Globalizing fibrations by schedules

by

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Abstract. For any open covering of a space \( B \), the paths of \( B \) can be continuously decomposed into subpaths each lying in an element of the covering. Such a "scheduling" into subpaths leads to a simple verification of the globalization theorem for Hurewicz fibrations, as well as to globalization results for more rigid kinds of fibrations.

The crucial method of establishing that a continuous function \( p: Y \to B \) is a fibration is the Globalization Theorem for Fibrations. It states, roughly, that if for sufficiently many open sets \( U \) in \( B \) the restriction of \( p \) over \( U \), \( p_U: Y_U \to U \) for \( Y_U = p^{-1}(U) \), is a fibration, then so is \( p \). The first general statement of this theorem appears in Hurewicz ([3]), a more detailed investigation is in Dold ([1]). It can be found in introductory topology textbooks; for example, Dugundji ([2]).

We present here yet another proof of this fundamental result, a proof which it is hoped clearly isolates the concepts used and which may well have further utility. This proof is based on an investigation of properties of the path space of \( B \) related to an open covering of \( B \). We obtain a "Schedule Theorem" which in a continuous manner decomposes each path into subpaths, each of which is in a prescribed element of the covering. This is a purely internal statement about the space \( B \) equipped with an open covering; it has nothing directly to do with any mapping \( p: Y \to B \) or any statement about fibrations.

The globalization results for fibrations are immediate consequences of the Schedule Theorem. A bonus of this approach is an immediate proof of a globalization theorem for a special class of fibrations which we call "inversible". These have lifting functions which define homeomorphisms between fibers. This is a much stronger geometrical statement than the usual homotopy equivalence assertion. For a space \( B \) which is both paracompact and locally contractible in the large, a mapping \( p: Y \to B \) is an inversive fibration if and only if there is a locally finite covering \( \{ U_a \ a \in A \} \) of \( B \) by numerable open sets such that each of the restrictions \( p_{U_a}: Y_{U_a} \to U_a \) is vertically homeomorphic to a projection map \( U_a \times F \to U_a \); i.e., \( p \) is an inversive fibration if and only if it is locally trivial. Simple examples show that fibrations need not be locally trivial.
§ 1. Schedules. For a set $A$ we denote by $A^*$ the free monoid generated by the elements of $A$. An element $s$ of $A^*$ is usually written as a word

$$s = a_1 \ldots a_n$$

with $a_1, \ldots, a_n \in A$. We define the word length

$$\# s = n.$$  

The unit element of $A^*$ is the "empty word" and is denoted by $A$ (regardless of what $A$ is). Of course, $\# A = 0$. It is useful to regard $\#$ as a morphism of monoids

$$\#: A^* \rightarrow \mathbb{N}$$

where $\mathbb{N}$ is the set of all integers $n \geq 0$ with addition as operation.

We denote by $T$ the monoid of all real numbers $t \geq 0$ with addition as operation. The monoid $T^*$ is also equipped with a length function

$$l: T^* \rightarrow T$$

defined for each word $v = t_1 \ldots t_n$ by

$$l(v) = t_1 + \ldots + t_n.$$  

This length is not to be confused with the word length $\# v$, which is $n$.

There is a right operation of $T$ on $T^*$ given by

$$vt = (t_1, t) \ldots (t_n, t) .$$

Clearly,

$$\# vt = \# v$$

and

$$l(vt) = l(v) + t .$$

The monoid $SA = (A \times T)^*$ is called the schedule monoid of the set $A$ and its elements are called schedules in $A$.

There are two monoid morphisms

$$p_1: SA \rightarrow A^*$$

and

$$p_2: SA \rightarrow T^*$$

defined for each generating schedule $(a, t)$ by

$$p_1(a, t) = a$$

and

$$p_2(a, t) = t .$$

For any schedule $s$ we have

$$\# p_1(s) = \# s = \# p_2(s) .$$

Given any $w \in A^*$ and $v \in T^*$ such that $\# w = \# v$, there is a unique $s \in SA$ such that $p_1(s) = w$ and $p_2(s) = v$. It will be convenient to write $s = (w, v)$. In this way $SA$ becomes identified with $A^* \times T^*$.

