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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Von Neumann's paradox with translations

by

M. Laczkovich (Budapest)

Abstract. Let I and J be intervals with $|I|<|J|<2|I|$. It is shown that there are disjoint decompositions $I=\bigcup_{i=1}^{4}A_i$, $J=\bigcup_{i=1}^{4}B_i$ and there is a strictly increasing contraction f from I into J such that $B_1=f(A_1)$ and B_i is a translated copy of A_i for $i=2,3,4$. This implies that von Neumann's paradox can be realized by using four pieces. Also, an upper estimate is given for the inner Lebesgue measure of the set $\bigcup_{i=1}^{\infty}f_i(H_i)$, where the sets $H_i\subset \boldsymbol{R}^n$ are pairwise disjoint and the maps $f_i\colon H_i\to \boldsymbol{R}^n$ are Lipschitz. Using this estimate, it is proved that von Neumann's paradox cannot be realized by using two pieces and that four pieces can be used only if $|J|<2|I|$.

1. Introduction. A subset B of the real line $\boldsymbol{R}$ is called *metrically smaller* than the set $A\subset\boldsymbol{R}$ if there is a bijection f of A onto B such that $|f(x)-f(y)|<|x-y|$ for every $x,y\in A$. The following theorem ([5, p. 115]; [11, p. 105]) is known as von Neumann's paradox. Let I,J be intervals with $|I|<|J|$. Then there are decompositions $I=\bigcup_{i=1}^{n}A_i$, $J=\bigcup_{i=1}^{n}B_i$ such that B_i is metrically smaller than A_i for every $i=1,\dots,n$. More exactly, von Neumann proves that $B_i=f_i(A_i)$, where f_i is a strictly increasing contraction on I ($i=1,\dots,n$). (A function $f\colon A\to\boldsymbol{R}$ is a *contraction* if $|f(x)-f(y)|\le q|x-y|$ holds for every $x,y\in A$ with a constant $q<1$.)

In this paper we present a similar paradoxical decomposition which uses only one contraction and three translations.

THEOREM 1. *Let I,J be intervals with $|J|<2|I|$. Then there are decompositions $I=\bigcup_{i=1}^{4}A_i$, $J=\bigcup_{i=1}^{4}B_i$ such that B_1 is metrically smaller than A_1 and B_i is congruent to A_i for $i=2,3,4$. More exactly, $B_1=f(A_1)$, where f is a strictly increasing contraction on I, and B_i is a translated copy of A_i for $i=2,3,4$.*

It is well known that if $|I|<|J|$ then there are no decompositions $I=\bigcup_{i=1}^{n}A_i$, $J=\bigcup_{i=1}^{n}B_i$ such that B_i is congruent to A_i for every $i=1,\dots,n$, i.e., J is not equivalent by finite decomposition to I. This is obvious by the existence of a Banach measure [2, p. 257], but can also be proved effectively, without using the axiom of choice ([9, p. 222] or [8, p. 72], see also [11]). This easily implies that the upper bound on $|J|$ in Teorem 1 is sharp, even if we use more parts to be translated.

THEOREM 2. *Let I, J be intervals, and suppose that there are decompositions $I = \bigcup_{i=1}^{n} A_i$, $J = \bigcup_{i=1}^{n} B_i$ such that B_1 is metrically smaller than A_1 and B_i is congruent to A_i for $i = 2, \ldots, n$. Then $|J| < 2|I|$.*

Indeed, let $J = [a, b]$ and let $[c, d]$ be the closed convex hull of B_1. Since B_1 is metrically smaller than A_1, and $\operatorname{diam} A_1 \leqslant |I|$, it is easy to see that $d-c < |I|$. Then $[a, c] \cup (d, b]$ is equivalent by finite decomposition to a subset of I, since $[a, c] \cup (d, b] \subset \bigcup_{i=2}^{n} B_i \cup \{c\}$ and there are congruences that map B_i into A_i $(i \geqslant 2)$ and c into a point of A_1. This implies that $[a, c+b-d]$ is equivalent by finite decomposition to a subset of I. Hence $c+b-d-a \leqslant |I|$ and $|J| = b-a \leqslant d-c+|I| < 2|I|$. ■

Theorem 1 implies that von Neumann's paradox can be realized by four-piece decompositions.

THEOREM 3. *Let I, J be intervals with $|J| < 2|I|$. Then there are decompositions $I = \bigcup_{i=1}^{4} A_i$, $J = \bigcup_{i=1}^{4} B_i$ such that B_i is metrically smaller than A_i for every $i = 1, 2, 3, 4$.*

Indeed, we can take an interval J' with $|J| < |J'| < 2|I|$, apply Theorem 1 for the intervals I and J' and then use a contraction which maps J' onto J. We remark that von Neumann's proof requires at least 33 pieces in the decompositions. Obviously, Theorem 3 is much weaker than Theorem 1 and, accordingly, is much easier to prove. In the next section we give an independent proof.

We also show that the upper bound on $|J|$ in Theorem 3 is sharp. This is an immediate corollary of the following, more general theorem. We denote by λ_n and $\underline{\lambda}_n$ the n-dimensional Lebesgue outer and inner measures, respectively.

