Globalizing fibrations by schedules

by

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Abstract. For any open covering of a space B, the paths of B can be continuously decomposed into subpaths each lying in an element of the covering. Such a "scheduling" into subpaths leads to a simple verification of the globalization theorem for Hurewicz fibrations, as well as to globalization results for more rigid kinds of fibrations.

The crucial method of establishing that a continuous function $p\colon Y \to B$ is a fibration is the Globalization Theorem for Fibrations. It states, roughly, that if for sufficiently many open sets U in B the restriction of p over U, $p_U\colon Y_U \to U$ for $Y_U = p^{-1}(U)$, is a fibration, then so is p. The first general statement of this theorem appears in Hurewicz ([3]); a more detailed investigation is in Dold ([1]). It can be found in introductory topology textbooks; for example, Dugundji ([2]).

We present here yet another proof of this fundamental result, a proof which it is hoped clearly isolates the concepts used and which may well have further utility. This proof is based on an investigation of properties of the path space of B related to an open covering of B. We obtain a "Schedule Theorem" which in a continuous manner decomposes each path into subpaths, each of which is in a prescribed element of the covering. This is a purely internal statement about the space B equipped with an open covering; it has nothing directly to do with any mapping $p\colon Y \to B$ or any statement about fibrations.

The globalization results for fibrations are immediate consequences of the Schedule Theorem. A bonus of this approach is an immediate proof of a globalization theorem for a special class of fibrations which we call "inversible". These have lifting functions which define homeomorphisms between fibers. This is a much stronger geometrical statement than the usual homotopy equivalence assertion. For a space B which is both paracompact and locally contractible in the large, a mapping $p\colon Y \to B$ is an inversible fibration if and only if there is a locally finite covering $\{U_a \mid a \in A\}$ of B by numerable open sets such that each of the restrictions $p_{U_a}\colon Y_{U_a} \to U_a$ is vertically homeomorphic to a projection map $U_a \times F \to U_a$; i.e., p is an inversible fibration if and only if it is locally trivial. Simple examples show that fibrations need not be locally trivial.

§ 1. Schedules. For a set A we denote by A^* the free monoid generated by the elements of A. An element s of A^* is usually written as a word

$$s = a_1 \dots a_n$$

with $a_1, \dots, a_n \in A$. We define the *word length*

$$\# s = n .$$

The unit element of A^* is the "empty word" and is denoted by Λ (regardless of what A is). Of course, $\# \Lambda = 0$. It is useful to regard $\#$ as a morphism of monoids

$$\#: A^* \to N$$

where N is the set of all integers $n \geqslant 0$ with addition as operation.

We denote by T the monoid of all real numbers $t \geqslant 0$ with addition as operation. The monoid T^* is also equipped with a *length function*

$$l: T^* \to T$$

defined for each word $v = t_1 \dots t_n$ by

$$l(v) = t_1 + \dots + t_n .$$

This length is not to be confused with the word length $\# v$, which is n.

There is a right operation of T on T^* given by

$$vt = (t_1 t) \dots (t_n t) .$$

Clearly,

$$\# vt = \# v \quad \text{and} \quad l(vt) = l(v)t .$$

The monoid

$$SA = (A \times T)^*$$

is called the *schedule monoid* of the set A and its elements are called *schedules* in A.

There are two monoid morphisms

$$p_1: SA \to A^* \quad \text{and} \quad p_2: SA \to T^*$$

defined for each generating schedule (a, t) by

$$p_1(a, t) = a \quad \text{and} \quad p_2(a, t) = t .$$

For any schedule s we have

$$\# p_1(s) = \# s = \# p_2(s) .$$

Given any $w \in A^*$ and $v \in T^*$ such that $\# w = \# v$, there is a unique $s \in SA$ such that $p_1(s) = w$ and $p_2(s) = v$. It will be convenient to write $s = (w, v)$. In this way SA becomes identified with $A^* \underset{N}{\times} T^*$.

The right operation of T on T^* extends to one of T on SA by defining $(w, v)t = (w, vt)$. We define $l: SA \to T$ to be the composite

$$SA \xrightarrow{p_2} T^* \xrightarrow{l} T .$$

Of course, $l(st) = l(s)t$ for $s \in SA$, $t \in T$.

A schedule s is said to be *reduced* if it is a product of pairs (a, t) in $A \times T$ with $t > 0$. The reduced schedules form a submonoid RSA of SA. There is also a retraction

$$\varrho: SA \to RSA .$$

It is the monoid morphism defined by

$$\varrho(a, t) = \begin{cases} \Lambda & \text{if } t = 0 , \\ (a, t) & \text{if } t > 0 . \end{cases}$$

Observe that

$$\# \varrho(s) \leqslant \# s$$

with equality holding if and only if $\varrho(s) = s$; i.e., if and only if s is reduced. Note however that

$$l(\varrho(s)) = l(s) .$$

In particular, $\varrho(s) = \Lambda$ if and only if $l(s) = 0$.

