Topological spaces with a linear basis

by

Dominikus Noll (Stuttgart)

Abstract. We prove that a separable metrizable connected and locally connected space is homeomorphic with one of the spaces $\{0\}$, $[0,1]$, $[0,1)$, $(0,1)$ or S^1 provided that each of its open connected subsets has at most two boundary points. More generally, we introduce the notion of a "linear basis", a concept which provides an axiomatic description of the intuitive observation that an ordered space has a basis consisting of sets with "two ends", the open intervals. We prove that every connected space admitting a linear basis may in turn be described by means of order terms. As a consequence we obtain topological characterizations of connected orderable spaces as well as a topological characterization of the sphere S^1.

Introduction. A topological space E is called *orderable* if its topology is induced by a linear order $<$ on E. E is called *suborderable* if it is a subspace of an orderable space. Various authors have deduced topological characterizations of orderable and suborderable spaces. See for instance Moore ([E, p. 460]), Eilenberg ([Ei]), Michael ([M]), Kowalski ([Ko]), Herrlich ([He$_1$], [He$_2$]), Lutzer ([Lu]), v. Dalen–Wattel ([DW]), Purisch ([Pu]), v. Mill–Wattel ([MW]).

In [He$_1$], Herrlich has proved, extending a result of Moore's, that a connected and locally connected T_1-space E is orderable provided that every connected subset C of E has at most two noncut points. (x is called a *noncut point of* C if $C\setminus\{x\}$ is connected). The starting point of our present investigation is the following question, closely related with Herrlich's result. Let E be a connected and locally connected T_1-space and suppose that every connected subset (resp. every open connected subset) C of E has at most two boundary points. Must E be orderable? The answer is in the negative, of course, since this property is shared by the unit sphere S^1. We will prove, however, that this is in fact the only possible exception, i.e. every connected and locally connected T_1-space E whose open connected subsets have at most two boundary points is either orderable or it is a generalized sphere. (We agree that E is called a *generalized sphere* if it arises from an ordered continuum by identifying the first and the last point.)

Intuitively, an ordered space has a basis consisting of "sets with two ends", the open intervals. As long as the space under consideration is connected and locally connected, the phenomenon of "having two ends" may be described by the fact that every connected open set has at most two boundary points. In the case where E is not necessarily locally connected, one may try to describe this phenomenon in a more abstract way. This leads to the following.

DEFINITION 1. Let E be a topological space and let $\mathfrak{B}$ be a basis for E. $\mathfrak{B}$ is called *linear* provided that it satisfies the following conditions:

(L1) Whenever $B, B' \in \mathfrak{B}$, $B \cap B' \neq \emptyset$, then $B \cup B' \in \mathfrak{B}$;

(L2) If $B, B_1, B_2 \in \mathfrak{B}$ are given with $B \cap B_i \neq \emptyset$, $B_i \not\subset B$, $i = 1, 2$, and $B_1 \cap B_2 = \emptyset$, then every $\tilde{B} \in \mathfrak{B}$ with $B \cap \tilde{B} \neq \emptyset$, $\tilde{B} \not\subset B$ intersects B_1 or B_2 (or both).

The purpose of this paper is now to examine the structure of the spaces admitting a linear basis. It turns out that under the assumption of connectedness, these spaces are really "linear" in the sense that their topology may be described by means of order terms. In the nonconnected case, strange things may happen, i.e. the notion of a linear basis seems to be no longer appropriate in this case to describe "linearity". Before stating our main result, let us consider some examples.

(1) Let E be a suborderable space. Then there exists a linear order $<$ on E such that E has a basis consisting of (not necessarily open) intervals (see [Lu]). Let $\mathfrak{B}$ denote the basis consisting of all finite unions $I_1 \cup \ldots \cup I_n$ of such intervals having $I_j \cap I_{j+1} \neq \emptyset$. Clearly every $B \in \mathfrak{B}$ is an interval and so $\mathfrak{B}$ is a linear basis.

(2) Let E be a connected and locally connected T_1-space and let $\mathfrak{B}$ denote the basis consisting of all open connected subsets of E. Then $\mathfrak{B}$ is linear if and only if every $B \in \mathfrak{B}$ has at most two boundary points. To see this, it is sufficient to observe that for $B, B_1 \in \mathfrak{B}$, $B \cap B_1 \neq \emptyset$, $B_1 \not\subset B$, B_1 must contain a boundary point of B.

(3) Clearly the open "intervals" on the sphere S^1 constitute a linear basis for S^1.

(4) Let E be metrizable and strongly zero-dimensional ($\dim(E) = 0$). There exists a sequence $(\mathfrak{B}_n)_{n=1}^{\infty}$ of disjoint open coverings of E such that $\mathfrak{B}_{n+1}$ refines $\mathfrak{B}_n$ and $\mathfrak{B} = \bigcup \{\mathfrak{B}_n : n \in N\}$ is a basis for E. Clearly $\mathfrak{B}$ is linear since for $B, B' \in \mathfrak{B}$, $B \cap B' \neq \emptyset$ we must have $B \subset B'$ or $B' \subset B$.

