. Therefore $A_0'$ is an essential annulus with 

$$\partial A_0' \cap \partial A_0 = \partial A_0 \cap \partial A_0$$

and with $A_0' \cap \partial A_0 = A_0 \cap \partial A_0 = S$.

Thus we have eliminated a 1-sphere from $A_0 \cap \partial A_0$.

**Case 2.** $\bar{A} = A''$.

Then $A_0 \cap A_0' = S$ and $\partial(A_0 \cap \bar{A}) = A'$. Let $A^{**}$ be the one of the preceding four annuli with $A^{**} \cap A^* = \emptyset$. Then $A^{**} = A^*$ and $\partial A^{**} = A^*$.

Define the proper annulus 

$$A_0'' = (V \cap A'') \cup A^* \cup \bar{A} = (V \cap \bar{A})$$

and with no fixed points of $i$ on $A_0 \cap A_0'$.

Next assume that $S \neq S$.

Let $V$ be a standard regular neighborhood of $S$ in $M$ with $V \cap V = \emptyset$. Then $\partial(A_0 \cap V)$, $\partial(A_0 \cap V)$ subdivide $\partial V$ into four annuli. Again let $A^*$ be the annulus with one component of $\partial A^*$ in $\bar{A}$ and the other component in $A_0 - A'$. Let $A^{**}$ be the annulus with $A^{**} \cap A^* = \emptyset$.

Define the proper annulus 

$$A_0'' = A_0 - (A' \cup \bar{A} \cup (V \cap A_0) \cup (V \cap A_0)) \cup A^* \cup \bar{A} - (V \cap \bar{A})$$

and with $\partial A_0'' \cap \partial A_0 = \partial A_0 \cup \partial A_0$.

There is an ambient isotopy on $M$ that maps $A_0$ to $A_0'$. Therefore $A_0'$ is an essential annulus with $\partial A_0' = \partial A_0$ and with $A_0' \cap \partial A_0 = A_0 \cap \partial A_0 - (S \cup S)$.

We have eliminated two 1-spheres from $A_0 \cap \partial A_0$.

Since one boundary component of each is a boundary component of $A_0$, they are also incompressible, and therefore essential.

Step 3. Both proper annuli $A' \cup \bar{A}$ and $A'' \cup \bar{A}$ are not boundary parallel.

We may assume that $i \bar{A} \subset A'$.

If $i \bar{A} \subset A'$, define $A'' = A' \cup \bar{A}$.

If $i \bar{A} = S$, define as in Step 2, Case 1, the essential annulus 

$$A_0'' = A_0' - (V \cap A'') \cup A^* \cup \bar{A} - (V \cap \bar{A})$$

Then $A_0''$ satisfies 

$$\partial A_0'' \cap \partial A_0 = \partial A_0 \cup \partial A_0$$

and $A_0 \cap A_0'' = A_0 \cap \partial A_0 - (S \cup S)$.

After a finite number of steps we arrive at the annulus $A$ or the annuli $A_1$, $A_2$ with the desired properties.

Note that a proper annulus in $M$ is 2-sided (1-sided) if and only if its boundary is 2-sided (1-sided) in $\partial M$. **
If \( tA_i = A_i \), \( l = 1, 2 \), applying Lemma 2.4 there is a regular neighbourhood \( W_1 \) of \( A_1 \), \( l = 1, 2 \), with \( W_1 = W_1 \) and with no fixed points if \( \iota \) on \( W_1 \). If \( A_i \) is 1-sided then \( W_1 \) is a solid Klein bottle (see proof of Proposition 3.4). But a solid Klein bottle does not admit a fixed point free involution. ■

If \( A_2 \) is 2-sided in \( M \) and if the components of \( \partial A_2 \) are parallel to \( \partial M \), the generalized loop theorem of [7] proves also Theorem 3.2 as follows: Pushing off \( A_2 \) from the fixed points of \( \iota \) and cutting out an \( \iota \)-invariant regular neighbourhood of the fixed point set, we may assume that \( \iota \) has no fixed points. Let \( p : M \rightarrow M/\iota \) be the natural projection. Then \( p(A_2) = M/\iota \) is a singular annulus. It can be replaced by an essential annulus. This lifts to an essential annulus \( A \) in \( M \) with \( A \cap \partial A = \emptyset \) (this argument was pointed out by the referee).