The right operation of $T$ on $T^*$ extends to one of $T$ on $SA$ by defining

$$(w, v)t = (w, vt) .$$

We define $l: SA \rightarrow T$ to be the composite

$$SA \rightarrow T^* \rightarrow T .$$

Of course, $l(st) = l(s) + l(t)$ for $s \in SA$, $t \in T$.

A schedule $s$ is said to be reduced if it is a product of pairs $(a, t)$ in $A \times T$ with $t > 0$. The reduced schedules form a submonoid $RSA$ of $SA$. There is also a retraction

$$q: SA \rightarrow RSA .$$

It is the monoid morphism defined by

$$q(a, t) = \begin{cases} a & \text{if } t = 0 , \\ (a, t) & \text{if } t > 0 . \end{cases}$$

Observe that

$$\# q(s) \leq \# s$$

with equality holding if and only if $q(s) = s$; i.e., if and only if $s$ is reduced. Note however that

$$l(q(s)) = l(s) .$$

In particular, $q(a) = a$ if and only if $l(a) = 0$.

We now introduce topologies on $SA$ and on $RSA$. For each $w \in A^*$ consider

$$D_w = p_2^{-1}(w) .$$

The function $p_2: SA \rightarrow T^*$ defines a bijection between $D_w$ and $T^w$. We use this bijection to define the topology of $D_w$. The topology of $SA$ is defined to be that of the coproduct of $(D_w)_{w \in A^*}$. Thus $U$ is open in $SA$ if and only if $U \cap D_w$ is open in $D_w$ for every $w \in A^*$.

When it comes to topologizing $RSA$ we have in principle the choice of regarding $RSA$ as a submonoid or as a quotient monoid of $SA$. These two topologies are different. It is important that we use the second one. Thus a subset $U$ of $RSA$ is open if and only if the set $q^{-1}(U)$ is open in $SA$.

§ 2. The Schedule Theorem. The path space $PX$ of a topological space $X$ is the subspace of $T \times (T, X)$ given by

$$PX = \{ x = (t, u) \mid u(t) = u(l) \text{ for } l \leq t \} .$$

Here $(T, X)$ is the space of continuous functions $T \rightarrow X$ with the compact-open topology. The number $l$ is the length of the path $u$ and projection to the first coordinate is a continuous function $l \mid PX \rightarrow T$. We have the source and target morphisms

$$\sigma: PX \rightarrow X$$

and

$$\tau: PX \rightarrow X$$

defined by

$$\sigma(a) = u(0)$$

and

$$\tau(a) = u(l) .$$
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of continuous functions such that

(1) for each \( \alpha \in F_a \), \( \alpha \| f_a(a) \) and
(2) for each \( \alpha \in F_a \cap F_b \), \( \alpha f_a(a) = \alpha f_b(b) \).

**Corollary.** There exists a continuous function

\[ h: PX \to R \times \{0\} \]

such that

\[ \alpha \| h(a) \quad \text{if } l(a) > 0 \quad \text{and} \quad h(a) = \lambda \quad \text{if } l(a) = 0. \]

We shall see that the Globalization Theorem for Fibrations is an easy consequence of the Schedule Theorem. The more elegant corollary seems adequate for proving globalization results only when the base space is Hausdorff.

Our proof of the Schedule Theorem is presented in Section 5.

§ 3. Globalization of fibrations. The usual definition of a mapping \( p: X \to B \) being a (Hurewicz) fibration involves lifting arbitrary homotopies into \( B \) extending given lifts of their sources. A completely equivalent definition is the following: the mapping \( p: X \to B \) is a fibration if and only if there is a continuous function (called an action)

\[ \star: Y \times_{x} PB \to Y, \]

to be written \( y \star x \), which satisfies the conditions

\[ p(y \star x) = p(x) \quad \text{and} \quad y \star 0_{P(a)} = y. \]

The space \( Y \times_{x} PB \) is the subspace of \( Y \times PB \) of all pairs \( (y, a) \) such that \( p(y) = \sigma(a) \). A useful stronger notion is that of an inverse fibration. The mapping \( p: Y \to B \) is an inverse fibration provided there exist two actions

\[ \star, \star': Y \times_{x} PB \to Y \]

(called a reciprocal pair of actions) which in addition to satisfying the conditions imposed by each of their being an action also satisfy the conditions

\[ (y \star x) * \star' = y = (y' \star' x) * \star (x') \]

for all \( (y, a) \in Y \times_{x} PB \).