THEOREM 4. *Let $A_1, A_2, \ldots$ be a finite or infinite sequence of pairwise disjoint subsets of $\mathbf{R}^n$, and let, for every i, $f_i\colon A_i \to \mathbf{R}^n$ be a Lipschitz function with Lipschitz constant M_i. Then*

$$\underline{\lambda}_n\Big(\bigcup_i f_i(A_i)\Big) \leqslant M \lambda_n\Big(\bigcup_i A_i\Big),$$

where

$$M = \max\Big(\tfrac{1}{2}\sum_i M_i^n,\ \sup_i M_i^n\Big).$$

Now suppose that I, J are intervals and there are decompositions $I = \bigcup_{i=1}^{4} A_i$, $J = \bigcup_{i=1}^{4} B_i$ such that each B_i is metrically smaller than A_i. Then $B_i = f_i(A_i)$, where

f_i is a Lipschitz function with Lipschitz constant 1. Hence, by Theorem 4,

$$|J| = \underline{\lambda}_1(J) \leqslant \tfrac{1}{2} 4 \lambda_1(I) = 2|I|.$$

Also, if one of the maps is a contraction then the inequality is strict, since then there exists $q < 1$ such that

$$|J| \leqslant \tfrac{1}{2}(q+3)|I| < 2|I|.$$

Taking $M_1 = M_2 = 1$ in Theorem 4 we obtain the following corollary.

THEOREM 5. *Let A_1, A_2 be disjoint subsets of $\mathbf{R}^n$ and let $f_i\colon A_i \to \mathbf{R}^n$ be maps such that $|f_i(x)-f_i(y)| \leqslant |x-y|$ $(x, y \in A_i,\ i = 1, 2)$. Then*

$$\underline{\lambda}_n(f_1(A_1) \cup f_2(A_2)) \leqslant \lambda_n(A_1 \cup A_2).$$

In particular, von Neumann's paradox cannot be realized using two-piece decompositions. We do not know, however, if a three-piece von Neumann paradox exists.

We shall use the following notation. We denote by $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ the closed complex plane. By a *linear fractional transformation* we mean a function $\varrho(x) = \dfrac{ax+b}{cx+d}$ $(x \in \overline{\mathbf{C}})$, where $a, b, c, d \in \mathbf{C}$ and $ad-bc \neq 0$. The set of all linear fractional transformations is denoted by L. Each $\varrho \in L$ is a permutation of $\overline{\mathbf{C}}$ (i.e., a bijection of $\overline{\mathbf{C}}$ onto itself), and under the operation of composition L forms a group. The unit element of L (the identity map on $\overline{\mathbf{C}}$) will be denoted by j. If the coefficients of $\varrho \in L$ are real, then ϱ is also a permutation of the extended reals $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$. (Note that $\overline{\mathbf{R}}$ only contains one infinite element.) We denote by L_1 the set of linear fractional transformations $\varrho(x) = \dfrac{ax+b}{cx+d}$ with $ad-bc = 1$. Then L_1 is a subgroup of L. The composition of the maps α and β will be denoted by $\alpha\beta$, so that $\alpha\beta(x) = \alpha(\beta(x))$. By a decomposition we mean a union of pairwise disjoint sets.

2. Proof of Theorem 3. The proof is based on the following theorem of Robinson ([7, p. 254]; [11, p. 46]). Suppose that $\alpha, \beta, \gamma, \delta$ are independent rotations of the unit sphere S (i.e., they are free generators of a free subgroup of the group of rotations of S). Then there is a decomposition $S = \bigcup_{k=1}^{4} S_k$ such that

$$\alpha(S_1 \cup S_2) = S_1, \quad \beta(S_1 \cup S_2) = S_2, \quad \gamma(S_3 \cup S_4) = S_3, \quad \delta(S_3 \cup S_4) = S_4.$$

In the next lemma we transform this decomposition into $\overline{\mathbf{R}}$ and from that we infer a four-piece von Neumann paradox.

LEMMA 1. *Suppose that the real numbers a_k, b_k, c_k, d_k $(k = 1, 2, 3, 4)$ are algebraically independent over the rationals and let $\alpha_k(x) = \dfrac{a_k x + b_k}{c_k x + d_k}$ $(k = 1, 2, 3, 4)$.*

Then there is a decomposition $\overline{\boldsymbol{R}} = \bigcup_{k=1}^{4} H_k$ *such that*

$$(1)\quad \alpha_1(H_1\cup H_2) = H_1,\quad \alpha_2(H_1\cup H_2) = H_2,\quad \alpha_3(H_3\cup H_4) = H_3,\quad \alpha_4(H_3\cup H_4) = H_4.$$

Proof. The transformations $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ generate a free subgroup of L (see [5, p. 107]).

Let

$$A_k = a_k+ib_k,\qquad B_k = c_k+id_k,$$
$$C_k = -c_k+id_k,\qquad D_k = a_k-ib_k\qquad (k = 1, 2, 3, 4).$$

Then the system of numbers A_k, B_k, C_k, D_k $(k = 1, 2, 3, 4)$ is algebraically independent over $\boldsymbol{Q}$. Indeed, each of the numbers a_k, b_k, c_k, d_k is algebraically dependent on this system. Hence if this was algebraically dependent over $\boldsymbol{Q}$ then the degree of transcendence of the system a_k, b_k, c_k, d_k $(k = 1, 2, 3, 4)$ would be less than 16 which is impossible [10, p. 201]. Therefore there is a field automorphism φ of $\boldsymbol{C}$ such that

$$\varphi(a_k) = A_k,\quad \varphi(b_k) = B_k,\quad \varphi(c_k) = C_k,\quad \varphi(d_k) = D_k\quad (k = 1, 2, 3, 4).$$

We define $\varphi(\infty) = \infty$.