**Corollary 3.3** If in addition \( A_2 \) is \( \iota \)-incompressible in \( M \) we may assume that \( A, A_1, A_2 \) respectively are \( \iota \)-incompressible in \( M \).

**Proof.** The same four steps of the proof of Theorem 3.2 may be applied with the following modifications.

Step 1. Suppose there are annuli \( A \subset A_2 \) and \( A' \subset A_2 \) with \( \partial A \cap A' = \partial A = \partial A' \) and suppose further that there is a 2-cell \( D \subset M \) with \( D \cap A_2 = \partial D \cap A' = I \) is a nonseparating proper 1-cell in \( A' \) and \( D \cap \partial A = \partial D - I \).

Step 2. Suppose \( A' \cup A = \partial A \) is not \( \iota \)-incompressible.

Then the annuli \( A_1, A_2, A_3 \) constructed in the four steps will be incompressible and \( \iota \)-incompressible.

**Proposition 3.4.** Let \( M \) be an irreducible 3-manifold that is not \( P^2 \times [0, 1] \), \( P^2 \) the projective plane, and let \( A \subset M \) be a 1-sided incompressible annulus such that the components of \( \partial M \) that contain \( \partial A \) are incompressible. Let \( W \) be a regular neighborhood of \( A \) in \( M \). Then \( \hat{A} = \partial W - (W \cap \partial M) \) is a 2-sided essential annulus in \( M \).

**Proof.** Let \( M/O = [0, 1] \times \{ -1, 1 \} \cup \{ 0, \infty \} \) be a Möbius band and let \( S^1 = [0, 1] \times 0 \cup (0, 0) \sim (1, 0) = M/O \).

Then \((\hat{W}, \hat{A}) = (MO \times [0, 1]) \cup (M \times \{ 0, 1 \}), S^1 \times \{ 0, 1 \} \) and \( W \cap \partial M = MO \times 0 \cup MO \times 1 \).

Suppose that \( \hat{A} \) is not incompressible. Since \( M \) is irreducible and since the components of \( \partial M \) that contain \( \partial A \) are incompressible, these components must be two distinct projective planes and we must have \( M = P^2 \times [0, 1] \), which is excluded.

Suppose \( \hat{A} \) is boundary parallel. Then \( M \) is a solid Klein bottle and \( \partial M \) is not incompressible, a contradiction. ■

**Proposition 3.5.** \( P^2 \times [0, 1] \) does not admit an involution.

**Proof.** Suppose \( \iota : P^2 \times [0, 1] \rightarrow P^2 \times [0, 1] \) is an involution. Then \( 4: \partial(P^2 \times [0, 1]) \rightarrow \partial(P^2 \times [0, 1]) \) is fixed point free. Since a projective plane does not admit a fixed point free involution, we must have \( \iota(P^2 \times 0) = P^2 \times 1 \). If \( \iota \) has isolated fixed points \( x_1, \ldots, x_k \), let \( C_1, \ldots, C_k \subset \partial(P^2 \times [0, 1]) \) be disjoint 3-cells with \( x_i \in \partial C_i \) and \( \iota C_i = C_i, i = 1, \ldots, k \) (Lemma 2.4). Let \( M = P^2 \times [0, 1] - (C_1 \cup \ldots \cup C_2) \).

Then \( M \rightarrow M/\iota \) is a 2-sheeted covering. Since \( \pi_1(M) = Z_2 \), the order of \( \pi_1(M/\iota) \) must be 4. It follows from the Lefschetz Fixed Point Theorem that \( \iota \) must have fixed points. Since \( \partial(M/\iota) \) contains the projective planes \( P^2 \times 0 \) and \( \partial C_i \), we conclude that \( \pi_1(M/\iota) = Z_2 \oplus Z_2 \). A contradiction: \( Z_2 \oplus Z_2 \) cannot be the fundamental group of a 3-manifold ([1], Theorem 9.13). ■

**Theorem 3.6.** Let \( M \) be a 3-manifold and let \( \iota : M \rightarrow M \) be an involution. Suppose that \( A \subset M \) is a \( \iota \)-incompressible essential annulus.

Then there is a \( \iota \)-incompressible essential annulus \( A \subset M \) with either \( A \cap \partial A = \emptyset \) or \( A \cap A = \emptyset \) and with no fixed points of \( \iota \) on \( A \).