We now introduce topologies on SA and on RSA. For each $w \in A^*$ consider

$$D_w = p_1^{-1}(w) .$$

The function $p_2: SA \to T^*$ defines a bijection between D_w and $T^{\# w}$. We use this bijection to define the topology of D_w. The topology of SA is defined to be that of the coproduct of $\{D_w | \; w \in A^*\}$. Thus U is open in SA if and only if $U \cap D_w$ is open in D_w for every $w \in A^*$.

When it comes to topologizing RSA we have in principle the choice of regarding RSA as a submonoid or as a quotient monoid of SA. These two topologies are different. It is important that we use the *second* one. Thus a subset U of RSA is open if and only if the set $\varrho^{-1}(U)$ is open in SA.

§ 2. The Schedule Theorem. The *path space* PX of a topological space X is the subspace of $T \times (T, X)$ given by

$$PX = \{\alpha = (l, u) | \; u(t) = u(l) \text{ for } l \leqslant t\} .$$

Here (T, X) is the space of continuous functions $T \to X$ with the compact-open topology. The number l is the *length of the path* α and projection to the first coordinate is a continuous function $l: PX \to T$. We have the *source* and *target* morphisms

$$\sigma: PX \to X \quad \text{and} \quad \tau: PX \to X$$

defined by

$$\sigma(\alpha) = u(0) \quad \text{and} \quad \tau(\alpha) = u(l) .$$

Given paths $\alpha = (l, u)$ and $\alpha' = (l', u')$ with $\tau(\alpha) = \sigma(\alpha')$, the *sum* $\alpha+\alpha' = (l+l', v)$ is defined by

$$v(t) = \begin{cases} u(t) & 0 \leqslant t \leqslant l, \\ u'(t-l) & l \leqslant t. \end{cases}$$

We have the morphism $0\colon X \to PX$ assigning to each $x \in X$ the path $0_x = (0, \bar{x})$, where $\bar{x}$ is the constant map of T to x. And there is the morphism

$$-\colon PX \to PX$$

which assigns to each path $\alpha = (l, u)$ the path $-\alpha = (l, \hat{u})$ with $\hat{u}$ given by

$$\hat{u}(t) = \begin{cases} u(l-t), & 0 \leqslant t \leqslant l, \\ u(0), & l \leqslant t. \end{cases}$$

Let $\mathscr{U} = \{U_a | \ a \in A\}$ be a family of subsets of the space X. Let $\alpha \in PX$ and $s = (a_1 \dots a_n, t_1, \dots, t_n) \in SA$. We shall say that the *path α fits the schedule s*, written $\alpha || s$, provided $l(\alpha) = l(s)$ and for the decomposition

$$\alpha = \alpha_1 + \dots + \alpha_n \quad \text{with } l(\alpha_i) = t_i,$$

it is true that $\alpha_i \in PU_{a_i}$, $i = 1, \dots, n$. The relation $\alpha || s$ is characterized by the following three properties:

(i) $\alpha || (a, t)$ if and only if $l(\alpha) = t$ and $\alpha \in PU_a$,
(ii) if $\alpha_1 || s_1$, $\alpha_2 || s_2$ and $\tau(\alpha_1) = \sigma(\alpha_2)$, then $\alpha_1+\alpha_2 || s_1 s_2$, and
(iii) if $\alpha_1+\alpha_2 || s_1 s_2$, $l(\alpha_1) = l(s_1)$, and $l(\alpha_2) = l(s_2)$, then $\alpha_1 || s_1$ and $\alpha_2 || s_2$.

We have the evident

PROPOSITION. *If $\alpha || s$ and $l(\alpha) > 0$, then $\alpha || \varrho(s)$.* ∎

The converse of this proposition is false.

A subset V of the topological space X is called *numerable* if there is a continuous function $g\colon X \to T$ such that $g(x) > 0$ if and only if $x \in V$. Such a function g is called a *numeration* of the set V. The indexed collection $\mathscr{U} = \{U_a | \ a \in A\}$ of subsets of X is called *locally finite* if there exists an open covering $\mathscr{W}$ of X such that for each $W \in \mathscr{W}$ all of the intersections $W \cap U_a$ are empty except for finitely many indices $a \in A$.