Let

$$\beta_k(x) = \frac{A_k x+B_k}{C_k x+D_k}\qquad (k = 1, 2, 3, 4).$$

Since $D_k = \overline{A}_k$ and $C_k = -\overline{B}_k$, each β_k represents a rotation of the Riemann sphere S through the stereographic projection. That is, if $\pi\colon S\to\overline{\boldsymbol{C}}$ is the stereographic projection then $\varrho_k = \pi^{-1}\beta_k\pi$ is a rotation of S for every $k = 1, 2, 3, 4$ [6, p. 55]. We have for every $x\in\overline{\boldsymbol{C}}$ and $k = 1, 2, 3, 4$ $\beta_k\varphi(x) = \dfrac{A_k\varphi(x)+B_k}{C_k\varphi(x)+D_k} = \varphi\left[\dfrac{a_k x+b_k}{c_k x+d_k}\right]$ $= \varphi\alpha_k(x)$ hence $\beta_k = \varphi\alpha_k\varphi^{-1}$ and $\varrho_k = \pi^{-1}\varphi\alpha_k\varphi^{-1}\pi$. This shows that the rotations ϱ_k are independent, because if $\varrho_{i_1}^{n_1}\ldots\varrho_{i_s}^{n_s} = j$, where $i_1, \ldots, i_s = 1, 2, 3, 4$ and $n_1, \ldots, n_s\in\boldsymbol{Z}$ then $\alpha_{i_1}^{n_1}\ldots\alpha_{i_s}^{n_s} = \varphi^{-1}\pi j\pi^{-1}\varphi = j$. Therefore, by Robinson's theorem, there is a decomposition $S = \bigcup_{k=1}^{4} S_k$ such that

$$\varrho_1(S_1\cup S_2) = S_1,\quad \varrho_2(S_1\cup S_2) = S_2,\quad \varrho_3(S_3\cup S_4) = S_3,\quad \varrho_4(S_3\cup S_4) = S_4.$$

We put

$$H_k = (\varphi^{-1}\pi(S_k))\cap\overline{\boldsymbol{R}}\qquad (k = 1, 2, 3, 4).$$

Using the facts that all the maps involved are bijections and $\alpha_k(\overline{\boldsymbol{R}}) = \overline{\boldsymbol{R}}$, it is easy to check that (1) holds. ■

Now we turn to the proof of Theorem 3. Let I, J be intervals with $|J| < 2|I|$. We may assume that $J = [0, 2]$. Let $\varepsilon > 0$ be fixed such that $|I| > \dfrac{1+2\varepsilon}{1-\varepsilon}$. Then there are algebraically independent real numbers a_k, b_k, c_k, d_k $(k = 1, 2, 3, 4)$ such that the functions $\alpha_k(x) = \dfrac{a_k x+b_k}{c_k x+d_k}$ have the following properties:

$$|\alpha_k'(x)-1| < \varepsilon\quad (x\in[0, 1],\quad k = 1, 2, 3, 4),\quad\text{and}$$
$$\alpha_k([0, 1])\subset\alpha_1([0, 1])\subset[-\varepsilon, 1+\varepsilon]\quad (k = 2, 3, 4).$$

(One has to choose a_k and d_k close to 1, b_k and c_k close to zero. Also, the coefficients of α_1 have to be chosen first such that $[0, 1]\subset\operatorname{int}[\alpha_1([0, 1])]$ and then we pick the coefficients of $\alpha_2, \alpha_3, \alpha_4$.)

By Lemma 1, there is a decomposition $\overline{\boldsymbol{R}} = \bigcup_{k=1}^{4} H_k$ such that (1) holds. Let β denote the translation $\beta(x) = x+1$.

We define

$$g(x) = \begin{cases}\alpha_1(x) & \text{if } x\in(H_1\cup H_2)\cap[0, 1],\\ \alpha_3(x) & \text{if } x\in(H_3\cup H_4)\cap[0, 1],\\ \alpha_2\beta^{-1}(x) & \text{if } x\in\beta(H_1\cup H_2)\cap(1, 2],\\ \alpha_4\beta^{-1}(x) & \text{if } x\in\beta(H_3\cup H_4)\cap(1, 2].\end{cases}$$

Then g is a one-to-one map from $J = [0, 2]$ into $\alpha_1([0, 1])$. Let $h(x) = \alpha_1^{-1}(x)$ for $x\in\alpha_1([0, 1])$; then h is a one-to-one map from $\alpha_1([0, 1])$ into J. By Banach's theorem [1] there is a decomposition $J = P\cup Q$ such that

$$s(x) = \begin{cases} g(x) & \text{if } x\in P,\\ h^{-1}(x) = \alpha_1(x) & \text{if } x\in Q\end{cases}$$

is a bijection from J onto $\alpha_1([0, 1])$. Let

$$B_1 = [(H_1\cup H_2)\cap[0, 1]]\cup Q,\qquad B_2 = \beta(H_1\cup H_2)\cap(1, 2]\cap P,$$
$$B_3 = (H_3\cup H_4)\cap[0, 1]\cap P\quad\text{and}\quad B_4 = \beta(H_3\cup H_4)\cap(1, 2]\cap P.$$

Then $J = \bigcup_{k=1}^{4} B_k$ is a decomposition. Also, $s(x) = \alpha_k(x)$ for $x\in B_k$ and $k = 1, 3$ and $s(x) = \alpha_k\beta^{-1}(x)$ for $x\in B_k$ and $k = 2, 4$.

Let γ be a linear function such that $\gamma(\alpha_1([0, 1])) = I$. Since $|I| > \dfrac{1+2\varepsilon}{1-\varepsilon}$ and $|\alpha_1([0, 1])| < 1+2\varepsilon$, we have $\gamma' > \dfrac{1}{1-\varepsilon}$. Then γs is a bijection between J and I. Let $A_k = \gamma s(B_k)$ $(k = 1, 2, 3, 4)$, $f_k = (\gamma\alpha_k)^{-1}$ $(k = 1, 3)$, and $f_k = (\gamma\alpha_k\beta^{-1})^{-1}$ $(k = 2, 4)$, then $f_k(A_k) = B_k$ $(k = 1, 2, 3, 4)$. Since $(\gamma\alpha_k)'(x) > \dfrac{1}{1-\varepsilon}(1-\varepsilon) = 1$ for every $x\in[0, 1]$, we have $0 < f_k'(x) < 1$ for every $x\in I$, and hence each f_k is a contraction. This completes the proof. ■