If \( \iota A = A \), then \( A \) is 2-sided in \( M \).

If \( M \) is irreducible and if the components of \( \partial M \) that contain \( \partial A \) are incompressible, we may also assume that \( A \) is 2-sided in \( M \).

Furthermore, if \( A \) is a given neighborhood of \( A_0 \cup A_0 \) in \( M \), we may assume that \( A \subset \partial \).

**Proof.** By Proposition 2.5 we may assume that there are no fixed points of \( \iota \) on \( A_0 \), that \( A_0 \) and \( \iota A_0 \) intersect transversally, that there are no 1-spheres in \( A_0 \), and that there are no 1-cells in \( A_0 \cap \iota A_0 \) separating \( A_0 \). Consequently, \( A_0 \cap \iota A_0 \) consists either of 1-spheres that decompose \( A_0 \) into annuli, or of nonseparating 1-cells that decompose \( A_0 \) into 2-cells. In the first case we must have \( \partial A_0 \cap \iota \partial A_0 = \emptyset \). Corollary 3.3 proves the theorem in this case. Hence we may assume that \( A_0 \cap \iota A_0 \) consists of nonseparating proper 1-cells. Our goal is to successively eliminate the 1-cells from \( A_0 \cap \iota A_0 \).

Note that if \( \iota \subset A_0 \cap \iota A_0 \) is a 1-cell, we must have \( \iota 

\rightcup \emptyset \). Therefore if \( A_0 \cap \iota A_0 \subset \emptyset \), it must consist of more than one cell.

Let \( D = A_0 \) be a 2-cell with \( D \cap A_0 = I_1 \cup I_2, I_1, I_2 \) proper 1-cells in \( A_0 \) and \( \iota A_0 \), and with \( D \cap \partial M = \partial D - (I_2 \cup I_2) \). The cells \( I_1, I_2 \) decompose \( A_0 \) into two 2-cells \( D' \) and \( D'' \). Let \( A_0 = D' \cup D'' \) with \( D' \cap D'' = \partial D' = I_1 \cup I_2 \).

Then \( A' = D'/D'' \) and \( A'' = D''/D' \) are either both proper annuli or both proper Möbius bands. Since \( A_0 \) is \( \iota \)-incompressible, both \( A' \) and \( A'' \) must be \( \iota \)-incompressible. Since \( A_0 \) is incompressible, at least one of \( A', A'' \) must be incompressible.

Step 1. Suppose both \( A' \) and \( A'' \) are proper annuli. We may assume that \( A' \) is incompressible. Therefore \( A' \) is a \( \iota \)-incompressible essential annulus.
Case 1. \( u(I_1 \cup I_2) = I_1 \cup I_2 \) and \( tD = D' \).
Define \( A = A' \). Then \( A \) is a \( \delta \)-incompressible essential annulus with \( \lambda A = A \).

Case 2. \( u(I_1 \cup I_2) = I_1 \cup I_2 \) and \( tD = D'' \).
Let \( C_1, C_2 \) be disjoint regular neighborhoods of \( I_1, I_2 \) in \( M \) respectively with \( uC_1 = C_1 \) and with \( uA_0 \cap C_1, uA_0 \cap C_2 \) are proper 2-cells in the 3-cell \( C_0 \). Let \( D'' = D \cap \partial M \) and \( A'' = D \cap (C_1 \cap \partial M) \) is subdivided by the four proper 1-cells \( (A_0 \cup A_1) \cap A'' \) into four 2-cells. Let \( B^+ = \partial C_1 \) be the one of these four 2-cells that meets \( D'' \) and \( D \). See Figure 5.

Define the annulus
\[
A_1 = D'' - (D'' \cap (C_1 \cup C_2)) \cup B^+ \cup B^+ \cup D' - (D' \cap (C_1 \cup C_2)).
\]
Then \( A_1 \cap A_1 = A_0 \cap A_0 - (I_1 \cup I_2) \).

There is an ambient isotopy on \( M \) that maps \( A' \) onto \( A_1 \). We have removed two 1-cells from \( A_0 \cap A_0 \).

Note that it is not possible that \( (I_1 \cup I_2) \cap (I_1 \cup I_2) \) is a single 1-cell, say \( I_1 \). Namely, then necessarily \( uI_1 = I_1 \), and consequently there must be a fixed point of \( u \) on \( I_1 \subseteq A_0 \), a contradiction.