1. The main result. In this section we obtain a characterization of the connected T_2-spaces admitting a linear basis.

THEOREM 1. *Let E be a connected T_2-space with a linear basis $\mathfrak{B}$. Then E is either orderable or a generalized sphere.*

The proof of this result will be divided into a sequence of lemmata.

LEMMA 1. *E is regular.*

Proof. Let $x \in E$ be fixed. Let $\mathfrak{F}$ be the filter of neighbourhoods of x. We have to prove that $\overline{\mathfrak{F}} = \{\overline{U}: U \in \mathfrak{F}\}$ converges to x. Assume the contrary and choose a neighbourhood $B \in \mathfrak{B}$ of x having $\overline{U} \not\subset B$ for all $U \in \mathfrak{F}$. Let $\mathfrak{F}_0$ be the set of $U \in \mathfrak{F}$ having $U \subset B$. For $U \in \mathfrak{F}_0$ let x_U be chosen such that $x_U \in \overline{U}$, $x_U \notin B$. Clearly, every x_U is a boundary point of B. But note that B has at most two boundary points as a consequence of the fact that $\mathfrak{B}$ is linear. This proves that the net $(x_U : U \in \mathfrak{F}_0)$ is eventually constant. This, however, contradicts the fact that E is a Hausdorff space. ∎

For the remainder of the proof we will need some preparations. We start with two definitions. Let $\mathfrak{V}$ be an open cover of E, $\mathfrak{V} \subset \mathfrak{B}$. A sequence $(V_1, \ldots, V_n)$ of elements of $\mathfrak{V}$ is called a chain in $\mathfrak{V}$ if $V_i \cap V_j \neq \emptyset$ is satisfied if and only if $|i-j| \leqslant 1$. n is called the *length* of the chain. If $x \in V_1$, $y \in V_n$, then $(V_1, \ldots, V_n)$ is called a *chain from x to y*. A sequence $(V_1, \ldots, V_n)$ of elements of $\mathfrak{V}$ is called a *cycle* in $\mathfrak{V}$ if $n \geqslant 4$ and $(V_1, \ldots, V_{n-1})$, $(V_2, \ldots, V_n)$ are chains in $\mathfrak{V}$ and, moreover, $V_1 \cap V_n \neq \emptyset$. Clearly, if $(V_1, \ldots, V_n)$ is a cycle, then so is $(V_i, \ldots, V_n, V_1, \ldots, V_{i-1})$ for every i. If a cycle exists in $\mathfrak{V}$, then $\mathfrak{V}$ is called a *cyclic cover*, otherwise $\mathfrak{V}$ is called *acyclic*.

The following two possibilities may arise for the linear basis $\mathfrak{B}$ on E:

(α) There exists a cover $\mathfrak{V}_0$ of E, $\mathfrak{V}_0 \subset \mathfrak{B}$, such that every cover $\mathfrak{W}$ of E refining $\mathfrak{V}_0$ and having $\mathfrak{W} \subset \mathfrak{B}$ is acyclic.

(β) For every cover $\mathfrak{V}$ of E, $\mathfrak{V} \subset \mathfrak{B}$, there exists a cover $\mathfrak{W}$ of E refining $\mathfrak{V}$ and having $\mathfrak{W} \subset \mathfrak{B}$ such that $\mathfrak{W}$ is cyclic.

We prove that in case (α) the space E is orderable while in case (β) it is a generalized sphere. We start with the treatment of case (α). Let $\mathfrak{B}_0$ be the base of all $B \in \mathfrak{B}$ which are contained in some element of $\mathfrak{V}_0$. So $\mathfrak{B}_0$ contains no cycles in view of (α). Since the case $|E| = 1$ is trivial, we may assume that there exist two points $a, b \in E$, $a \neq b$. Now let U_a, U_b be open neighbourhoods of a, b resp. such that $\overline{U}_a \cap \overline{U}_b = \emptyset$ (Lemma 1). Let $\mathfrak{V}_1$ be the cover of all $B \in \mathfrak{B}_0$ with the property that $a \in B$ implies $B \subset U_a$, $b \in B$ implies $B \subset U_b$ such that $B \in \mathfrak{V}_1$ does not intersect both U_a and U_b. Consequently, any chain in $\mathfrak{V}_1$ connecting a and b must have length $\geqslant 4$, and of course the same is true in any cover $\mathfrak{V}$ refining $\mathfrak{V}_1$.