3. Proof of Theorem 1. The idea of the proof is that we find a linear fractional transformation α_1 and translations $\alpha_2, \alpha_3, \alpha_4$ such that a decomposition theorem similar to Lemma 1 holds. Unfortunately, the complete analogue of Lemma 1 cannot hold. Indeed, if $\overline{R} = \bigcup_{i=1}^{4} H_i$ is a decomposition such that (1) is satisfied then $\alpha_3\alpha_4(H_3 \cup H_4) \subset H_3$ and $\alpha_4\alpha_3(H_3 \cup H_4) \subset H_4$. However, if α_3 and α_4 are translations then $\alpha_3\alpha_4 = \alpha_4\alpha_3$, and hence $H_3 \cup H_4 = \emptyset$. Therefore, $H_1 \cup H_2 = \overline{R}$, and $H_1 = \alpha_1(\overline{R}) = \overline{R}$, $H_2 = \alpha_2(\overline{R}) = \overline{R}$, which is impossible. Therefore we have to replace Lemma 1 by a weaker statement.

Lemma 2. *Let the real numbers a, c, d, e be algebraically independent over the rationals and put $b = \frac{ad-1}{c}$. Let $\alpha, \beta \in L_1$ be defined by $\alpha(x) = \frac{ax+b}{cx+d}$ and $\beta(x) = x+e$. Then there are decompositions $\overline{R} = X_1 \cup X_2$ and $\overline{R} = X_3 \cup X_4$ such that the sets $\alpha(X_1)$, $\beta(X_2)$, X_3, $\beta^2(X_4)$ are pairwise disjoint.*

This is an immediate consequence of Lemmas 3 and 4 below. In the proof of Lemma 3 (which is a variant of [11, Theorem 4.5, p. 37]) we use an idea of Robinson. He observed that in a free group of rotations of S, the subgroup of those rotations which leave a given point fixed is commutative and hence it is cyclic. Unfortunately, two linear fractional transformations do not necessarily commute if they have a common fixed point. They do commute, however if they have two common fixed points. What we have to prove in Lemma 4 is that, in the group generated by α and β, if two elements have a common fixed point, then they have two common fixed points.

Let X be a non-empty set and let S_X denote the group of permutations of X. We say that a subgroup $G \subset S_X$ is *locally commutative* provided that whenever two elements of G have a common fixed point then they commute.

Lemma 3. *Let X be a non-empty set, let the group $G \subset S_X$ be locally commutative, and suppose that G is freely generated by the elements $a, b \in G$. Then there are decompositions $X = X_1 \cup X_2$ and $X = X_3 \cup X_4$ such that the sets $a(X_1)$, $b(X_2)$, X_3 and $b^2(X_4)$ are pairwise disjoint.*

Proof. Let i denote the unit element of G (the identity map on X). Every $r \in G$, $r \neq i$ has a unique representation $r = a^{k_1}b^{n_1} \ldots a^{k_p}b^{n_p}$, where the exponents are integers, $k_t \neq 0$ for $t = 2, \ldots, p$ and $n_t \neq 0$ for $t = 1, \ldots, p-1$. This will be called the *canonical representation of* r. The number $|k_1|+|n_1|+\ldots+|k_p|+|n_p|$ is the length of the representation. Putting $x \sim y$ if $y = r(x)$ for some $r \in G$, we define an equivalence relation on X. Let E be an arbitrary equivalence class. Since $r(E) = E$ for every $r \in G$, it is enough to prove that there are decompositions $E = E_1 \cup E_2$ and $E = E_3 \cup E_4$ such that the sets $a(E_1)$, $b(E_2)$, E_3 and $b^2(E_4)$ are pairwise disjoint.

Suppose first that, for every $x \in E$ and $r \in G \setminus \{i\}$, $r(x) \neq x$.

Let an element $v \in E$ be chosen. Then for every $x \in E$ there is a unique $r_x \in G$ such that $r_x(v) = x$. The elements of the sets E_i will be selected according to the values of k_1 and n_1 in the canonical representation of r_x. We put the element x into E_1 if $k_1 = 0$ and $n_1 \neq 0$, into E_2 if $r_x = i$ or $k_1 \neq 0$, into E_3 if $k_1 = 0$ and $n_1 < 0$, and into E_4 if $r_x = i$, or $k_1 \neq 0$, or $k_1 = 0$ and $n_1 > 0$. It is easy to check that $E = E_1 \cup E_2 = E_3 \cup E_4$. If $x \in a(E_1)$, then the canonical representation of r_x has $k_1 = 1$. Also, if $x \in b(E_2)$ then we have $k_1 = 0$ and $n_1 = 1$, if $x \in E_3$ then $k_1 = 0$ and $n_1 < 0$ and, finally, if $x \in b^2(E_4)$ then $k_1 = 0$ and $n_1 \geqslant 2$. This proves that $a(E_1)$, $b(E_2)$, E_3 and $b^2(E_4)$ are pairwise disjoint.