Case 3. \( (I_1 \cup I_2) \cap (I_1 \cup I_2) = \emptyset \).

If \( tD \subset \text{int} \ D'' \), define \( A_1 = A' \).

Next suppose \( tD \subset \text{int} \ D' \). Let \( D \times [0, \varepsilon] \) be a sufficiently thin collar of \( D = D \times 0 \) in \( M \) with \( D \times [0, \varepsilon] \cap A_0 = \partial D \times [0, \varepsilon], I_1 \times [0, \varepsilon] \subset D' \), and \( D \times [0, \varepsilon] \cap \partial M = \partial D - (I_1 \cup I_2) \times [0, \varepsilon] \).

Then \( D \times [0, \varepsilon] \cap \partial M = \emptyset \). Define the proper annulus
\[
A_1 = \begin{cases} (D' - \partial D \times [0, \varepsilon]) \cup D \times \varepsilon, & \text{if } D \times \varepsilon \cap D'' = \emptyset, \\ (D' - I_1 \times [0, \varepsilon]) \cup D \times \varepsilon \cup I_2 \times \varepsilon, & \text{if } D \times \varepsilon \cap D'' = A_2 \times \varepsilon. \end{cases}
\]

Then \( A_1 \cap A_1 = A_0 \cap A_0 - (I_1 \cup I_2) \).

There is an ambient isotopy on \( M \) that maps \( A' \) onto \( A_1 \). At least two 1-cells have been removed from \( A_0 \cap A_0 \).

Step 2. We may now assume that for all 2-cells \( D \subset A_0 \), both \( A' \) and \( A'' \) are Möbius bands.

\( A_0 \cap A_0 \) must contain more than two 1-cells. Namely, otherwise, there is a \( D \subset A_0 \) with \( uD = D \) and \( uD \) is a fixed point free involution on \( D \cup D \).
Since a Möbius band does not admit a fixed point free involution, \( D \cap D \) must be an annulus, and \( A' \) and \( A'' \) determined by \( D \) are annuli, a contradiction.

Let \( D_1, D_2 \subset A_0 \) be two consecutive 2-cells with \( D_1 \cap A_0 = I_1 \cup I_2 \) and \( D_2 \cap A_0 = I_1 \cup I_3, I_2 \cup I_3 \) 1-cells. The three 1-cells \( I_1, I_2, I_3 \) decompose \( A_0 \) into three 2-cells. Let \( D' \subset A_0 \) be the 2-cell with \( I_1, I_2 \subset \partial D' \) and \( D' \cap I_3 = \emptyset \). See Figure 6.

Define \( A' = D' \cup D_1 \cup D_2 \). Since \( A_0 \) is \( \delta \)-incompressible, \( A' \) is \( \delta \)-incompressible. Since both \( D_1 \) and \( D_2 \) define Möbius bands, \( A' \) must be an annulus.

Claim. \( A' \) is incompressible.

Proof of Claim. Suppose \( A' \) is not incompressible. Then \( A' \) has a compressing 2-cell \( B \) such that \( B \cap A' = \partial B \) is a nonseparating 1-sphere on \( A' \). We may assume that \( A' \) intersects \( A_0 \) transversely. There is a 2-cell \( B_0 \subset B \) such that
\[
B_0 \cap A_0 = \partial B_0 \cap A_0 = \varepsilon\]
is a 1-cell and the 1-cell \( B_0 \cap A_0 = \partial B_0 - \varepsilon \) is contained either in \( D_1 \) or in \( D_2 \). We conclude that either \( D_1 \) or \( D_2 \) respectively determine annuli instead of Möbius bands, a contradiction.

Thus \( A' \) is a \( \delta \) incompressible essential annulus.

Define \( D = D_1 \cup D_2 \). The construction of Step 1 can be applied again.

Case 1. \( u(I_1 \cup I_3) = I_1 \cup I_3 \) and \( uI_2 \subset D' \).

Define \( A = A' \). Then \( A \) is a \( \delta \) incompressible essential annulus with \( \lambda A = A \).
Case 2. \((I_1 \cup I_2) = I_1 \cup I_3\) and \(U_2 \subset A_0 - D^2\).