LEMMA 2. *Let $\mathfrak{V}$ be any cover of E having $\mathfrak{V} \subset \mathfrak{B}$ and refining $\mathfrak{V}_1$. Let $x \in E$ and let $\varkappa_{\mathfrak{V}}(a, b) = (V_1, \ldots, V_n)$ be a shortest chain in $\mathfrak{V}$ connecting a and b, $\varkappa_{\mathfrak{V}}(a, x) = (W_1, \ldots, W_m)$ a shortest chain in $\mathfrak{V}$ connecting a and x and let $\varkappa_{\mathfrak{V}}(b, x) = (U_1, \ldots, U_k)$ be a shortest chain in $\mathfrak{V}$ connecting b and x. Suppose we have $m, k \geqslant 3$. Then precisely one of the following statements is true:*

$$(1)\ a \in \bigcup_{i=1}^{k} U_i, \quad (2)\ b \in \bigcup_{i=1}^{m} W_i, \quad (3)\ x \in \bigcup_{i=1}^{n} V_i.$$

Proof. We first prove that one of the following statements

$$(1')\ a \in \bigcup_{i=1}^{k} \operatorname{st}(U_i, \mathfrak{V}), \quad (2')\ b \in \bigcup_{i=1}^{m} \operatorname{st}(W_i, \mathfrak{V}), \quad (3')\ x \in \bigcup_{i=1}^{n} \operatorname{st}(V_i, \mathfrak{V})$$

is satisfied. Assume in the contrary that none of the statements (1′)–(3′) is true. Then we have $V_n \cap \bigcup_{i=1}^{m} W_i = \emptyset$. Let i denote the first index having $V_i \cap \bigcup_{j=1}^{m} W_j = \emptyset$. We claim that $V_s \cap \bigcup_{j=1}^{m} W_j = \emptyset$ holds for all indices s having $i \leqslant s \leqslant n$, for otherwise we might construct a cycle in $\mathfrak{B}$ using $V_{i-1}, V_i, \ldots, V_s$ and appropriate elements of $\varkappa_{\mathfrak{B}}(a, x)$ connecting V_{i-1} and V_s.

Let j be the largest index having $V_{i-1} \cap W_j \neq \emptyset$. Then

$$\lambda = (V_n, \ldots, V_i, V_{i-1}, W_j, \ldots, W_m)$$

is a chain in $\mathfrak{B}$ from b to x. Since $\mathfrak{B}$ has no cycles, we deduce that there exists a first index s having $U_s \cap V_{i-1} \neq \emptyset$. We claim that $V_{i-2} \cap \bigcup_{r=1}^{k} U_r \neq \emptyset$. Assume the contrary. Let t denote the largest index satisfying $U_t \cap V_{i-1} \neq \emptyset$. Obviously, $t < k$. Now define the set $B = U_s \cup \ldots \cup U_t \cup V_{i-1} \in \mathfrak{B}$. We obtain three mutually disjoint sets $U_{s-1}, U_{t+1}, V_{i-2}$ of $\mathfrak{B}$ which all intersect B but are not contained in B, contradicting the linearity of $\mathfrak{B}$. Therefore V_{i-2} must intersect some U_r. Repeating this argument with V_{i-1} replaced by V_{i-2} finally proves that V_1 must intersect some U_r. This contradicts our assumption and proves that one of the statements (1')–(3') is true.

Suppose now statement (3') above is satisfied. We prove that in fact (3) must be true. Let i be the smallest index having $x \in \mathrm{st}(V_i, \mathfrak{B})$ and choose $U \in \mathfrak{B}_1$ having $x \in U$, $U \cap V_i \neq \emptyset$. Since $(V_1, \ldots, V_i, U)$ is a chain in $\mathfrak{B}$ from a to x it must have length $\geqslant 3$. This implies $i > 1$. Let j denote the largest index having $U \cap V_j \neq \emptyset$, then $j < n$ since $(U, V_j, \ldots, V_n)$ is a chain in $\mathfrak{B}$ from x to b and hence has length $\geqslant 3$ by assumption. Now let $B = V_i \cup \ldots \cup V_j \in \mathfrak{B}$, then V_{i-1}, V_{j+1} and U are mutually disjoint sets in $\mathfrak{B}$ which intersect B but are not contained in B, a contradiction. This proves (3). ■

Let $\mathfrak{B}$ be a cover of E refining $\mathfrak{B}_1$ and having $\mathfrak{B} \subset \mathfrak{B}$. We shall say that two points $x, y \in E$ are separated by $\mathfrak{B}$ if a shortest chain in $\mathfrak{B}$ connecting x and y has length $\geqslant 3$. Clearly since E is Hausdorff, it is possible to find a cover $\mathfrak{B}$ of this type separating x, y, whenever $x \neq y$.

Let $\mathfrak{B}$ be a cover of E refining $\mathfrak{B}_1$ and having $\mathfrak{B} \subset \mathfrak{B}$. To every x in E which is separated from a and b by $\mathfrak{B}$ we assign an integer $\varphi(\mathfrak{B}, x)$. Let $\varkappa_{\mathfrak{B}}(a, b)$, $\varkappa_{\mathfrak{B}}(a, x)$, $\varkappa_{\mathfrak{B}}(b, x)$ be shortest chains in $\mathfrak{B}$ connecting a with b, a with x, b with x respectively. By Lemma 2, precisely one of the following constellations occurs:

(1) x is contained in an element of $\varkappa_{\mathfrak{B}}(a, b)$,
(2) b is contained in an element of $\varkappa_{\mathfrak{B}}(a, x)$,
(3) a is contained in an element of $\varkappa_{\mathfrak{B}}(b, x)$.