This type of fibration is much more rigid than the usual type. For \( A \to B \) denote by \( Y \) the subspace \( p^{-1}(A) \) of \( Y \). In an inverse fibration the mapping

\[ \star: Y \to Y \]

is a homeomorphism for every \( a \in PB \), with \( \star' = \star' \) as inverse.

**Globalization Theorem for Fibrations.** Let \( p: Y \to B \) be a continuous function. Suppose that \( \mathcal{U} = \{ U_a \mid a \in A \} \) is a locally finite covering of \( B \) by numerable open sets of \( Y \). Then there exists a covering \( F \) of \( PX \) and a family

\[ f_a: F_a \to SA \]
open sets and that for each \( a \in A \) the restriction of \( p \) over \( U_a, P_a: Y_a \to U_a \) where \( Y_a = p^{-1}(U_a) \) is a fibration. Then \( p \) is a fibration.

More specifically, for each \( a \in A \) let an action \( \ast: Y_a \times P_a U_a \to Y_a \) be given and let \( \mathcal{F} = \{ F_c, c \in C \} \) be a local covering of \( PB \) and \( \{ f_c: F_c \to SA, c \in C \} \) be a family of continuous functions as in the Schedule Theorem. Then the assignment

\[
y \ast a = y \ast_{a_1} \ast_{a_2} \ast \ldots \ast_{a_n}
\]

is an action \( \ast: Y_a \times P_a B \to Y_a \) for \( y \), where \( a \in F_c \), \( f_c(a) = (a_1, \ldots, a_n, t_1, \ldots, t_n) \) and \( a = a_1 + \ldots + a_n \) with \( I(a_i) = t_i \) for \( i = 1, \ldots, n \).

Proof. First we note that the collection \( \{ Y_a \times P_a F_c, c \in C \} \) is a local covering of \( Y_a \times P_a B \). The action \( \ast \) is given and continuous on each member of this collection. Condition (2) of the Schedule Theorem implies that the action is a globally defined function. Its continuity is an immediate consequence of the definition of local covering. That the function is an action is clear.

Globalization Theorem for Inversive Fibrations. If, additionally, each \( p_a \) is an inversible fibration, then so is \( p \).

More specifically, if for each \( a \in A \) a reciprocal pair \((\ast', \ast')\) of actions for \( p_a \) is given, then \( \ast \) as defined explicitly above, together with \( \ast' \), defined below, are a reciprocal pair of actions for \( p \).

For \((y, a') \in Y_a \times P_a B\) with \(-a' \in F_c\) and \( f_c(-a') = (b_1, \ldots, b_{n-1}, s_1, \ldots, s_n)\) define

\[
y \ast' a' = y \ast'_{b_1} \ast'_{b_2} \ast' \ldots \ast'_{b_{n-1}} \ast'_{s_1} \ast'_{s_2} \ast' \ldots \ast'_{s_n} (-a_1)
\]

where \(-a' = b_1 + \ldots + b_{n-1}\) with \( I(b_i) = s_i \) for \( i = 1, \ldots, k \).

Proof. We prove first that \( \ast' \) as defined is an action for \( p \). For a subset \( K \) of \( PB \) denote by \(-K\) the collection \( \{ a \in PB, \text{ } a \in K \} \). Since \(-: PB \to PB\) is an involutory homeomorphism, the covering \( -\mathcal{F} \) is also a local covering of \( PB \). The proof that \( \ast' \) is an action for \( p \) is just that of the Globalization Theorem for Fibrations, except we use the local covering \( \{ Y_K \times -F_c, c \in C \} \) of \( Y_a \times P_a B \).