Suppose now that $g(u) = u$ for some $u \in E$ and $g \in G$, $g \neq i$. Then every $x \in E$ is a fixed point of some $r \in G$, $r \neq i$. Indeed, if $x = d(u)$ then $dgd^{-1}(x) = x$. By assumption, each group $G_x = \{s \in G: s(x) = x\}$ is commutative. Also, as subgroups of the free group G, they are free [4, p. 96]. Therefore G_x is cyclic for every $x \in E$. Let s_x denote one of the generators of G_x. Let $v \in E$ be such that s_v has the smallest length among the elements s_x $(x \in E)$, and put $s = s_v$. Then the canonical representation of s is such that the product ss does not cancel. Because if it did then one of the elements asa^{-1}, $a^{-1}sa$, bsb^{-1}, $b^{-1}sb$ would have smaller length than that of s which is impossible since they are (one of) the generators of the groups G_{av}, $G_{a^{-1}v}$, G_{bv} and $G_{b^{-1}v}$, respectively. For every $x \in E$ there is an $r \in G$ such that $x = r(v)$. This r is not unique since $rs^n(v) = r(v) = x$ for every n. However, we can select an r_x such that $x = r_x(v)$ and the product $r_x s$ does not cancel. Indeed, if $x = r(v)$ then rs^n will have this property for n large enough, since it ends with the same factor as s. Having selected r_x for every $x \in E$, we define the sets E_i $(i = 1, 2, 3, 4)$ in the same way as above. Then obviously $E = E_1 \cup E_2 = E_3 \cup E_4$. We prove that $a(E_1) \cap b(E_2) = \emptyset$. Suppose that $a(x_1) = b(x_2)$ for some $x_1 \in E_1$ and $x_2 \in E_2$. Then $ar_{x_1}(v) = br_{x_2}(v)$ and hence $(br_{x_2})^{-1}(ar_{x_1})(v) = v$. Since s generates G_v, this implies that $(br_{x_2})^{-1}(ar_{x_1}) = s^n$ for some integer n and thus $ar_{x_1} = br_{x_2}s^n$. Suppose that $n \geqslant 0$. Since $r_{x_2}s$ does not cancel, the canonical representation of the right hand side begins with a b, while that of ar_{x_1} begins with an a, which is impossible. If $n < 0$ then we write $ar_{x_1}s^{-n} = br_{x_2}$ and get the same contradiction. Similar arguments show that $a(E_1)$, $b(E_2)$, E_3 and $b^2(E_4)$ are pairwise disjoint. ■

Lemma 4. *Let the numbers a, c, d, e be algebraically independent over the rationals and put $b = \frac{ad-1}{c}$. Let $\alpha, \beta \in L_1$ be defined by $\alpha(x) = \frac{ax+b}{cx+d}$ and $\beta(x) = x+e$ $(x \in \overline{C})$, and let G denote the group generated by α and β. Then G is freely generated by α and β, and G is locally commutative.*

Proof. $\boldsymbol{Z}[x_1, \ldots, x_n]$ will denote the ring generated by $\boldsymbol{Z}$ and the numbers $x_1, \ldots, x_n$. The field generated by $\boldsymbol{Q}$ and the numbers $x_1, \ldots, x_n$ will be denoted by $\boldsymbol{Q}(x_1, \ldots, x_n)$. It is easy to check that if $\varrho \in G$ then $\varrho(x) = \frac{Ax+B}{Cx+D}$, where $A, B, C, D \in \boldsymbol{Z}[a, b, c, d, e]$ and $AD-BC = 1$. Substituting $b = \frac{ad-1}{c}$, we can see that each of A, B, C, D can be written in the form of $\frac{P}{c^k}$, where $P \in \mathbf{Z}[a, c, d, e]$ and k is a non-negative integer. Let $\varrho = \alpha^{k_1}\beta^{n_1} \dots \alpha^{k_p}\beta^{n_p}$, where k_i, n_i are integers, $k_i \neq 0$ for $i = 2, \dots, p$ and $n_i \neq 0$ for $i = 1, \dots, p-1$ if $p > 1$, and $k_1 \neq 0$ or $n_1 \neq 0$ if $p = 1$. Let $\varrho(x) = \frac{Ax+B}{Cx+D}$ as above. We show that $C = 0$ implies $p = 1$ and $\varrho = \beta^{n_1}$. Indeed, if $C = 0$ then this must be an identity since a, c, d, e are algebraically independent over $\mathbf{Q}$. That is, for every $\gamma \in L_1$ and for every translation $\delta(x) = x+u$, the map $\gamma^{k_1}\delta^{n_1} \dots \gamma^{k_p}\delta^{n_p}$ is of the form $\frac{A'x+B'}{D'}$, where $A'D' = 1$. In particular, the absolute value of this function at x tends to infinity, as $x \to \infty$. Let $\omega(x) = \frac{-1}{x}$ and $\delta(x) = x+2$. Putting $\gamma = \omega\delta\omega$, we obtain $\gamma^k = \omega\delta^k\omega$ for every $k \in \mathbf{Z}$, and hence

$$\gamma^{k_1}\delta^{n_1} \dots \gamma^{k_p}\delta^{n_p} = \omega\delta^{k_1}\omega\delta^{n_1}\omega\delta^{k_2} \dots \omega\delta^{n_p}.$$

Now it is easy to check that if $p > 1$ or $p = 1$ and $k_1 \neq 0$ then the value of the right-hand side at x has a finite limit as $x \to \infty$. (This argument is due to von Neumann [5, p. 107].) This contradiction shows that $C = 0$ implies $\varrho = \beta^{n_1}$ indeed.

Since either $C \neq 0$ or $\varrho = \beta^{n_1}$ with $n_1 \neq 0$ implies that ϱ is not the identity map, we have proved that G is freely generated by α and β.

Now let $\varrho, \tau \in G$, $\varrho(x) = \frac{Ax+B}{Cx+D}$, $\tau(x) = \frac{A'x+B'}{C'x+D'}$, and suppose that ϱ and τ have a common fixed point u. We may assume that neither ϱ nor τ is the identity. Suppose first that $u = \infty$. Then $C = C' = 0$ and hence $\varrho = \beta^{n_1}$ and $\tau = \beta^{m_1}$. Thus ϱ and τ are translations and then they commute.