Let \(C_1, C_2\) be disjoint regular neighborhoods of \(I_1, I_3\) in \(M\) respectively, with \(\partial C_1 = C_1^0\) and with \(A_0 \cap C_2, A_0 \cap C_1\) are proper 2-cells in the 3-cell \(C_i, i = 1, 3\). The annulus \(A_i^2 = \partial C_i \cap (\partial M)\) is subdivided by the four proper 1-cells \(A_0 \cup A_i \cap A_i^2\) into four 2-cells. Let \(B_i \subset \partial C_i\) be one of these four 2-cells that meets \(D^2\) and \(D\). Define the annulus

\[
A_i = D^2 - \left(D^2 \cap \left(\partial C_1 \cup C_2\right)\right) \cup B_i^0 \cup B_i \cup D - \left(\partial D \cap (C_1 \cup C_2)\right).
\]

Then \(A_1 \cap A_2 = A_0 \cap A_0 - (I_1 \cup I_2 \cup I_3)\). There is an ambient isotopy on \(M\) that maps \(I_2^0\) onto \(A_1\). We have removed three 1-cells from \(A_0 \cap A_0\).

Again it is not possible that \((I_1 \cup I_3) \cap (I_1 \cup I_3)\) is a single 1-cell.

Case 3. \((I_1 \cup I_3) \cap (I_1 \cup I_3) = \emptyset\).

It is not possible that \(U_2 = I_2\). Otherwise \(D_1 \cap D_1 = \partial D_1 = \partial D_1\) and \(\partial D \cap \partial D\)

defines a fixed point free involution on \(D_1 \cup D_2\). Since a Moebius band does not admit a fixed point free involution, \(D_1 \cup D_2\) must be an annulus. Therefore \(D_1\) does not determine Moebius bands, a contradiction. Similarly, \(U_2 = I_2\) is not possible. Consequently, \(\partial S = D^1\) or \(\partial S = A_0 - D^2\).

Next suppose \(\partial S = D^1\). Let \(D \times [0, \varepsilon]\) be a sufficiently thin collar of \(D = D \times 0\) in \(M\) with \(D \times [0, \varepsilon] \cap A_0 = \partial D \times [0, \varepsilon], I_1 \times [0, \varepsilon] \subset D^1\), and \(D \times [0, \varepsilon] \cap \partial M = \partial D \times (I_2 \cup I_3) \times [0, \varepsilon]\).

Then \(D \times \varepsilon \cap (D \times \varepsilon) = \emptyset\). Define the proper annulus

\[
A_1 = \begin{cases} 
D^2 - \partial D \times [0, \varepsilon] \cup D \times \varepsilon, & \text{if } D \times \varepsilon \cap D^1 = \emptyset, \\
D^2 - I_1 \times [0, \varepsilon] \cup D \times \varepsilon \cup I_2 \times \varepsilon, & \text{if } D \times \varepsilon \cap D^1 = I_2 \times \varepsilon.
\end{cases}
\]

Then \(A_1 \cap A_2 = A_0 \cap A_0 - (I_1 \cup I_2)\).

There is an ambient isotopy on \(M\) that maps \(I_1^0\) onto \(A_1\). At least two 1-cells have been removed from \(A_0 \cap A_0\).

Therefore, in a finite number of steps we arrive at the \(\partial\)-incompressible essential annulus \(A\) with either \(A \cap A = \emptyset\) or \(A = A\) and with no fixed points of \(\iota\) on \(A\).

If \(\iota = A = A\) must be 2-sided in \(M\). Namely, otherwise, there is a regular neighborhood \(W\) of \(A\) in \(M\) with \(W \cap W = W\) and with no fixed points of \(\iota\) on \(W\). But \(W\) is a solid Klein bottle. It does not admit a fixed point free involution, a contradiction.

Suppose \(A \cap A = \emptyset\) and \(A\) is 1-sided in \(M\). If \(M\) is irreducible and if the components of \(\partial M\) that contain \(\partial A\) are incompressible, let \(W\) be a regular neighborhood of \(A\) in \(M\) with \(W \cap W = \emptyset\). By Proposition 3.4, \(\tilde{A} = \overline{W} - (W \cap \partial M)\) is a 2-sided essential annulus in \(M\) with \(\tilde{A} \cap A = \emptyset\). Since \(A\) is \(\partial\)-incompressible, so is \(\tilde{A}\). Replace \(A\) by \(\tilde{A}\).