To see that \( (\ast', \ast') \) is a reciprocal pair of actions for \( p \) we must verify for \((y, a) \in Y_a \times P_a B\) that

\[
y \ast a' \ast' \ast (-a) = y \ast' a \ast (-a) = y
\]

Assume \( a \in F_c \) with \( f_c(a) = (a_1, \ldots, a_n, t_1, \ldots, t_n) \) and write \( a = a_1 + \ldots + a_n \) with \( I(a_i) = t_i \) for \( i = 1, \ldots, n \). Let \( y = y \ast a \ast' \) and, in the notation of the definition of \( \ast' \), write \( a' = -a \). Then \( -a' = a \in F_c \) and \( f_c(-a') = f_c(a) \). Also, \(-a' = a_1 + \ldots + a_n \), thus

\[
y \ast a' \ast' (-a) = (y \ast_{a_1} \ast_{a_2} \ast \ldots \ast_{a_n}) \ast' (-a) = (y \ast_{a_1} \ast_{a_2} \ast \ldots \ast_{a_n} \ast' (-a_1) \ast' \ldots \ast' (-a_n)) = y
\]

Also,

\[
y \ast' a \ast (-a) = y \ast'_{b_1} \ast'_{b_2} \ast' \ldots \ast'_{b_{n-1}} \ast'_{s_1} \ast'_{s_2} \ast' \ldots \ast'_{s_n} (-a_1)
\]

where \(-a' \in F_c\) with \( f_c(-a') = (b_1, \ldots, b_{n-1}, s_1, \ldots, s_n) \) and \(-a' = b_1 + \ldots + b_{n-1}\) with \( I(b_i) = s_i \) for \( i = 1, \ldots, k \).

The explicit forms of the actions obtained are convenient for globalizing additional structure. For example, let \( G \) be a set and \( m: Y \times G \to Y \) be a function. We shall write \( yg \) in place of \( m(y, g) \) and shall think of \( m \) as defining a right "action" of \( G \) on the space \( Y \). We require nothing more of \( G \) or the action \( m \), although clearly an interesting and important case is that in which \( G \) is a topological group and the action is continuous, associative and unitary.

Corollary. Continuing with the previous notation, suppose there is a right action \( Y \times G \to Y \) of \( G \) on \( Y \) such that \( p(yg) = p(y) \) for all \((y, g) \in Y \times G \). Suppose also that for each \( a \in A \)

\[\text{(3) } (yg) \ast a = (y \ast a)g.g.\]

Then for the globally defined \( \ast \) in the statement of the Globalization Theorem, (3) is also true.

Moreover, if for each \( a \in A \) there is a reciprocal pair \((\ast', \ast')\) of actions for \( p_a \) each of which satisfies (3), then the same is true of their globalizations.

In particular, if \( G \) is a topological group, \( m \) is a right actioin of \( G \) on \( Y \), and each \( p_a: Y_a \to U_a \) is a right action of \( G \) on \( Y_a \), then \( p_a \) is a principal right \( G \) bundle. Moreover, there is a reciprocal pair \((\ast', \ast')\) of actions for \( p \) each of which satisfies (3).

In the globalization statements above, we required the open covering of \( B \) to be locally finite and each of its elements to be numerable. These conditions can be dropped if one assumes \( B \) to be paracompact hausdorff. For in this case any open covering of \( B \) has a refinement satisfying the additional conditions.

§ 4. Local triviality. The mapping \( p: Y \to B \) is called trivial if there exists a (vertical) homomorphism \( h: Y \to B \times F \) for some space \( F \) such that the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{h} & B \times F \\
\downarrow^{p} & & \downarrow^{p_B} \\
B & \xrightarrow{\pi} & F
\end{array}
\]

commutes. Evidently, any such mapping is an inversible fibration.

We say that \( p: Y \to B \) is locally trivial if for some covering \( \mathcal{W} = \{ W \} \) of \( B \) by open sets, each of the restrictions \( p_W: Y_W \to W \) is trivial.

The space \( B \) is called locally contractible in the large if there is an open covering \( \mathcal{W} = \{ W \} \) of \( B \) such that each \( W \) is contractible to a point in \( B \).
**Proposition.** If \( p : Y \to B \) is an invertible fibration and \( B \) is locally contractible in the large, then \( p \) is locally trivial.

This result is an immediate consequence of the

**Lemma.** If \( p : Y \to B \) is an invertible fibration and the subset \( A \) of \( B \) is contractible to a point in \( B \), then \( p_A : Y_A \to A \) is trivial.