A covering $\mathscr{F} = \{F_c | \ c \in C\}$ of the space X is called *local* if each set F_c is closed in X and each point of X is contained in the interior of the union of some finite subcollection of $\mathscr{F}$. Such a covering need not be locally finite. It is clear however that given a function $f\colon X \to Y$ such that $f | F_c$ is continuous for every $c \in C$, we can conclude that f itself is continuous.

THE SCHEDULE THEOREM. *Let $\mathscr{U} = \{U_a | \ a \in A\}$ be a locally finite covering of the space X by numerable open sets. Then there exists a local covering $\mathscr{F} = \{F_c | \ c \in C\}$ of PX and a family*

$$f_c\colon F_c \to SA$$

of continuous functions such that

(1) *for each $\alpha \in F_c$, $\alpha || f_c(\alpha)$ and*
(2) *for each $\alpha \in F_c \cap F_{c'}$, $\varrho(f_c(\alpha)) = \varrho(f_{c'}(\alpha))$.*

COROLLARY. *There exists a continuous function*

$$h\colon PX \to RSA$$

such that

$$\alpha || h(\alpha) \quad \text{if } l(\alpha) > 0 \quad \text{and}$$
$$h(\alpha) = \Lambda \quad \text{if } l(\alpha) = 0.$$ ∎

We shall see that the Globalization Theorem for Fibrations is an easy consequence of the Schedule Theorem. The more elegant corollary seems adequate for proving globalization results only when the base space is Hausdorff.

Our proof of the Schedule Theorem is presented in Section 5.

§ 3. Globalization of fibrations. The usual definition of a mapping $p\colon Y \to B$ being a (Hurewicz) fibration involves lifting arbitrary homotopies into B extending given lifts of their sources. A completely equivalent definition is the following: the mapping $p\colon Y \to B$ is a *fibration* if and only if there is a continuous function (called an *action*)

$$*\colon Y_p \times_\sigma PB \to Y,$$

to be written $y * \alpha$, which satisfies the conditions

$$p(y * \alpha) = \tau(\alpha) \quad \text{and} \quad y * 0_{p(y)} = y.$$

The space $Y_p \times_\sigma PB$ is the subspace of $Y \times PB$ of all pairs (y, α) such that $p(y) = \sigma(\alpha)$.

A useful stronger notion is that of an inversible fibration. The mapping $p\colon Y \to B$ is an *inversible fibration* provided there exist two actions

$$*, *'\colon Y_p \times_\sigma PB \to Y$$

(called a *reciprocal pair of actions*) which in addition to satisfying the conditions imposed by each of their being an action also satisfy the conditions

$$(y * \alpha) *' (-\alpha) = y = (y *' \alpha) * (-\alpha)$$

for all $(y, \alpha) \in Y_p \times_\sigma PB$.

This type of fibration is much more rigid than the usual type. For $A \subset B$ denote by Y_A the subspace $p^{-1}(A)$ of Y. In an inversible fibration the mapping

$$\_ * \alpha\colon Y_{\sigma(\alpha)} \to Y_{\tau(\alpha)}$$

is a homeomorphism for every $\alpha \in PB$, with $\_ *'(-\alpha)$ as inverse.

GLOBALIZATION THEOREM FOR FIBRATIONS. *Let $p\colon Y \to B$ be a continuous function. Suppose that $\mathscr{U} = \{U_a | \ a \in A\}$ is a locally finite covering of B by numerable* *open sets and that for each* $a \in A$ *the restriction of* p *over* U_a, $p_a\colon Y_a \to U_a$ *where* $Y_a = p^{-1}(U_a)$, *is a fibration. Then* p *is a fibration.*

More specifically, for each $a \in A$ *let an action* $\underset{a}{*}\colon Y_p \times_\sigma PU_a \to Y_a$ *be given and let* $\mathscr{F} = \{F_c |\ c \in C\}$ *be a local covering of* PB *and* $\{f_c\colon F_c \to SA |\ c \in C\}$ *be a family of continuous functions as in the Schedule Theorem. Then the assignment*

$$y * \alpha = y \underset{a_1}{*} \alpha_1 \underset{a_2}{*} \alpha_2 \ldots \underset{a_n}{*} \alpha_n$$

is an action $*\colon Y_p \times_\sigma PB \to Y$ *for* p, *where* $\alpha \in F_c$, $f_c(\alpha) = (a_1 \ldots a_n, t_1, \ldots, t_n)$ *and* $\alpha = \alpha_1 + \ldots + \alpha_n$ *with* $l(\alpha_i) = t_i$ *for* $i = 1, \ldots, n$.