Next let u be finite. Then ϱ and τ cannot be translations and hence, by our preceding argument, $C \neq 0$ and $C' \neq 0$. Thus from $\varrho(u) = u$ we obtain $u = \frac{A-D+\sqrt{\Delta}}{2C}$, where $\Delta = (D-A)^2+4BC = (D-A)^2+4AD-4 = (A+D)^2-4$, and the value of $\sqrt{\Delta}$ is chosen appropriately. Similarly, $u = \frac{A'-D'+\sqrt{\Delta'}}{2C'}$, where $\Delta' = (A'+D')^2-4$. This implies that $C'\sqrt{\Delta}-C\sqrt{\Delta'} = r \in \mathbf{Q}(a, c, d, e)$ and hence, computing r^2, we obtain $\sqrt{\Delta}\sqrt{\Delta'} = s \in \mathbf{Q}(a, c, d, e)$. Suppose first that $\sqrt{\Delta} \notin \mathbf{Q}(a, c, d, e)$. Then $\sqrt{\Delta'} = \frac{s}{\sqrt{\Delta}} = t\sqrt{\Delta}$, where $t \in \mathbf{Q}(a, c, d, e)$.

Since $u = \frac{A-D+\sqrt{\Delta}}{2C} = \frac{A'-D'+t\sqrt{\Delta}}{2C'}$, we have $\frac{A-D}{C} = \frac{A'-D'}{C'}$ and $\frac{1}{C} = \frac{t}{C'}$. Hence $\frac{A-D-\sqrt{\Delta}}{2C} = \frac{A'-D'-\sqrt{\Delta'}}{2C'}$ which shows that the other fixed points of ϱ and τ also coincide. This implies ϱ and τ commute. Indeed, let u' be the other fixed point and let $\xi(x) = \frac{x-u}{x-u'}$. Then the fixed points of $\eta = \xi\varrho\xi^{-1}$ and $\theta = \xi\tau\xi^{-1}$ are 0 and ∞. Therefore $\eta(x) = c_1x$ and $\theta(x) = c_2x$. In particular, η and θ commute and hence so do ϱ and τ.

Now let $\sqrt{\Delta} \in \mathbf{Q}(a, c, d, e)$. We prove that in this case $\Delta = 0$. Indeed, let $\Delta = (A+D)^2-4 = T^2$, where $T \in \mathbf{Q}(a, c, d, e)$. As we saw in the beginning of the proof, $A+D = \frac{P}{c^k}$, where $P \in \mathbf{Z}[a, c, d, e]$ and $k \geqslant 0$. Then $P^2-4c^{2k} = T^2c^{2k}$, and hence $P^2-4c^{2k} = S^2$, where $S \in \mathbf{Z}[a, c, d, e]$. Thus $(P-S)(P+S) = 4c^{2k}$, from which $P-S = zc^n$, $P+S = \frac{4}{z}c^m$, where $z = \pm 1, \pm 2$, or ± 4, $n, m \geqslant 0$ and $n+m = 2k$. Therefore $P = \frac{z}{2}c^n+\frac{2}{z}c^m$, and $A+D = \frac{z}{2}c^{n-k}+\frac{2}{z}c^{k-n}$. Again, this must be an identity. In other words, if $\varrho = \alpha^{k_1}\beta^{n_1} \dots \alpha^{k_p}\beta^{n_p}$, then for every $\gamma(x) = \frac{a_1x+b_1}{c_1x+d_1} \in L_1$ and for every translation $\delta(x) = x+e_1$, we have $\pi(x) = \gamma^{k_1}\delta^{n_1} \dots \gamma^{k_p}\delta^{n_p}(x) = \frac{A_1x+B_1}{C_1x+D_1}$, where $A_1D_1-B_1C_1 = 1$ and $A_1+D_1 = \frac{z}{2}c_1^{n-k}+\frac{2}{z}c_1^{k-n}$. Let c_1 be an arbitrary non-zero real number and put $a_1 = d_1 = e_1 = 0$ and $b_1 = -\frac{1}{c_1}$. Then $\gamma(x) = -\frac{1}{c_1^2x}$, $\delta(x) = x$ and hence $\pi(x) = x$ or $\pi(x) = -\frac{1}{c_1^2x}$.

In the former case we have $A_1 = D_1 = 1$ or $A_1 = D_1 = -1$, while in the latter case $A_1 = D_1 = 0$. Since the value of A_1+D_1 must be equal to $\frac{z}{2}c_1^{n-k}+\frac{2}{z}c_1^{k-n}$ and c_1 was arbitrary, we obtain that $\frac{z}{2}c^{n-k}+\frac{2}{z}c^{k-n} = 2$, -2 or 0 for every $c \neq 0$. Hence $A+D = 2$, -2 or 0. Now $A+D = 0$ is impossible. Indeed, $A+D = 0$ would imply $\sqrt{\Delta} = \pm 2i \in \mathbf{Q}(a, c, d, e)$ contradicting the assumption that a, c, d, e are algebraically independent over $\mathbf{Q}$. Therefore $A+D = \pm 2$ and $\Delta = 0$.

Since $C'\sqrt{\Delta}-C\sqrt{\Delta'} \in \mathbf{Q}(a, c, d, e)$, $\sqrt{\Delta'}$ lies in $\mathbf{Q}(a, c, d, e)$ and hence $\Delta' = 0$, too. Therefore both ϱ and τ have only one fixed point, u. This, again, implies that they commute. Indeed, let $\lambda(x) = \frac{1}{x-u}$, and put $\eta = \lambda\varrho\lambda^{-1}$, $\vartheta = \lambda\tau\lambda^{-1}$. Then the only fixed point of η and ϑ is ∞ and hence they are translations. Thus η and ϑ commute, whence ϱ and τ commute as well. This completes the proof of Lemma 4. ■

Now we turn to the proof of Theorem 1. Since the assertion of the theorem is obvious if $|J| \leqslant |I|$ (take $A_1 = I$ $B_1 = J$ if $|J| < |I|$ and $A_1 = B_1 = \emptyset$ if $|J| = |I|$), we may suppose $|I| < |J|$. Also, we may assume that $I = [0, u]$ and $J = [0, 2v]$, where $0 < \frac{u}{2} < v < u$.