COROLLARY 3.7. Let \(M\) be an irreducible 3-manifold and let \(\iota: M \to M\) be an involution. Suppose that \(A_0 \subset M\) is an essential annulus such that the components of \(\partial M\) that contain \(\partial A\) are incompressible.

Then there is a 2-sided essential annulus \(A \subset M\) with either \(A \cap A = \emptyset\) or \(A = A\) and with no fixed points of \(\iota\) on \(A\).

Furthermore, if \(O\) is a given neighborhood of \(A_0 \cup A_0\) in \(M\), we may assume that \(A \cap O = \emptyset\).

Proof. Proposition 3.1 and Theorem 3.6. ■

Note also that if the 3-manifold \(M\) is orientable each proper annulus in \(M\) is 2-sided.

§ 4. Equivariant surgery on essential Moebius bands with respect to involutions.

Again note that a \(\partial\)-incompressible, incompressible proper Moebius band in a 3-manifold is essential. Also again we have the following converse.

PROPOSITION 4.1. Let \(M\) be an irreducible 3-manifold and let \(MO \subset M\) be an essential Moebius band. Suppose that \(\partial MO \subset R, R = \partial M\) is an incompressible component.

Then \(MO\) is \(\partial\)-incompressible in \(M\).

Proof. If \(MO\) is not \(\partial\)-incompressible, then as in the proof of Proposition 3.1 it follows that \(MO\) is boundary parallel, a contradiction. ■

Again if \(M\) is not assumed to be irreducible or if \(R\) is not assumed to be incompressible, it cannot be concluded that \(MO\) is \(\partial\)-incompressible. If for example \(V = [0, 1] \times [-1, 1] \times [-1, 1]\{0, x, y\} \sim (1, -x, -y)\) is the solid torus, then \([0, 1] \times [-1, 1] \times [0, x, y] \sim (1, -x, 0)\) is an essential Moebius band in \(V\) that is not \(\partial\)-incompressible. Similairly as in the case of the annulus examples can be constructed of essential Moebius bands \(MO\) in 3-manifolds \(M\) that are not irreducible and with \(R\) incompressible such that \(MO\) is not \(\partial\)-incompressible.

THEOREM 4.2. Let \(M\) be a 3-manifold and let \(\iota: M \to M\) be an involution. Suppose that \(MO_0 \subset M\) is an essential Moebius band with \(\partial MO_0 \cap \partial MO_0 = \emptyset\).

Then one of the following three properties holds.

(i) Either there is an essential Moebius band \(MO_0 \subset M\) with \(MO \cap MO = \emptyset\) or there is a 2-sided essential annulus \(A \subset M\) with \(A = A\) and with no fixed points of \(\iota\) on \(A\). If \(MO_0\) is 2-sided (1-sided), so is \(MO\).

(ii) \(MO_0\) is 1-sided in \(M\) and there is a 2-sided essential annulus \(A \subset M\) with \(\partial A = \partial MO_0 \cup \partial MO_0\) and \(A \cap A = \emptyset\).

(iii) \(MO_0\) is a solid torus. There is an essential Moebius bundle \(MO_0 \subset M\) with \(\partial MO_0 = \partial MO_0\) and \(MO_0 \cap MO_0 = S^1\) is a single nonseparating 1-sphere in \(MO_0\).

Proof. We apply Proposition 2.5. We may assume that there are no fixed points of \(\iota\) on \(MO_0\), that \(MO_0\) and \(\partial MO_0\) intersect transversally, and that there are no 1-spheres in \(MO_0 \cap \partial MO_0\) that bound 2-cells in \(MO_0\).
Therefore $MO_0 \cap iMO_0$ consists of disjoint 1-spheres. If $MO_0$ is 2-sided in $M$ then each 1-sphere separates $MO_0$ into an annulus and a Möbius band (Lemma 2.6). If $MO_0$ is 1-sided in $M$ then it is not possible that a 1-sphere separates one and not the other of $MO_0$ and $iMO_0$ (otherwise $MO_0$ must be 2-sided in $M$, a contradiction).

Our goal is to successively eliminate the 1-spheres from $MO_0 \cap iMO_0$.

Step 1. Suppose there is a 1-sphere in $MO_0 \cap iMO_0$ that separates $MO_0$.