Proof. First we note that the collection $\{Y_p \times_\sigma F_c |\ c \in C\}$ is a local covering of $Y_p \times_\sigma PB$. The action as given is defined and continuous on each member of this collection. Condition (2) of the Schedule Theorem implies that the action is a globally defined function. Its continuity is an immediate consequence of the definition of local covering. That the function is an action is clear. ■

Globalization Theorem for Inversible Fibrations. *If, additionally, each* p_a *is an inversible fibration, then so is* p.

More specifically, if for each $a \in A$ *a reciprocal pair* $(\underset{a}{*}, \underset{a}{*}')$ *of actions for* p_a *is given, then* $*$ *as defined explicitly above, together with* $*'$, *defined below, are a reciprocal pair of actions for* p.

For $(y', \alpha') \in Y_p \times_\sigma PB$ *with* $-\alpha' \in F_{c'}$ *and* $f_{c'}(-\alpha') = (b_1 \ldots b_k, s_1, \ldots, s_k)$ *define*

$$y' *' \alpha' = y' \underset{b_k}{*}{}' (-\beta_k) \underset{b_{k-1}}{*}{}' (-\beta_{k-1}) \ldots \underset{b_1}{*}{}' (-\beta_1)$$

where $-\alpha' = \beta_1 + \ldots + \beta_k$ *with* $l(\beta_i) = s_i$ *for* $i = 1, \ldots, k$.

Proof. We prove first that $*'$ as defined is an action for p. For a subset K of PB denote by $-K$ the collection $\{\alpha \in PB |\ -\alpha \in K\}$. Since $-\colon PB \to PB$ is an involutory homeomorphism, the covering $-\mathscr{F} = \{-F_c |\ c \in C\}$ is also a local covering of PB. The proof that $*'$ is an action for p is just that of the Globalization Theorem for Fibrations, except we use the local covering $\{Y_p \times_\sigma (-F_c) |\ c \in C\}$ of $Y_p \times_\sigma PB$.

To see that $(*, *')$ is a reciprocal pair of actions for p we must verify for $(y, \alpha) \in Y_p \times_\sigma PB$ that

$$y * \alpha *' (-\alpha) = y = y *' \alpha * (-\alpha).$$

Assume $\alpha \in F_c$ with $f_c(\alpha) = (a_1, \ldots, a_n, t_1, \ldots, t_n)$, and write $\alpha = \alpha_1 + \ldots + \alpha_n$ with $l(\alpha_i) = t_i$ for $i = 1, \ldots, n$. Let $y' = y * \alpha$ and, in the notation of the definition of $*'$, write $\alpha' = -\alpha$. Then $-\alpha' = \alpha \in F_c$ and $f_c(-\alpha') = f_c(\alpha)$. Also, $-\alpha' = \alpha_1 + \ldots + \alpha_n$. Thus,

$$y * \alpha *'(-\alpha) = (y \underset{a_1}{*} \alpha_1 \underset{a_2}{*} \alpha_2 \ldots \underset{a_n}{*} \alpha_n) *' (-\alpha)$$
$$= y \underset{a_1}{*} \alpha_1 \ldots \underset{a_n}{*} \alpha_n \underset{a_n}{*}{}' (-\alpha_n) \ldots \underset{a_1}{*}{}' (-\alpha_1) = y.$$

Also,

$$y *' \alpha * (-\alpha) = y \underset{b_k}{*}{}' (-\beta_k) \ldots \underset{b_1}{*}{}' (-\beta_1) \underset{b_1}{*} \beta_1 \ldots \underset{b_k}{*} \beta_k = y$$

where $-\alpha \in F_{c'}$ with $f_{c'}(-\alpha) = (b_1 \ldots b_k, s_1, \ldots, s_k)$ and $-\alpha = \beta_1 + .. + \beta_k$ with $l(\beta_i) = s_i$ for $i = 1, .., k$. ■

The explicit forms of the actions obtained are convenient for globalizing additional structure. For example, let G be a set and $m\colon Y \times G \to Y$ be a function. We shall write yg in place of $m(y, g)$ and shall think of m as defining a right "action" of G on the space Y. We require nothing more of G or of the action m, although clearly an interesting and important case is that in which G is a topological group and the action is continuous, associative and unitary.

Corollary. *Continuing with the previous notation, suppose there is a right action* $Y \times G \to Y$ *of* G *on* Y *such that* $p(yg) = p(y)$ *for all* $(y, g) \in Y \times G$. *Suppose also that for each* $a \in A$

(3) $(yg) \underset{a}{*} \alpha = (y \underset{a}{*} \alpha) g$.