**Proof.** Let \( H : I \times A \to B \) be a homotopy contracting \( A \) to the point \( b \) of \( B \); thus, \( H(0, a) = a \) and \( H(1, a) = b \) for all \( a \in A \). Let \( h : A \to (I, B) \) be the adjoint of \( H \): for each \( a \in A \), \( h(a) \) is a path of length 1 in \( B \) with \( \sigma(h(a)) = a \) and \( \tau(h(a)) = b \).

Let \((*, *)\) be a reciprocal pair of actions for the invertible fibration \( p \). And define

\[
f : Y_A \to A \times Y_b
\]

by \( f(y) = (p(y), y \cdot h(p(y))) \). This mapping has inverse given by \((a, w) \mapsto w \cdot (\gamma(a)) \). Clearly, \( p_s f(y) = p(y) \); and so, \( f \) is a vertical homeomorphism. ■

**Corollary.** Assume that the space \( B \) is paracompact, Hausdorff and locally contractible in the large. Then \( p : Y \to B \) is an invertible fibration if and only if \( p \) is locally trivial.

The proposition implies that if \( p \) is an invertible fibration, then it is locally trivial. Conversely, if \( p \) is locally trivial, then since \( B \) is paracompact there is a locally finite covering \( \mathcal{U} = \{U_a\ a \in A\} \) of \( B \) by numerable open sets such that each \( p_U : Y_U \to U \) is trivial. Since \( p_U \) is then an invertible fibration, the Globalization Theorem implies that \( p \) itself is an invertible fibration.

The hypotheses of the corollary are known to be satisfied if \( B \) is a CW-complex.

Since the property of being locally contractible in the large is an invariant of homotopy type, it follows that the hypotheses of the corollary hold for paracompact, Hausdorff spaces having the homotopy type of a CW-complex. We do not know if the conclusion of the corollary is valid for a space \( B \) which has the homotopy type of a CW-complex. In particular the following question is open.

**Question.** Let \( p : Y \to B \) be a locally trivial mapping and suppose that \( B \) is contractible. Does it follow that \( p \) is trivial?

There is an example due to P. T. McAnulty (private communication, April 1980) of a mapping which is locally trivial but is not a fibration. We describe her example here.

Let \( L \) be the "long line". (This space can be defined as follows. Let \( \omega \) be an uncountable well-ordered set in which each term has only countable many predecessors. In \( \omega \times [0, 1] \) introduce the linear order \((x, s) < (x', s')\) if and only if either \( x < x' \) or \( x = x' \) and \( s < s' \). The space \( L \) has underlying set \( \omega \times [0, 1] \) and is topologized by the order topology.) Let \( Y \) be the subspace of \( L \times 0 \) of all pairs \((l, l')\) with \( l < l' \), let \( B = L \), and define \( p : Y \to B \) to be projection onto the first factor.

For any two points \( l < l' \) of \( B \), the open segment \( \langle l, l' \rangle = \{t \in B \mid l < t < l'\} \) is homeomorphic to the open interval \((0, 1) \times (L - 0) \). Moreover, the restriction \( p_{\langle l, l' \rangle} \) is trivial; \( Y_{\langle l, l' \rangle} \) is vertically homeomorphic to the product \( (l, l') \times (L - 0) \), where \( 0 \) denotes the least element of \( L \). The mapping \( p \) is thus seen to be locally trivial.

However, it is impossible to define an action for \( p \). In fact, \( p \) is not even a "delay fibration" — this being a weaker notion than fibration, due to Dold ([1]), in which the lifting of homotopies is allowed an initial delay during which it moves vertically.

§ 5. Proof of the Schedule Theorem. This proof will use two lemmas; their proofs are given in the next section.

**Lemma 1.** Let \( \mathcal{U} = \{U_a \ a \in A\} \) be a locally finite covering of the space \( X \) by numerable open sets. Then there exists a collection \( \{\mathcal{U}_s : a \in A\} \) of numerations of the sets \( U_a \) which is also a partition of \( 1 \).