If (1) or (2) hold, we define $\varphi(\mathfrak{B}, x) = |\varkappa_{\mathfrak{B}}(a, x)|$, in case (3) we define $\varphi(\mathfrak{B}, x) = -|\varkappa_{\mathfrak{B}}(a, x)|$. Here $|\lambda|$ denotes the length of the chain λ. We may in addition define $\varphi(\mathfrak{B}, a) = 0$ and $\varphi(\mathfrak{B}, b) = |\varkappa_{\mathfrak{B}}(a, b)|$. Clearly our intention is to define a linear order on E by means of the rank functions $\varphi(\mathfrak{B}, \ )$. This requires some sort of compatibility of the functions $\varphi(\mathfrak{B}, \ )$. This will be established by the next two lemmata.

Lemma 3. *Let x, y, z be different points in E and let $\mathfrak{B}$ be a cover of E having $\mathfrak{B} \subset \mathfrak{B}_0$. Suppose x, y, z are separated from each other by $\mathfrak{B}$. Let $\mathfrak{W}$ be a cover of E refining $\mathfrak{B}$ and having $\mathfrak{W} \subset \mathfrak{B}$. Let $\varkappa = (V_1, \ldots, V_n)$, $\lambda = (W_1, \ldots, W_m)$ be shortest chains in $\mathfrak{B}$ resp. $\mathfrak{W}$ joining x and z and suppose y is contained in some element V_i of $\varkappa$. Then y is as well contained in some element W_j of λ.*

Proof. First observe that the set V_i must intersect some of the W_j, $1 \leqslant j \leqslant m$. Indeed, otherwise we might construct a cycle within $\mathfrak{B}_0$, since both the chains $\varkappa$ and λ join x and z.

Now let $j(1)$ be the smallest index j having $V_i \cap W_j \neq \emptyset$ and let $j(2)$ denote the largest index with this property. Observe that $j(1) > 1$. Indeed, otherwise V_i would intersect W_1. Choosing $V \in \mathfrak{B}$ such that $W_1 \subset V$ now provides a chain (V, V_i) of length 2 in $\mathfrak{B}$ from x to y, a contradiction. Using the same argument one finds that $j(2) < n$.

Let $B := W_{j(1)} \cup \ldots \cup W_{j(2)} \in \mathfrak{B}$. Consider the sets $W_{j(1)-1}, W_{j(2)+1}, V_i$. By the definition of $j(1)$, $j(2)$, these are mutually disjoint. Moreover, $W_{j(1)-1}, W_{j(2)+1}$ are not contained in but intersect B. Since V_i by construction, also intersects B, we conclude using (L2) that V_i must be contained in B. This proves the result since $y \in V_i$. ■

With the aid of Lemma 3 we are now able to establish the compatibility of the rank functions $\varphi(\mathfrak{B}, x)$.

Lemma 4. *Let $\mathfrak{B}$ be a cover of E refining $\mathfrak{B}_1$ having $\mathfrak{B} \subset \mathfrak{B}$ and let $\mathfrak{W}$ be a cover of E refining $\mathfrak{B}$ having $\mathfrak{W} \subset \mathfrak{B}$. Let $x, y \in E$ be given such that a, b, x, y are separated from each other by $\mathfrak{B}$ and suppose $\varphi(\mathfrak{B}, x) < \varphi(\mathfrak{B}, y)$ is satisfied. Then $\varphi(\mathfrak{W}, x) < \varphi(\mathfrak{W}, y)$ holds as well.*

Proof. There are six different constellations from which the inequality $\varphi(\mathfrak{B}, x) < \varphi(\mathfrak{B}, y)$ may arise. Let us exemplary assume that b is contained in an element of $\varkappa_{\mathfrak{B}}(a, x)$ and in an element of $\varkappa_{\mathfrak{B}}(a, y)$ and that the length of $\varkappa_{\mathfrak{B}}(a, y)$ exceeds the length of $\varkappa_{\mathfrak{B}}(a, x)$. But both these chains connect a and b, hence every V in $\varkappa_{\mathfrak{B}}(a, x)$ intersects some V' in $\varkappa_{\mathfrak{B}}(a, y)$. The argument used in the proof of Lemma 2 now implies that x is actually contained in some element of $\varkappa_{\mathfrak{B}}(a, y)$. Using Lemma 3, we see that the situation is precisely the same for the corresponding chains in $\mathfrak{W}$, i.e. x, y are both contained in certain elements of $\varkappa_{\mathfrak{W}}(a, b)$ and x is contained in an element of $\varkappa_{\mathfrak{W}}(a, y)$. Since x, y, a, b are as well separated by $\mathfrak{W}$, this gives $\varphi(\mathfrak{W}, x) < \varphi(\mathfrak{W}, y)$ in our special situation. Since the remaining cases may be treated analogously, this proves the lemma. ■