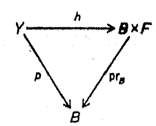
Then for the globally defined $*$ *in the statement of the Globalization Theorem,* (3) *is also true.*

Moreover, if for each $a \in A$ *there is a reciprocal pair* $(\underset{a}{*}, \underset{a}{*}')$ *of actions for* p_a *each of which satisfies* (3), *then the same is true of their globalizations.*

In particular, if G *is a topological group,* m *is a right action of* G *on* Y, *and each* $p_a\colon Y_a \to U_a$ *is vertically right* G *isomorphic to the projection* $\mathrm{pr}_1\colon U_a \times G \to U_a$ *(the action of* G *on* $U_a \times G$ *being given by the group in* G: $(u, g)g' = (u, gg')$*), then* p *is a principal right* G *bundle. Moreover, there is a reciprocal pair* $(*, *')$ *of actions for* p *each of which satisfies* (3). ■

In the globalization statements above, we required the open covering of B to be locally finite and each of its elements to be numerable. These conditions can be dropped if one assumes B is paracompact and Hausdorff. For in this case any open covering of B has a refinement satisfying the additional conditions.

§ 4. Local triviality. The mapping $p\colon Y \to B$ is called *trivial* if there exists a (vertical) homeomorphism $h\colon Y \to B \times F$ for some space F such that the diagram



commutes. Evidently, any such mapping is an inversible fibration.

We say that $p\colon Y \to B$ is *locally trivial* if for some covering $\mathscr{W} = \{W\}$ of B by open sets, each of the restrictions $p_W\colon Y_W \to W$ is trivial.

The space B is called *locally contractible in the large* if there is an open covering $\mathscr{W} = \{W\}$ of B such that each W is contractible to a point in B.

PROPOSITION. *If $p\colon Y\to B$ is an inversible fibration and B is locally contractible in the large, then p is locally trivial.*

This result is an immediate consequence of the

LEMMA. *If $p\colon Y\to B$ is an inversible fibration and the subset A of B is contractible to a point in B, then $p_A\colon Y_A\to A$ is trivial.*

Proof. Let $H\colon I\times A\to B$ be a homotopy contracting A to the point b of B; thus, $H(0,a)=a$ and $H(1,a)=b$ for all $a\in A$. Let $h\colon A\to(I,B)$ be the adjoint of H: for each $a\in A$, $h(a)$ is a path of length 1 in B with $\sigma(h(a))=a$ and $\tau(h(a))=b$.

Let $(*,*')$ be a reciprocal pair of actions for the inversible fibration p. And define

$$f\colon Y_A\to A\times Y_b$$

by $f(y)=\big(p(y),\ y*h(p(y))\big)$. This mapping has inverse given by $(a,w)\mapsto w*'(-h(a))$. Clearly, $\mathrm{pr}_A(f(y))=p(y)$; and so, f is a vertical homeomorphism. ■

COROLLARY. *Assume that the space B is paracompact, Hausdorff and locally contractible in the large. Then $p\colon Y\to B$ is an inversible fibration if and only if p is locally trivial.*

The proposition implies that if p is an inversible fibration, then it is locally trivial. Conversely, if p is locally trivial, then since B is paracompact there is a locally finite covering $\mathscr{U}=\{U\}$ of B by numerable open sets such that each $p_U\colon Y_U\to U$ is trivial. Since p_U is then an inversible fibration, the Globalization Theorem implies that p itself is an inversible fibration.

The hypotheses of the corollary are known to be satisfied if B is a CW-complex. Since the property of being locally contractible in the large is an invariant of homotopy type, it follows that the hypotheses of the corollary hold for paracompact, Hausdorff spaces having the homotopy type of a CW-complex. We do not know if the conclusion of the corollary is valid for a space B which has the homotopy type of CW-complex. In particular the following question is open.

QUESTION. *Let $p\colon Y\to B$ be a locally trivial mapping and suppose that B is contractible. Does it follow that p is trivial?*

There is an example due to P. T. McAuley (private communication, April 1980) of a mapping which is locally trivial but is not a fibration. We describe her example here.

Let L be the "long line". (This space can be defined as follows. Let ω be an uncountable well-ordered set in which each term has only countable many predecessors. In $\omega\times[0,1)$ introduce the linear order $(x,s)<(x',s')$ if and only if either $x<x'$ or $x=x'$ and $s<s'$. The space L has underlying set $\omega\times[0,1)$ and is topologized by the order topology.) Let Y be the subspace of $L\times L$ of all pairs (l,l') with $l<l'$, let $B=L$, and define $p\colon Y\to B$ to be projection onto the first factor.