Let \( \mathcal{U} = \{U_a \ a \in A\} \) be a collection of subsets of the space \( X \). For \( s = a_1 \ldots a_n \in A^* \) and \( x \in PX \) we shall say that \( x \) evenly fits \( s \), and write \( x \mid s \), if for the equidecomposition \( x = x_1 \ldots x_n \) of \( x \) into \( n \) parts of equal length, \( a_i \in P_U \) for \( i = 1, \ldots, n \). We adopt the convention that no path evenly fits \( A \).

**Lemma 2.** Let \( \mathcal{U} = \{U_a \ a \in A\} \) be a locally finite covering of the space \( X \) by numerable open sets. Then there exists a locally finite covering \( \mathcal{W} = \{W_x \ x \in A^*\} \) of \( PX \) by numerable open sets such that for each \( x \in W_x \), \( x \mid s \).

**Proof of the Schedule Theorem.** Let \( \mathcal{W} \) be a covering of \( PX \) as in Lemma 2 for the covering \( \mathcal{U} \) of \( X \) of the hypothesis of the Schedule Theorem. Let \( \{q_x : x \in A^*\} \) be a partition of \( 1 \) numerating the elements \( W_x \) of \( \mathcal{W} \).

Let \( \mathcal{B} \) be the collection of all finite subsets of \( A^* \setminus \{A\} \). For \( b \in \mathcal{B} \) define

\[
D_b = \{x \in PX \mid \exists s \in \mathcal{B} \ q_s(x) = 1 \}
\]

Notice that

\[
D_b = \{x \in PX \mid q_s(x) = 0 \text{ for all } s \text{ not in } b \}
\]

The collection \( \{D_b : b \in \mathcal{B} \} \) is a covering of \( PX \) by closed sets. By the local finiteness of \( \mathcal{W} \), for each \( x \in PX \) there exists \( s \in \mathcal{B} \) and open set \( V \) in \( PX \) containing \( x \) such that \( q_s(x) = 0 \) for all \( s \) not in \( b \). Thus for all \( b \in \mathcal{B} \), \( \sum_{s \in \mathcal{B}} q_s(x) = 1 \); hence, \( V \subseteq D_b \).

Totally order the elements of \( A^* \). Then each \( b \in \mathcal{B} \) is displayed with an order

\[
b = s_1 < \ldots < s_n
\]

Given \( a \in D_s \), define

\[
Q_s(a) = \sum_{s \mid a} q_s(a)
\]

Then

\[
0 = Q_s \leq Q_1 \leq \ldots \leq Q_n = 1
\]
For $b \in B$ consider 2k-tuples $e = (l_1, r_1, \ldots, l_k, r_k)$ of integers satisfying
\[ 1 \leq l_i \leq r_i \leq \# s_i. \]

Define
\[ D_{(b,e)} = \left\{ c \in D_b : \frac{b_{l_i} - 1}{\# s_i} \leq Q_{l_i}(c) \leq \frac{l_i}{\# s_i} \quad \text{and} \quad \frac{r_i - 1}{\# s_i} \leq Q_{r_i}(c) \leq \frac{r_i}{\# s_i} \right\}. \]

The set $D_{(b,e)}$ is closed in $D_b$ and the collection $\{D_{(b,e)} : b \in B\}$ is a finite cover of $D_b$.

Let $C$ be the collection of all pairs $c = (b, e)$ and let $F = D_{(b,e)}$. Then $\mathcal{G} = \{F_c : c \in C\}$ is a local covering of $PX$ by closed sets.

For $c = (b, e)$ as above, define
\[ f_c : F_c \to SA \]
as follows: for $a \in F_c = D_{(b,e)}$,
\[ f_c(a) = a_{11} \ldots a_{k1} l(a) \]
where $a_i$ is the schedule with
\[ \# a_i = r_i - l_i + 1 \quad \text{and} \quad l(a_i) = g_{a_i}(c), \]
and for $a_i = a_i \ldots a_i$, $a_i$ is the product of basic schedules
\[ a_i = \left( a_{i1}, \frac{1}{n} - Q_{l_i}(c) \right) \left( a_{i1}, \frac{1}{n} - \right) \ldots \left( a_{i1}, \frac{1}{n} - Q_{r_i}(c) \right) \]
provided $l_i < r_i$. For $l_i = r_i$, set $a_i = (a_{i1}, Q_{r_i}(c) - Q_{l_i}(c))$.