Let us now define a linear order $<$ on E. Let $x < y$ be satisfied if and only if there exists a cover $\mathfrak{B}$ of E refining $\mathfrak{B}_1$ and having $\mathfrak{B} \subset \mathfrak{B}$ such that x, y are separated by $\mathfrak{B}$ and $\varphi(\mathfrak{B}, x) < \varphi(\mathfrak{B}, y)$ is satisfied. $<$ is actually a linear ordering on E. Indeed, suppose we had $\varphi(\mathfrak{B}, x) < \varphi(\mathfrak{B}, y)$ and $\varphi(\mathfrak{W}, x) > \varphi(\mathfrak{W}, y)$ for certain covers $\mathfrak{B}$, $\mathfrak{W}$. Choosing a common refinement $\mathfrak{U}$ of $\mathfrak{B}$ and $\mathfrak{W}$, by Lemma 4, implies $\varphi(\mathfrak{U}, x) < \varphi(\mathfrak{U}, y) < \varphi(\mathfrak{U}, x)$, which is absurd.

It remains to prove that the order topology arising from $<$ coincides with the original topology on E.

LEMMA 5. *The original topology on E is finer than the order topology.*

Proof. Let $x, y, z \in E$ be given such that $x<y<z$. Since E is regular, there exists a cover $\mathfrak{B}$ of E such that $\varphi(\mathfrak{B}, x) < \varphi(\mathfrak{B}, y)$ and every chain in $\mathfrak{B}$ from x to y has length $\geqslant 4$. Now having regard of the six possible constellations from which $\varphi(\mathfrak{B}, x) < \varphi(\mathfrak{B}, y)$ may arise, we can find a chain $(V_1, \ldots, V_n)$ in $\mathfrak{B}$ from x to y which "increases in positive direction". Since $n \geqslant 4$, we see that V_n is contained in $(x, \rightarrow)$. Indeed, for every $v \in V_n$ a shortest chain in $\mathfrak{B}$ from x to v must have length $\geqslant 3$. This yields $\varphi(\mathfrak{B}, x) < \varphi(\mathfrak{B}, v)$. Consequently, $y \in V_n \subset (x, \rightarrow)$. Using a similar argument, we find an open set W satisfying $y \in W \subset (\leftarrow, z)$. This proves the lemma. ■

LEMMA 6. *The order topology is finer than the original topology.*

Proof. Let $B \in \mathfrak{B}_0$ be contained in some element of $\mathfrak{B}_1$ and let $x \in B$. Let $z \in B \setminus \{x\}$, $z < x$, say. We prove that $[z, x]$ is contained in $\bar{B}$. Assume the contrary. Then there exists $y \notin \bar{B}$, $z<y<x$. Choose $V \in \mathfrak{B}_0$ having $y \in V$ and $V \cap B = \emptyset$. Now there exists a chain $(B_1, \ldots, B_n)$ in $\mathfrak{B}_0$ such that $z \in B_1$, $x \in B_n$, $B_i \subset V$ for some i. But this gives rise to a cycle in $\mathfrak{B}_0$ since $B_1 \cap B \neq \emptyset$, $B_n \cap B \neq \emptyset$ and $B_i \cap B = \emptyset$, a contradiction.

Using the same argument, we prove that for $z' \in B \setminus \{x\}$, $x<z'$, the interval $[x, z']$ is contained in $\bar{B}$. Suppose now that x is either the first or the last point of E. Then, in view of the fact that E is regular with respect to the original topology, the situation is sufficient. Suppose on the other hand that x is an internal point of E. We have to prove that there exist $z, z' \in B$ such that $z<x<z'$ holds. Suppose that for all $z \in B \setminus \{x\}$ we had $z<x$. This means that x is not an accumulation point of $(x, \rightarrow)$ with respect to the original topology. But note that $\overline{(x, \rightarrow)} = [x, \rightarrow)$ with respect to the order topology, hence in view of Lemma 5, $(x, \rightarrow)$ is closed in the original topology. On the other hand, $(\leftarrow, x]$ is closed in E with respect to the order topology and hence with respect to the original topology, too. This provides a contradiction with the fact that E is connected. ■

Lemma 6 ends the proof of case (α). We will now proceed towards a proof of case (β).

LEMMA 7. *Let $(V_1, \ldots, V_n)$ be a cycle in $\mathfrak{B}$. Then $E = \bigcup_{i=1}^{n} \bar{V}_i$.*

Proof. In view of the axiom (L1) for $\mathfrak{B}$ we may restrict ourselves to the case $n = 4$. Assume $x \notin \bar{V}_i$, $i = 1, \ldots, 4$ for some $x \in E$. Let $\mathfrak{B}'$ denote the set of all $B \in \mathfrak{B}$ such that $x \in B$ implies $B \cap V_i = \emptyset$, $i = 1, \ldots, 4$ and such that B does not contain any of the V_i, $i = 1, \ldots, 4$. Clearly $\mathfrak{B}'$ is a basis for E. E being connected, there exists a shortest chain $(B_1, \ldots, B_k)$ in $\mathfrak{B}'$ joining x and some fixed $y \in V_1$. Let n be the first index having $B_n \cap V_i \neq \emptyset$ for some i. By the definition of $\mathfrak{B}'$ we have $n > 1$. Suppose B_n intersects precisely one of the V_i, say V_1. Then V_2, V_4 and B_{n-1} are mutually disjoint elements of $\mathfrak{B}$ which are not contained in but intersect $B_n \cup V_1 \in \mathfrak{B}$, a contradiction. So B_n intersects at least two of the V_i.