For any two points $l<l'$ of B, the open segment $\langle l,l'\rangle=\{l''\in B|\ l<l''<l'\}$ is homeomorphic to the open interval $(0,1)\subset I$. Moreover, the restriction $p_{\langle l,l'\rangle}$ is trivial; $Y_{\langle l,l'\rangle}$ is vertically homeomorphic to the product $\langle l,l'\rangle\times(L-0)$, where 0 denotes the least element of L. The mapping p is thus seen to be locally trivial.

However, it is impossible to define an action for p. In fact, p is not even a "delay fibration" — this being a weaker notion than fibration, due to Dold ([1]), in which the lifting of homotopies is allowed an initial delay during which it moves vertically.

§ 5. Proof of the Schedule Theorem. This proof will use two lemmas; their proofs are given in the next section.

LEMMA 1. *Let $\mathscr{U}=\{U_a|\ a\in A\}$ be a locally finite covering of the space X by numerable open sets. Then there exists a collection $\{p_a|\ a\in A\}$ of numerations of the sets U_a which is also a partition of 1.*

Let $\mathscr{U}=\{U_a|\ a\in A\}$ be a collection of subsets of the space X. For $s=a_1\dots a_n\in A^*$ and $\alpha\in PX$ we shall say that α *evenly fits* s, and write $\alpha\|_e s$, if for the equidecomposition $\alpha=\alpha_1+\dots+\alpha_n$ of α into n parts of equal length, $\alpha_i\in PU_{a_i}$ for $i=1,\dots,n$. We adopt the convention that no path evenly fits Λ.

LEMMA 2. *Let $\mathscr{U}=\{U_a|\ a\in A\}$ be a locally finite covering of the space X by numerable open sets. Then there exists a locally finite covering $\mathscr{W}=\{W_s|\ s\in A^*\}$ of PX by numerable open sets such that for each $\alpha\in W_s$, $\alpha\|_e s$.*

Proof of the Schedule Theorem. Let $\mathscr{W}$ be a covering of PX as in Lemma 2 for the covering $\mathscr{U}$ of X of the hypothesis of the Schedule Theorem. Let $\{q_s|\ s\in A^*\}$ be a partition of 1 numerating the elements W_s of $\mathscr{W}$.

Let $\mathscr{B}$ be the collection of all finite subsets of $A^*-\{\Lambda\}$. For $b\in\mathscr{B}$ define

$$D_b=\{\alpha\in PX|\ \sum_{s\in b}q_s(\alpha)=1\}\,.$$

Notice that

$$D_b=\{\alpha\in PX|\ q_s(\alpha)=0 \text{ for all } s \text{ not in } b\}\,.$$

The collection $\{D_b|\ b\in\mathscr{B}\}$ is a covering of PX by closed sets. By the local finiteness of $\mathscr{W}$, for each $\alpha\in PX$ there exist some $b\in\mathscr{B}$ and open set V in PX containing α such that $q_s|V=0$ for all s not in b. Thus for all $\beta\in V$, $\sum_{s\in b}q_s(\beta)=1$; hence, $V\subset D_b$.

Totally order the elements of A^*. Then each $b\in\mathscr{B}$ is displayed with an order

$$b=s_1<\dots<s_k\,.$$

Given $\alpha\in D_b$, define

$$Q_i(\alpha)=\sum_{j=1}^{i}q_{s_j}(\alpha)\,.$$

Then

$$0\equiv Q_0\leqslant Q_1\leqslant\dots\leqslant Q_k=1\,.$$

For $b \in \mathscr{B}$ consider $2k$-tuples $e = (l_1, r_1, \ldots, l_k, r_k)$ of integers satisfying

$$1 \leqslant l_i \leqslant r_i \leqslant \# s_i .$$

Define

$$D_{(b,e)} = \left\{ \alpha \in D_b \middle| \frac{l_i-1}{\# s_i} \leqslant Q_{i-1}(\alpha) \leqslant \frac{l_i}{\# s_i} \quad \text{and} \quad \frac{r_i-1}{\# s_i} \leqslant Q_i(\alpha) \leqslant \frac{r_i}{\# s_i} \right\}.$$

The set $D_{(b,e)}$ is closed in D_b and the collection $\{D_{(b,e)}\}$ is a finite cover of D_b.

Let C be the collection of all pairs $c = (b, e)$ and let $F_c = D_{(b,e)}$. Then $\mathscr{F} = \{F_c | \ c \in C\}$ is a local covering of PX by closed sets.