The function $f_c$ is continuous. Also for $a \in F_c$, $a$ fits $f_c(a)$ and for $a \in F_c \cap F_r$, $\omega(f_c(a)) = \omega(f_c(a))$.

6. Proofs of lemmas.

Proof of Lemma 1. Let $g_a : X \to T$ be a numeration of $U_a$. For $x \in X$ define $g_a(x) = \sum_{a \in A} g_{a纸上}. We claim the function $g_a$ is continuous and never zero. Let $W \subseteq X$ be an open covering of $X$ as in the definition of locally finite; for $W \subseteq \mathcal{G}$, $X \cap U_a = \emptyset$ except for finitely many $a \in A$. Thus $g_W$ is the zero function except for the same exceptional $a$'s and $g_{W}$ is a finite sum of continuous functions. Hence $g_a$ is continuous. Any $x \in X$ is in some $U_a$; thus $g_a(x) > 0$ and $g(x) > g_a(x) > 0$.

Since $g_a$ is continuous and never zero, $1/g_a$ is also continuous. Define $p_a : X \to I$ to be $g_a g$. Then $p_a$ is a numeration of $U_a$. Clearly $\sum_{a \in A} p_a = 1$.

Proof of Lemma 2. Suppose we have a locally finite covering $\mathcal{G} = \{V_s : s \in \mathcal{A}^*\}$ of $(I, X)$ by numerable open sets such that for each $x \in (I, X)$, $\omega(x)|s|$. Then letting $W = r^{-1}(V_s)$, we obtain a covering $\mathcal{G} = \{W_s : s \in \mathcal{A}^*\}$ satisfying the conclusion of Lemma 2. Thus Lemma 2 is a consequence of the following lemma.

Lemma. Let $\mathcal{U} = \{U_a : a \in \mathcal{A}\}$ be a locally finite covering of the space $X$ by numerable open sets. Then there is a locally finite covering $\mathcal{G} = \{V_s : s \in \mathcal{A}^*\}$ of $(I, X)$ by numerable open sets such that for each $x \in V_s$, $\omega(x)|s|_n$.

Proof. Let $\{p_a : a \in \mathcal{A}\}$ be a partition of 1 numerating the elements of the collection $\mathcal{U}$. For $s = a_1 \ldots a_k \in \mathcal{A}^*$ and $x \in (I, X)$, define
\[ f_s(x) = \prod_{i=1}^n \inf \left\{ p_{a_i}(\omega_i(t)) : \frac{1}{n} - 1 \leq t \leq \frac{1}{n} \right\}. \]

The function $f_s : (I, X) \to I$ is continuous and numerates the open set
\[ E_{\omega(s)} = \{x \in (I, X) : \omega(x)|s| \} \]
Denote by $A_{\mathcal{Y}}$ the set of those words $s \in \mathcal{A}^*$ with $\# s < n$; and define $E_{\omega(s)} = \{x \in (I, X) : \omega(x)|s| \}$.

We claim the collection $E_{\omega(s)}$ is locally finite. For $x \in (I, X)$ we shall find an open set $U$ in $(I, X)$ which contains $x$ and intersects only finitely many elements of $E_{\omega(s)}$. For each $i \in I$ there is an open set $W_i$ in $X$ which contains $a(t)$ and intersects only finitely many elements of $\mathcal{U}$. By compactness of $X$, there is a finite set $\{t_1, \ldots, t_k\} \subseteq I$ such that $\omega(I)$ is contained in $W = W_{t_1} \cup \ldots \cup W_{t_k}$. Then open set $W$ intersects only finitely many elements of $\mathcal{U}$.