Suppose B_n intersects V_1 and V_3. Then V_1, V_3 and B_{n-1} are mutually disjoint elements of $\mathfrak{B}$ which intersect B_n but are not contained in B_n, a contradiction. Hence B_n must intersect V_1, V_2. But note that in this case V_2, V_4 and B_{n-1} are mutually disjoint sets not contained in but intersecting $B_n \cup V_1$, a contradiction once more.

Suppose B_n intersects three of the V_i, V_1, V_2, V_3, say. Then V_1, V_3 and B_{n-1} are mutually disjoint sets not contained in but intersecting B_n, a contradiction. This proves the lemma. ■

As a consequence of Lemma 7 we derive that in case (β), E must be a compact space. Indeed, given any open cover $\mathfrak{U}$ of E, there exists an open cover $\mathfrak{B}$ of E, refining $\mathfrak{U}$, having $\mathfrak{B} \subset \mathfrak{B}$ such that $\mathfrak{B}$ has a cycle. $\mathfrak{B}$ being a linear basis, we deduce that every $V \in \mathfrak{B}$ has at most two boundary points. Hence by Lemma 7, $\mathfrak{B}$ has a finite subcover, and consequently so has $\mathfrak{U}$.

LEMMA 8. *For fixed $x_0 \in E$ the subspace $E \setminus \{x_0\}$ is connected.*

Proof. Let $x, y \in E \setminus \{x_0\}$, $x \neq y$ and let $\mathfrak{B} \subset \mathfrak{B}$ be a cover of $E \setminus \{x_0\}$ such that $V \subset E \setminus \{x_0\}$ for every $V \in \mathfrak{B}$. We have to establish the existence of a chain in $\mathfrak{B}$ connecting x and y. Let $\mathfrak{W}$ denote the set of all $B \in \mathfrak{B}$ for which either $x_0 \notin \bar{B}$ and B is contained in some $V \in \mathfrak{B}$, or $x_0 \in B$ and $x, y \notin \bar{B}$. Clearly $\mathfrak{W}$ is a base for E and therefore has a cycle $(B_1, \ldots, B_k)$ by (β). By Lemma 7 we have $x_0 \in \bar{B}_1$, say. By the definition of $\mathfrak{W}$ this implies $x, y \in \bar{B}_2 \cup \ldots \cup \bar{B}_k$, $x_0 \notin \bar{B}_i$, $i = 2, \ldots, k$. Choosing $B_x, B_y \in \mathfrak{W}$, $x \in B_x$, $y \in B_y$, $B_x \cap B_i \neq \emptyset$ for some i, $B_y \cap B_j \neq \emptyset$ for some j, $2 \leqslant i$, $j \leqslant k$, both B_x, B_y contained in certain elements of $\mathfrak{B}$, provides a chain in $\mathfrak{W}$ connecting x, y within $E \setminus \{x_0\}$. This chain may be used to obtain the desired chain in $\mathfrak{B}$ from x to y. ■

Our intention is to prove that $E \setminus \{x_0\}$ is orderable. So let $\mathfrak{B}_0$ denote the linear basis for $E \setminus \{x_0\}$ consisting of all $B \in \mathfrak{B}$ having $x_0 \notin \bar{B}$. We claim that (α) is true for $\mathfrak{B}_0$.

LEMMA 9. *$\mathfrak{B}_0$ has no cycles.*

Proof. Indeed, every cycle $(B_1, \ldots, B_k)$ in $\mathfrak{B}_0$ is as well a cycle in $\mathfrak{B}$, hence by Lemma 7 we have $x_0 \in \bar{B}_i$ for some i, which is absurd. ■

From part (α) of the proof of theorem 1 we deduce that $E \setminus \{x_0\}$ is orderable. Since it is not compact, we either have $E \setminus \{x_0\} \cong [a, b)$ or $E \setminus \{x_0\} \cong (a, b)$. But note that E is the one-point compactification of $E \setminus \{x_0\}$ while $[a, b]$ is the one-point compactification of $[a, b)$, so $E \setminus \{x_0\} \cong [a, b)$ would imply $E \cong [a, b]$, a contradiction with Lemma 8 since for $a<c<b$, $[a, b] \setminus \{c\}$ is not connected. So we deduce $E \setminus \{x_0\} \cong (a, b)$. But now it is clear that E is the quotient of $[a, b]$ arising from identifying a with b, since this space is the one-point compactification of (a, b). This completes the proof of Theorem 1.