For $c = (b, e)$ as above, define

$$f_c\colon F_c \to SA$$

as follows: for $\alpha \in F_c = D_{(b,e)}$,

$$f_c(\alpha) = \sigma_1 \ldots \sigma_k l(\alpha)$$

where σ_i is the schedule with

$$\# \sigma_i = r_i - l_i + 1 \quad \text{and} \quad l(\sigma_i) = q_{s_i}(\alpha) ,$$

and for $s_i = a_1 \ldots a_n$, σ_i is the product of basic schedules

$$\sigma_i = \left(a_{l_i}, \frac{l_i}{n} - Q_{i-1}(\alpha)\right)\left(a_{l_i+1}, \frac{1}{n}\right) \ldots \left(a_{r_i-1}, \frac{1}{n}\right)\left(a_{r_i}, Q_i(\alpha) - \frac{r_i-1}{n}\right)$$

provided $l_i < r_i$. For $l_i = r_i$, set $\sigma_i = (a_{l_i}, Q_i(\alpha) - Q_{i-1}(\alpha))$.

The function f_c is continuous. Also for $\alpha \in F_c$, α fits $f_c(\alpha)$ and for $\alpha \in F_c \cap F_{c'}$, $\varrho(f_c(\alpha)) = \varrho(f_{c'}(\alpha))$. ■

6. Proofs of lemmas.

Proof of Lemma 1. Let $g_a\colon X \to T$ be a numeration of U_a. For $x \in X$ define $g(x) = \sum_{a \in A} g_a(x)$. We claim the function g is continuous and never zero. Let $\mathscr{W}$ be an open covering of X as in the definition of locally finite; for $W \in \mathscr{W}$, $W \cap U_a = \emptyset$ except for finitely many $a \in A$. Thus $g_a | W$ is the zero function except for the same exceptional a's and $g|W$ is a finite sum of continuous functions. Hence g is continuous. Any $x \in X$ is in some U_a; thus $g_a(x) > 0$ and $g(x) \geqslant g_a(x) > 0$.

Since g is continuous and never zero, $1/g$ is also continuous. Define $p_a\colon X \to I$ to be g_a/g. Then p_a is a numeration of U_a. Clearly $\sum_{a \in A} p_a = 1$. ■

Proof of Lemma 2. The subspace of PX of those paths having length 1 is homeomorphic to the function space (I, X). Moreover there is a retraction $r\colon PX \to (I, X)$; one such is given by

$$r(\alpha) = \omega_a\colon I \to X$$

where for $\alpha = (l, u)$ and $0 \leqslant t \leqslant 1$, $\omega_a(t) = u(tl)$.

Suppose we have a locally finite covering $\mathscr{V} = \{V_s | \ s \in A^*\}$ of (I, X) by numerable open sets such that for each $\omega \in (I, X)$, $\omega ||_e s$. Then letting $W_s = r^{-1}(V_s)$, we obtain a covering $\mathscr{W} = \{W_s | \ s \in A^*\}$ satisfying the conclusion of Lemma 2. Thus Lemma 2 is a consequence of the

LEMMA. *Let $\mathscr{U} = \{U_a | \ a \in A\}$ be a locally finite covering of the space X by numerable open sets. Then there is a locally finite covering $\mathscr{V} = \{V_s | \ s \in A^*\}$ of (I, X) by numerable open sets such that for $\omega \in V_s$, $\omega ||_e s$.*

Proof. Let $\{p_a | \ a \in A\}$ be a partition of 1 numerating the elements of the collection $\mathscr{U}$. Bor $s = a_1 \ldots a_n \in A^*$ and $\omega \in (I, X)$, define

$$f_s(\omega) = \prod_{i=1}^{n} \inf \left\{ p_{a_i}(\omega(t)) \middle| \frac{i-1}{n} \leqslant t \leqslant \frac{i}{n} \right\}.$$

The function $f_s\colon (I, X) \to I$ is continuous and numerates the open set

$$EU_s = \{\omega \in (I, X) | \ \omega ||_e s\} .$$

Denote by $A^*_{[n]}$ the set of those words $s \in A^*$ with $\# s \leqslant n$; and define

$$E\mathscr{U}_{[n]} = \{EU_s | \ s \in A^*_{[n]}\} .$$

We claim the collection $E\mathscr{U}_{[n]}$ is locally finite. For $\omega \in (I, X)$ we shall find an open set S in (I, X) which contains ω and intersects only finitely many elements of $E\mathscr{U}_{[n]}$. For each $t \in I$ there is an open set W_t in X which contains $\omega(t)$ and intersects only finitely many elements of $\mathscr{U}$. By compactness of I, there is a finite set $\{t_1, \ldots, t_k\} \subset I$ such that $\omega(I)$ is contained in $W = W_{t_1} \cup \ldots \cup W_{t_k}$. Then open set W intersects only finitely many elements of $\mathscr{U}$.