Then there is a unique annulus $\tilde{A} \subset \overline{MO_0}$ with $\tilde{A} \cap MO_0 = \partial \tilde{A} \cap MO_0 = S$ a single 1-sphere and $\partial \tilde{A} = S \cup \partial MO_0$. Necessarily $\partial S = S$. Consider the annulus $A = \tilde{A} \cup i\tilde{A}$. Note $A$ is incompressible.

Case 1. The annulus $A$ is not boundary parallel. Then $A$ is an essential annulus with $\partial A = A$ and with no fixed points of $i$ on $A$. Further $A$ must be 2-sided in $M$ (see remark in proof of Theorem 3.6).

Case 2. The annulus $A$ is boundary parallel. Then the incompressible Möbius band $MO = \overline{MO_0 \cup \tilde{A}} \cup \tilde{A}$ is not boundary parallel and therefore essential. Applying Lemma 2.4, let $V$ be a regular neighborhood of $S$ in int $M$ such that

$$\partial V = V, \quad MO_0 \cap V, \quad iMO_0 \cap V$$

are proper annuli in $V$, and such that there are no fixed points of $i$ on $V$. Then $V$ is a solid torus and $\partial (MO_0 \cap V), \partial (iMO_0 \cap V)$ subdivide $\partial V$ into four annuli. Let $A^+ \subset \partial V$ be the one of the four annuli with one boundary component in $\tilde{A}$ and the other component in $\overline{MO_0 \cup \tilde{A}}$. Define the essential Möbius band

$$MO_0 = \overline{MO_0 \cup \tilde{A}} \setminus \partial MO_0.$$  

Then $MO_1 \cap MO_0 = MO_0 \cap iMO_0 = S$. One 1-sphere has been removed from $MO_1 \cap MO_0$. Note $\partial MO_1 = \partial iMO_0$.

Applying Step 1 a finite number of times we either arrive at property I of the theorem or at the following.

Step 2. $MO_0 \cap iMO_0 = S$ a single 1-sphere and $S$ does not separate $MO_0$ and not $iMO_0$.

Applying Lemma 2.4, let $V$ be a regular neighborhood of $S$ in int $M$ such that

$$\partial V = V, \quad MO_0 \cap V, \quad iMO_0 \cap V$$

are proper Möbius bands, and there are no fixed points of $i$ on $V$. Then $V$ is a solid torus and the two 1-spheres $\partial (MO_0 \cap V), \partial (iMO_0 \cap V)$ subdivide $\partial V$ into two annuli $A^+ \cup A^-$. Consider the incompressible 2-sided annuli $A' = MO_0 \setminus (V \cap MO_0) \cup A^+ \cup (MO_0 \setminus (V \cap iMO_0))$. If $A'$ is not boundary parallel it is easy to modify $A'$ to obtain the 2-sided essential annulus $A$ of property II of the theorem. If $A'$ is boundary parallel, it follows that $M$ is a solid torus and we have arrived at property III of the theorem.

**EXAMPLE.** Let $V = [0,1] \times [-1,1] \times [-1,1] \cup [0,1] \times [-1,1] \times (0,0,0) \sim (1,-x,-y)$ be the solid torus and let $MO = [0,1] \times [-1,1] \times (0,0,0) \sim (1,-x,0)$ be an essential Möbius band in $V$. Define the fixed point free involution $i: V \to V$ by

$$d(t,x,y) = (1-t,y,x).$$

Then $MO \cap iMO = S$ is a nonseparating 1-sphere in $MO$.

**Corollary 4.3.** If in addition $MO_0$ is $\partial$-incompressible in $M$ only properties (I) or (II) hold and we may assume that the annulus $A$ and the Möbius band $MO$ are $\partial$-incompressible.

**Proof.** In Step 1 of the proof of Theorem 4.2 either the annulus $A$ or the Möbius band $MO$ must be $\partial$-incompressible. In Step 2, the annulus $A$ must be $\partial$-incompressible. Property III cannot hold since a Möbius band in a solid torus is not $\partial$-incompressible.

**Theorem 4.4.** Let $M$ be a 3-manifold and let $i: M \to M$ be an involution. Suppose that $MO_0 \subset M$ is a $\partial$-incompressible essential Möbius band.

Then there is either a $\partial$-incompressible essential annulus or Möbius band $F \subset M$ with $\partial F \cap F = \emptyset$ or there is a 2-sided $\partial$-incompressible essential annulus $A \subset M$ with $iA = A$ and with no fixed points of $i$ on $A$.