We show that there are real numbers a, b, c, d, e such that a, c, d, e are algebraically independent over $\boldsymbol{Q}$, $ad - bc = 1$, and the transformations $\alpha(x) = \frac{ax+b}{cx+d}$ and $\beta(x) = x + e$ have the following properties: $\alpha([0, v]) \subset [0, u]$, $\beta([0, v]) \cup \beta^2([0, v]) \subset [0, u]$, and $0 < (\alpha^{-1})'(x) < 1$ for every $x \in [0, u]$.

First we choose $\delta \in (0, 1)$ such that $\frac{1-\delta}{\delta} > v$ and if $a, d \in (1-\delta, 1+\delta)$ and $b, c \in (-\delta, \delta)$ then $\frac{av+b}{cv+d} < u$. Next we find $\eta \in (0, \delta)$ such that if $a, d \in (1-\eta, 1+\eta)$ then $|ad-1| < \frac{\delta^2}{2}$. Then we choose a, c, d, e such that they are algebraically independent over $\boldsymbol{Q}$, and satisfy the inequalities $1 < a < 1 + \frac{\eta}{2}$, $-\delta < c < \frac{-\delta}{2}$, $1 - \eta < d < 1 - \frac{\eta}{2}$ and $0 < e < \frac{u-v}{2}$.

Then we put $b = \frac{ad-1}{c}$ and observe that $b > 0$ since $ad - 1 < \left(1 + \frac{\eta}{2}\right)\left(1 - \frac{\eta}{2}\right) - 1 < 0$ and $c < 0$, and also $b < \delta$ for $|ad-1| < \frac{\delta^2}{2}$, and $|c| > \frac{\delta}{2}$.

Since $\alpha'(x) = \frac{1}{(cx+d)^2}$ and $d + cv > (1-\delta) - \delta v > 0$, α is strictly increasing on $[0, v]$. Thus $0 < \alpha(0) = \frac{b}{d}$ and $\alpha(v) = \frac{av+b}{cv+d} < u$ imply $\alpha([0, v]) \subset [0, u]$. The inclusion $\beta([0, v]) \cup \beta^2([0, v]) \subset [0, u]$ is obvious by $e > 0$ and $v + 2e < u$. Finally, $\alpha^{-1}(x) = \frac{-dx+b}{cx-a}$ and $(\alpha^{-1})'(x) = \frac{1}{(a-cx)^2}$. If $x \geqslant 0$ then $a - cx \geqslant a > 1$, and hence $0 < (\alpha^{-1})'(x) < 1$.

By Lemma 2, there are decompositions $\overline{\boldsymbol{R}} = X_1 \cup X_2$ and $\overline{\boldsymbol{R}} = X_3 \cup X_4$ such that the sets $\alpha(X_1)$, $\beta(X_2)$, X_3 and $\beta^2(X_4)$ are pairwise disjoint. Let γ denote the translation $\gamma(x) = x + v$. We define

$$g(x) = \begin{cases} x & \text{if } x \in X_3 \cap [0, v], \\ \beta^2(x) & \text{if } x \in X_4 \cap [0, v], \\ \alpha\gamma^{-1}(x) & \text{if } x \in \gamma(X_1) \cap (v, 2v], \\ \beta\gamma^{-1}(x) & \text{if } x \in \gamma(X_2) \cap (v, 2v]. \end{cases}$$

Then g is a one-to-one map from $J = [0, 2v]$ into $I = [0, u]$. Since the identity is a one-to-one map from I into J, there is a decomposition $J = P \cup Q$ such that

$$s(x) = \begin{cases} x, & x \in P, \\ g(x), & x \in Q \end{cases}$$

is a bijection from J onto I. Now we put $B_1 = \gamma(X_1) \cap (v, 2v] \cap Q$, $B_2 = P \cup \cup [X_3 \cap [0, v]]$, $B_3 = X_4 \cap [0, v] \cap Q$, and $B_4 = \gamma(X_2) \cap (v, 2v] \cap Q$. Then $J = \bigcup_{k=1}^{4} B_k$ is a decomposition, $s|_{B_1} = \alpha\gamma^{-1}|_{B_1}$ and s is a translation on each B_i $(i = 2, 3, 4)$. Therefore the sets $A_i = s(B_i)$ $(i = 1, 2, 3, 4)$ and the function $f = \gamma\alpha^{-1}$ satisfy the requirements of Theorem 1. ■

4. Proof of Theorem 4. For an arbitrary function f we denote by $N(f, y)$ the number of elements (possibly infinite) of the set $f^{-1}(\{y\})$. It is well known that if $A \subset \boldsymbol{R}^n$ is a Borel set and if $f\colon A \to \boldsymbol{R}^n$ is a Lipschitz function with Lipschitz constant K then the function $N(f, y)$ is measurable and

$$\int_{\boldsymbol{R}^n} N(f, y)\, dy \leqslant K^n \lambda_n(A)$$

[3, 2.10.11, p. 176]. This is true for Lebesgue measurable sets as well since a Lipschitz function maps null sets into null sets.

Now let the sets A_i, functions f_i and numbers M_i and M be given as in Theorem 4. Suppose that the statement of the theorem is not true, that is,

$$\lambda_n\Big(\bigcup_i f_i(A_i)\Big) > M\lambda_n\Big(\bigcup_i A_i\Big).$$

Then $\lambda_n(\bigcup_i A_i) > 0$ since otherwise $\lambda_n(\bigcup_i f_i(A_i)) = 0$. Hence we have $M < \infty$. There are measurable sets A, B such that $\bigcup_i A_i \subset A$, $B \subset \bigcup_i f_i(A_i)$, and $M\lambda_n(A) < \lambda_n(B)$. In particular, $\lambda_n(A)$ is finite.