Denote by A_W the finite set $\{a \in A | \ W \cap U_a \neq \emptyset\}$. Then $A^*_W \cap A^*_{[n]}$ is a finite set. Let S be the subbasic open set $[I, W]$ in (I, X). We claim that if $S \cap EU_s \neq \emptyset$ for $EU_s \in \mathscr{U}_{[n]}$, then $s \in A^*_W$. Since $\# s \leqslant n$, it follows that s is in the finite set $A^*_W \cap A^*_{[n]}$; thus S is an open set in (I, X) satisfying the above requirements. Suppose $\lambda \in S \cap EU_s$ with $s = a_1 \ldots a_i$, $i \leqslant n$. Since $\lambda(I) \subset W$, each $a_j \in A_W$; i.e., $s \in A^*_W$.

Since $E\mathscr{U}_{[n]}$ is locally finite, the function

$$F_n = \sum_{s \in A^*_{[n]}} f_s\colon (I, X) \to T$$

is continuous. For $s \in A^*$ with $\# s = n$, define

$$g_s = \sup(0, f_s - nF_{n-1})\colon (I, X) \to I ,$$

and let $V_s = g_s^{-1}((0, 1])$.

We claim that $\mathscr{V} = \{V_s | \ s \in A^*\}$ is a covering of (I, X) as required. Since $g_s \leqslant f_s$, $V_s \subset EU_s$; i.e., for $\omega \in V_s$, $\omega ||_e s$. To see that $\mathscr{V}$ covers (I, X), for $\omega \in (I, X)$ let k be the least integer for which there is a word $s = a_1 \ldots a_k$ with $f_s(\omega) > 0$. Then $F_{k-1}(\omega) = 0$ and so $g_s(\omega) = f_s(\omega)$. Thus, $\omega \in V_s$.

Finally, to see that $\mathscr{V}$ is locally finite, let $\omega \in (I, X)$. For some integers n and k, $F_n(\omega) > 1/k$. Define

$$R = \{\lambda \in (I, X) |\ F_n(\lambda) > 1/k\} .$$

The set R is open in (I, X) and contains ω. We claim that for $m > \max(k, n)$ and $s \in A^*$ with $\# s = m$, $R \cap V_s = \emptyset$. This will suffice since we know already that the collection $E\mathscr{U}_{\max(k,n)}$ is locally finite. For $\lambda \in R$,

$$F_{m-1}(\lambda) \geqslant F_n(\lambda) > 1/k .$$

Thus

$$mF_{m-1}(\lambda) > m(1/k) > 1 .$$

Since $f_s(\lambda) \leqslant 1$, it follows that $g_s(\lambda) = 0$. Thus λ is not in V_s. ■

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Connections between different amoeba algebras

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Abstract. The "amoeba algebra" is the complete Boolean algebra which has the effect of making the union of all null Borel sets coded in the ground model have measure 0 in the corresponding Boolean extension. Six different versions of the amoeba algebra are studied, together with the localization algebra, and connections, in some cases isomorphism and in some cases forcing equivalence, are established between them.

§ 1. Introduction. A number of different versions of Martin and Solovay's original "amoeba" algebras have been considered. In their original application [5] the relevant set of conditions was taken to be the set of open subsets of the real line of measure less than a fixed ε, partially ordered by inclusion, approximating to an open set of measure ε. In [8] we took instead a "variable" ε. That is, a condition was a pair (p, ε) where p is an open subset of $\boldsymbol{R}$ of measure less than ε, giving the information about the generic open set X that $p \subseteq X$ and $\mu(X) < \varepsilon$. The main reason for this was to enable us to show that the amoeba set of conditions P satisfies $\mathrm{RO}(P) \cong \mathrm{RO}(P \times P)$ where $\mathrm{RO}(P)$ is the complete Boolean algebra associated with P (the "regular open" algebra). Whether this is true for Martin and Solovay's "fixed measure" case we still do not know. And then there are the amoeba algebras on compact intervals I (or equivalently on 2^{ω}) derived from the set of (relatively) open subsets of I of measure less than ε, which were used by Shelah in [7], and also by Miller and others in their investigations into the connections between measure and category on the real line.

What all these algebras $\boldsymbol{B}$ have in common is the following. In each case the Boolean value in $V^{\boldsymbol{B}}$ of the statement

$$\text{“}\mu\{x \in \boldsymbol{R}\colon x \text{ is not random over } V\} = 0\text{”}$$

is $\mathbf{1}$, where μ denotes Lebesgue measure. What ideally we would like to know is that this statement holds in an extension of V if and only if the extension contains a V-generic filter on $\boldsymbol{B}$. In the absence of this, however, the next best thing seems to be to show that as many as possible of the known versions of the amoeba algebra are isomorphic, or at any rate, are equivalent in the sense of forcing. This was in