Denote by $A_{\mathcal{Y}}$ the finite set $\{a \in \mathcal{A} : W \cap U_a = \emptyset\}$. Then $\mathcal{A}^* \cap A_{\mathcal{Y}}$ is a finite set. Let $S$ be the subbasic open set $[I, W]$ in $(I, X)$. We claim that if $S \subseteq \mathcal{U} \subseteq \emptyset$ for $E_{\omega(s)} \subseteq \mathcal{Y}_{\alpha(s)}$, then $x \in \mathcal{A}^*$. Since $\# s < n$, it follows that $x$ is in the finite set $A_{\mathcal{Y}} \cap A_{\mathcal{Y}}$, thus $x$ is an open set in $(I, X)$ satisfying the above requirements. Suppose $x \in S \cap \mathcal{Y}_{\alpha(s)}$ with $x = a_1 \ldots a_k, i \leq n$. Since $(I, X)$ is $\mathcal{A}$-regular, each $a_j \in \mathcal{A}^*$; i.e.,
\[ s \in \mathcal{A}^* \] since $E_{\omega(s)}$ is locally finite, the function
\[ F_{s} = \sum_{s \in \mathcal{A}^*} f_s : (I, X) \to T \]
is continuous. For $x \in \mathcal{A}^*$ with $\# x = n$, define
\[ g_x = \sup(0, f_{s} - \# f_{s - 1} : (I, X) \to I, \]
and let $V_s = g_{s}((0, 1))$. We claim that $\mathcal{F} = \{V_s : s \in \mathcal{A}^*\}$ is a covering of $(I, X)$ as required. Since $g_s \leq f_s, V_s \subseteq E_{\omega(s)}$; i.e., for $x \in V_s, \omega(x)|s|$. To see that $\mathcal{F}$ covers $(I, X)$, for $x \in (I, X)$ let $k$ be the least integer for which there is a word $s = a_1 \ldots a_k$ with $f_s(x) > 0$. Then $F_{s - 1}(x) = 0$ and so $f_s(x) = f_s(x)$. Thus, $x \in V_s$.
Finally, to see that $\forall^*$ is locally finite, let $\omega \in (I, X)$. For some integers $n$ and $k$, $F_\omega(a) > 1/k$. Define

$$R = \{ \lambda \in (I, X) : F_\lambda(a) > 1/k \}.$$ 

The set $R$ is open in $(I, X)$ and contains $\omega$. We claim that for $m > \max(k, n)$ and $s \in A^*$ with $s = m$, $R \cap V_s = \emptyset$. This will suffice since we know already that the collection $E_{\max(n, m)}$ is locally finite. For $\lambda \in R$,

$$F_{m-1}(\lambda) > F_\lambda(a) = 1/k.$$ 

Thus

$$mF_{m-1}(\lambda) > m(1/k) > 1.$$ 

Since $f_\lambda(1) \leq 1$, it follows that $g_\lambda(1) = 0$. Thus $\lambda$ is not in $V_s$. $\blacksquare$

References


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Connections between different amoeba algebras

by

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Abstract. The "amoeba algebra" is the complete Boolean algebra which has the effect of making the union of all null Borel sets coded in the ground model have measure 0 in the corresponding Boolean extension. Six different versions of the amoeba algebra are studied, together with the localization algebra, and connections, in some cases isomorphism and in some cases forcing equivalence, are established between them.

§1. Introduction. A number of different versions of Martin and Solovay's original "amoeba" algebras have been considered. In their original application, the relevant set of conditions was taken to be the set of open subsets of the real line of measure less than $\aleph$ and partially ordered by inclusion, approximating to an open set of measure $\aleph$. In [8] we took instead a "variable" $\aleph$. That is, a condition was a pair $(p, z)$ where $p$ is an open subset of $R$ of measure less than $\aleph$, giving the information about the generic open set $X$ that $p \subseteq X$ and $\mu(X) < \aleph$. The main reason for this was to enable us to show that the amoeba set of conditions $P$ satisfies $RO(P) = RO(P \times P)$ where $RO(P)$ is the complete Boolean algebra associated with $P$ (the "regular open" algebra). Whether this is true for Martin and Solovay's "fixed measure" case we still do not know. And then there are the amoeba algebras on compact intervals $I$ (or equivalently on $2^\omega$) derived from the set of (relatively) open subsets of $I$ of measure less than $\aleph$, which were used by Shelah in [7], and also by Miller and others in their investigations into the connections between measure and category on the real line.

What all these algebras $B$ have in common is the following. In each case the Boolean value in $\forall^*$ of the statement

$$\mu\{x \in R : x \text{ is not random over } V\} = 0$$

is 1, where $\mu$ denotes Lebesgue measure. What ideally we would like to know is that this statement holds in an extension of $V$ if and only if the extension contains a $V$-generic filter

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