2. Consequences. In this section we state and prove several consequences of the main result and finally consider several examples.

COROLLARY 1. *A connected T_2-space is orderable if and only if it has an acyclic linear basis.* ■

COROLLARY 2. *Let E be a separable metrizable connected space with a linear basis. Then E is homeomorphic with any one of the spaces $\{0\}$, $[0, 1]$, $[0, 1)$, $(0, 1)$ or S^1.* ■

COROLLARY 3. *Let E be a connected and locally connected T_1-space such that every open connected subset of E has at most two boundary points. Then E is either orderable or a generalized sphere.*

Proof. Let $\mathfrak{B}$ denote the basis consisting of all open connected subsets of E. Clearly $\mathfrak{B}$ is linear. It remains to prove that E is a Hausdorff space.

Let $x, x' \in E$, $x \neq x'$. Using T_1 choose $V, V' \in \mathfrak{B}$, $x \in V$, $x' \in V'$ such that $x' \notin V$, $x \notin V'$. Assume that for all $U, U' \in \mathfrak{B}$, $x \in U$, $x' \in U'$ we had $x \in \overline{U'}$, $x' \in \overline{U}$. Then $|\partial(V \cap V')| \leqslant 4$.

We claim that there is precisely one component G of $V \cap V'$ having $x \in \overline{G}$. Indeed, suppose we had two components G_1, G_2 of this type. Choose $U \in \mathfrak{B}$, $x \in U$, $G_i \not\subset U$, $i = 1, 2$. Then G_1, G_2 contain a boundary point of U. In view of the fact that $x' \in \partial U$, this gives $G_1 \cap G_2 \neq \emptyset$.

Let G be the component of $V \cap V'$ having $x \in \overline{G}$. Choose $U \in \mathfrak{B}$, $x \in U \subset V$, $G \not\subset U$. Let H be a component of $U \cap V'$ having $x \in \overline{H}$. Clearly we have $H \subset G$. But note that H is closed and open in G, hence $H = G$, a contradiction. This proves the result. ■

THEOREM 2 (Herrlich [He_1]). *Let E be a connected and locally connected T_1-space such that every connected subset of E has at most two noncut points. Then E is orderable.*

Proof. We claim that every connected open subset C of E has at most two boundary points. Indeed, since $\overline{C}$ is connected, it has at most two noncut points. Assume that C has three boundary points. So one of the boundary points, say x, must be a cut-point of $\overline{C}$, i.e. $\overline{C} \setminus \{x\}$ is not connected. But this is absurd in view of the fact that every B having $C \subset B \subset \overline{C}$ must be connected. This proves that E has a linear basis.

In view of Corollary 3, E must be orderable or a generalized sphere. Clearly the latter is impossible by our assumption. Hence E is in fact orderable. ■

Remarks. (1) Although T_2 may be replaced by T_1 in Corollary 3 and Theorem 2, this is not possible in Theorem 1. Indeed, let E be an infinite set with the cofinite topology. Clearly E has a linear basis but the statement of Theorem 1 is not true for E.

(2) A linear basis on a space E may contain cycles even when the space E is orderable. Take for instance $E = S^1 \setminus \{1\}$ and let $\mathfrak{B}$ denote the linear basis consisting of all sets $I \cap E$, where I varies over the open intervals of S^1.

The following result of Kowalsky [Ko] may be derived from the result of Herrlich (see [He_1]). We may as well obtain it as a consequence of our main theorem.

COROLLARY 4 (Kowalsky [Ko]). *Let E be a connected topological space. E is orderable if and only if it is a locally connected T_1-space such that given any three proper connected subsets of E we can always find two among which do not cover E.*

Proof. We have to prove the sufficiency of the condition. We prove that every open connected subset V of E has at most two boundary points. Assume that some open connected $V \subset E$ has three boundary points x_1, x_2, x_3. Let K_i be the component of $E \setminus \{x_i\}$ having $V \subset K_i$. We claim that for $i \neq j$, $K_i \cup K_j = E$, which provides a contradiction, since every K_i is a proper connected subset of E. Clearly $K_i \cup K_j$ is open in E since K_i is open in $E \setminus \{x_i\}$, $i = 1, 2, 3$. We claim that $K_i \cup K_j$ is as well closed in E. Indeed, K_i being closed in $E \setminus \{x_i\}$, there exist closed sets C_i in E such that $K_i = C_i \setminus \{x_i\}$. But note that $x_i \in K_j$, $x_j \in K_i$, hence we have $K_i \cup K_j = C_i \cup C_j$. Since E is connected, we deduce $K_i \cup K_j = E$.

By Corollary 3, E is either orderable or a generalized sphere. But clearly the latter is impossible in view of the assumption. This proves the result. ■

In [DW] van Dalen and Wattel have obtained a characterization of the suborderable spaces in terms of a subbase. Their result is valid without any assumptions on connectivity. In the connected case we may derive their result from our present investigation.