Let $\mathscr{H}$ denote the family of those measurable sets $X \subset A$ of positive measure for which $Y = \bigcup_i f_i(A_i \cap X)$ is measurable and $\lambda_n(Y) \leqslant M\lambda_n(X)$. Let $\mathscr{K}$ be a maximal disjoint subfamily of $\mathscr{H}$ and let X_0 be the union of the elements of $\mathscr{K}$. Then $\mathscr{K}$ is countable, and hence X_0 is measurable. Also, either $X_0 = \emptyset$ or $\lambda_n(X_0) > 0$ and in both cases, $Y_0 = \bigcup_i f_i(A_i \cap X_0)$ is measurable and $\lambda_n(Y_0) \leqslant M\lambda_n(X_0)$. Then we have

$$\lambda_n(B \setminus Y_0) \geqslant \lambda_n(B) - \lambda_n(Y_0) > M\lambda_n(A) - M\lambda_n(X_0) = M\lambda_n(A \setminus X_0).$$

Let g_i be an extension of f_i to $\boldsymbol{R}^n$ such that g_i is a Lipschitz function with Lipschitz constant M_i [3, 2.10.43, p. 201], and let $h_i = g_i|_{A \setminus X_0}$.

Since $B \setminus Y_0 \subset \bigcup_i f_i(A_i \setminus X_0) = \bigcup_i h_i(A_i \setminus X_0)$, we have $\sum_i N(h_i, y) \geqslant 1$ for every $y \in B \setminus Y_0$. On the other hand,

$$\int_{B \setminus Y_0} \sum_i N(h_i, y)\, dy \leqslant \sum_i \int_{\boldsymbol{R}^n} N(h_i, y)\, dy \leqslant \sum_i M_i^n \lambda_n(A \setminus X_0)$$
$$\leqslant 2M\lambda_n(A \setminus X_0) < 2\lambda_n(B \setminus Y_0),$$

and hence $\sum_i N(h_i, y) \leq 1$ on a positive measure subset of $B \setminus Y_0$. This implies that there is a k and there is a closed set $C \subset B \setminus Y_0$ such that $\lambda_n(C) > 0$ and for every $y \in C$ we have

$$N(h_i, y) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

We put $D = h_k^{-1}(C)$. Then D is measurable, and $\lambda_n(D) > 0$ since $\lambda_n(D) = 0$ would imply $\lambda_n(C) = \lambda_n(h_k(D)) = 0$.

We prove that $D \subset A_k \setminus X_0$. Obviously, $D \subset A \setminus X_0$ since $A \setminus X_0$ is the domain of h_k. Let $x \in D$ and suppose that $x \notin A_k \setminus X_0$. Since $h_k(x) \in C \subset B \setminus Y_0 \subset \bigcup_i h_i(A_i \setminus X_0)$, we have $h_k(x) = h_i(x_i)$ with some $x_i \in A_i \setminus X_0$. If $i = k$ then $x_i = x_k \in A_k \setminus X_0$, and hence $x \neq x_i$. Thus $N(h_k, h_k(x)) \geq 2$ which is impossible since $h_k(x) \in C$. If $i \neq k$, then we get $N(h_i, h_k(x)) \geq 1$ which also contradicts $h_k(x) \in C$.

Therefore $D \subset A_k \setminus X_0$ and, consequently, $D \cap A_i = \emptyset$ for $i \neq k$. This implies that $\bigcup_i f_i(A_i \cap D) = f_k(D) = h_k(D) = C$, where C is measurable and

$$0 < \lambda_n(C) \leq M_k^n \lambda_n(D) \leq M \lambda_n(D).$$

In other words, $D \in \mathscr{K}$. However, $D \cap X_0 = \emptyset$, and hence D is disjoint from the elements of $\mathscr{K}$ which contradicts the maximality of $\mathscr{K}$. This contradiction completes the proof. ■

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DEPARTMENT OF ANALYSIS
EOTVOS LORAND UNIVERSITY
Budapest, Muzeum Krt. 6-8
H-1088, Hungary

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An atriodic tree-like continuum with positive span which admits a monotone mapping to a chainable continuum

by

James F. Davis[1] (Richmond, Va.) and **W. T. Ingram** (Houston, Tex.)

Abstract. In this paper an example of an atriodic tree-like continuum with positive span is constructed. It is shown that there is a monotone mapping of this continuum onto a chainable continuum such that the only nondegenerate point inverse under the mapping is an arc.

1. Introduction. The following problems appear in the University of Houston Mathematics Problem Book. The first was raised by Howard Cook, the second by Cook and J. B. Fugate.

PROBLEM 92. If M is a continuum with positive span such that each of its proper subcontinua has span zero, does every nondegenerate, monotone, continuous image of M have positive span?

PROBLEM 105. Suppose M is an atriodic 1-dimensional continuum and G is an upper semi-continuous collection of continua filling up M such that M/G and every element of G are chainable. Is M chainable?

These problems also appeared as problems 163 and 15, respectively, in [9]. Several partial positive results concerning these problems have appeared ([2] and [8] for instance).

In this paper we construct an example which answers both questions in the negative. The example is constructed as an inverse limit of simple triods with a single bonding map and has positive span. It is similar in this respect to the examples constructed in [4, 5]. The inspiration for this example was an example of an attractor of a discrete dynamical system presented by Marcy Barge at the 1986 Spring Topology Conference at the University of Southwestern Louisiana [1]. However, this example is not the example he discussed.

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