Furthermore, if $O$ is a given neighborhood of $A \cup A_0$ in $M$, we may assume that $A \subset O$.

**Proof.** By Proposition 2.5 we may assume that there are no fixed points of $i$ on $MO_0$, that $MO_0$ and $iMO_0$ intersect transversely, that there are no 1-spheres in $MO_0 \cap iMO_0$ bounding 2-cells in $MO_0$ and that there are no 1-cells in $MO_0 \cap iMO_0$ separating $MO_0$. Consequently either $\partial MO_0 \cap \partial iMO_0 = \emptyset$ or $MO_0 \cap iMO_0$ consists of 1-spheres or $MO_0 \cap iMO_0$ consists of nonseparating 1-cells that decompose $MO_0$ into 2-cells.

In the first case Corollary 4.3 proves the theorem. If property II holds, let $W$ be a regular neighborhood of $A \cup A: A$ the 2-sided $\partial$-incompressible essential annulus with $\partial A \cup A = \partial A$, such that $iW = W$. Let $A' \subset \partial W$ be an annulus that is proper in $M$. Then $A'$ is a 2-sided $\partial$-incompressible essential annulus with $A' \cap A' = \emptyset$.

In the second case the proof of Theorem 3.6 applies. If $MO_0 \cap iMO_0 \neq \emptyset$ we actually can obtain an annulus with the required properties. Step 1 of the proof alone will do.

**Corollary 4.5.** Let $M$ be an irreducible 3-manifold and let $i: M \to M$ be an involution. Suppose that $MO_0 \subset M$ is an essential Möbius band such that the component of $\partial M$ that contains $\partial MO_0$ is incompressible.

Then there is either a 2-sided essential annulus or Möbius band $F \subset M$ with $\partial F \cap F = \emptyset$ or there is a 2-sided essential annulus $A \subset M$ with $iA = A$ and with no fixed points of $i$ on $A$.

Furthermore, if $O$ is a given neighborhood of $A \cup A_0$ in $M$, we may assume that $A \subset O$. 

Proof. Proposition 4.1 and Theorem 4.4. By Corollary 3.7 we may assume that if $F$ is an annulus it is 2-sided. If $F$ is a 1-sided plane, then $W$ be a regular neighborhood of $F$ in $M$ with $W_{\partial} = \emptyset$. Then $A = \partial W - (W_{\partial} \cap M)$ must be an essential annulus (since $\partial W \cap M$ consists of two 1-spheres it is incompressible), if it is boundary parallel, $M$ must be a solid torus and $\partial M$ is not incompressible, Thus $A$ is 2-sided and $A \cap M = \emptyset$.

On the Cauchy equation modulo $Z$

by

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Abstract. Assume that $X$ is a real linear topological space (which always is assumed to be Hausdorff) and let $f: X \to \mathbb{R}$ be a function such that

$$f(x+y)-f(x)-f(y) \in \mathbb{Z}$$

for all $x, y \in X$. Some conditions are established under which $f$ has the form $g+k$, where $g$ is a continuous linear functional on the space $X$ and the function $k$ takes integer values only. An application to the Cauchy equation

$$f(x+y) = f(x)+f(y)$$

for functions acting between linear topological spaces is also given.

Let a function $f: \mathbb{R} \to \mathbb{R}$ be given and assume that

$$f(x+y)-f(x)-f(y) \in \mathbb{Z}$$

for all $x, y \in \mathbb{R}$, where $\mathbb{Z}$ denotes the set of all integers. As follows from an example of G. Godini [6, Example 2], it is not generally true that such a function $f$ must be of the form $g+k$ where $g$ is an additive function and $k$ takes integer values only. However, the following theorem has been proved in paper [1]:

**Theorem 1.** If the Cauchy difference $f(x+y)-f(x)-f(y)$, as a function of two real variables, is Lebesgue measurable and takes integer values only, then there exists an additive function $g: \mathbb{R} \to \mathbb{R}$ and a Lebesgue measurable function $k: \mathbb{R} \to \mathbb{Z}$ such that

$$f = g+k.$$ 

In the present paper, the following theorem will be shown:

**Theorem 2.** Assume that $X$ is a real linear topological space.

If a function $f: X \to \mathbb{R}$ satisfies condition (1) for all $x, y \in X$ and there exists a set $E \subset X$ such that

$$0 \in \text{Int}(E - E)$$