COROLLARY 5 ([DW Cor. 2.3]). *A connected T_1-space E is orderable if and only if it has a subbase $\mathfrak{C}$ which admits a representation $\mathfrak{C} = \mathfrak{L} \cup \mathfrak{R}$ and $\mathfrak{L}$, $\mathfrak{R}$ are linearly ordered with respect to inclusion.*

Proof. We have to prove the sufficiency of the condition. So let $\mathfrak{B}$ denote the basis for E consisting of all sets $B = L \cap R$, $L \in \mathfrak{L}$, $R \in \mathfrak{R}$. We prove that $\mathfrak{B}$ is linear and has no cycles.

Let $B_1, B_2, B_3 \in \mathfrak{B}$ be fixed sets such that (B_1, B_2, B_3) is a chain and $B_1 \not\subset B_2$, $B_3 \not\subset B_2$. Suppose we have $B_i = L_i \cap R_i$, $i = 1, 2, 3$. Let $L_1 \subset L_2$. We claim that this implies $L_2 \subset L_3$ and $R_1 \supset R_2 \supset R_3$. Indeed, $R_1 \supset R_2$ follows from the fact that $R_1 \subset R_2$ would imply $B_1 \subset B_2$. Suppose we had $L_3 \subset L_1$. This gives $R_3 \cap L_3 = B_3 \subset R_3 \cap L_1 = \emptyset$ (in view of $B_1 \cap B_3 = \emptyset$), a contradiction since $B_3 \neq \emptyset$. So we have $L_1 \subset L_3$. We claim that $R_3 \subset R_2$. Suppose we had $R_2 \subset R_3$. Then $R_2 \cap L_1 \subset R_3 \cap L_1 = \emptyset$ would imply $B_1 \cap B_2 = \emptyset$, which is absurd. Finally, assume we had $L_3 \subset L_2$. This gives $L_3 \cap R_3 \subset L_2 \cap R_3$, hence $B_3 \subset B_2$, a contradiction. This proves the claim.

The above observation may be used to prove that $\mathfrak{B}$ is linear. Clearly (L1) is satisfied. We check (L2). Let $B, B_1, B_2, B_3 \in \mathfrak{B}$ be fixed, $B_i = L_i \cap R_i$, $B = L \cap R$. Suppose we have $B \cap B_i \neq \emptyset$, $B_i \not\subset B$, $B_1 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_3 = \emptyset$. Now the chains (B_1, B, B_2), (B_1, B, B_3) and (B_2, B, B_3) fulfil the requirements of the above situation. Suppose we have $L_1 \subset L$. This implies $L \subset L_2$, $R_1 \supset R \supset R_2$ and $L \subset L_3$, $R_1 \supset R \supset R_3$. Applying this to the last triplet yields $L_3 \subset L \subset L_2$, $R_3 \supset R \supset R_2$, hence $R_1 = R_2 = R_3 = R$, $L_1 = L_2 = L_3 = L$. This is impossible.

Using the same argument, one may prove that $\mathfrak{B}$ has no cycles. As a consequence of Corollary 1, we derive that E is orderable. ∎

We conclude our paper with two examples indicating that the notion of a linear basis is no longer appropriate to describe orderability resp. suborderability in the nonconnected setting.

EXAMPLES. (1) Let $E = N \times N \cup \{\infty\}$. For $n, m \in N$ let $\{(n, m)\}$ be an open set and for $n, m_i, i \in N$ let $U(n, (m_i)) = \bigcup_{j \geqslant n} \{j\} \times [m_j, \rightarrow) \cup \{\infty\}$ be open in E. These sets clearly constitute a linear basis for E without any cycles. However, E is not even suborderable. Indeed, if E would be suborderable, there would exist a well--ordered net $((n_\alpha, m_\alpha)\colon \alpha < \varkappa)$ in $N \times N$ converging to ∞ (see [He₃]). But then $\varkappa$ had to be countable, a contradiction since no sequence in $N \times N$ converges to ∞.

(2) We construct a compact subset of R^2 which has a linear basis without cycles and nevertheless is not suborderable. Let $E = E_1 \cup E_2$, where $E_1 = [-1, 1] \times \{0\}$, $E_2 = \{0\} \times \left\{\frac{1}{n} : n \in N\right\}$. A linear basis $\mathfrak{B}$ for E is obtained by choosing $\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_3$, where $\mathfrak{B}_1$ consists of the sets $I \times \{0\}$, I an open interval in $[-1, 1]$ not containing 0, $\mathfrak{B}_2$ the family of all sets $B(i, I)$, where $i \in N$ and I is an open interval in $[-1, 1]$ containing 0 and where

$$B(i, I) = (I \times \{0\}) \cup \left(\{0\} \times \left\{\frac{1}{n} \colon n \geqslant i\right\}\right),$$

and where finally $\mathfrak{B}_3$ consists of all singletons $\left\{\left(0, \frac{1}{n}\right)\right\}$, $n \in N$.

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UNIVERSITÄT STUTTGART
MATHEMATISCHES INSTITUT B
Pfaffenwaldring
7000 Stuttgart 80
BRD

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