A resolution of the square of a determinantal ideal associated to a symmetric matrix

by

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Abstract. In this paper we construct a free resolution of the square of the ideal of submaximal minors of a generic symmetric matrix. We use methods of the theory of Schur functors.

1. Introduction. Let $R = K[x_{ij}]_{1\leqslant i\leqslant j\leqslant n}$ be a ring of polynomials in $n(n+1)/2$ indeterminats x_{ij} over a field K of characteristic zero and let $X = (x_{ij})$ denote the $n\times n$ symmetric matrix where we put $x_{ij} = x_{ji}$ for $i>j$.

The *determinantal ideal* $I_p(X)$ is the ideal in R which is generated by all $p\times p$-minors of X.

These ideals appear in the classical invariant theory [W], [K]. Kutz in [K] proved that depth $I_p(X) = (n-p+1)(n-p+2)/2$ and the ideal $I_p(X)$ is perfect, i.e. depth $I_p(X) = \mathrm{pd}_R R/I_p(X)$. This means that $R/I_p(X)$ is Cohen–Macaulay. The minimal free resolutions of $R/I_p(X)$ over R were described in [J], [L], [J–P–W] in terms of Schur functors. In this paper we construct a minimal free resolution of $R/I^2_{n-1}(X)$ over R. The ideal $I^2_{n-1}(X)$ is no longer perfect, its depth is 3 and the length of a resolution is 6.

2. Preliminaries. In the proof of the acyclicity of our complex we use the following lemma.

LEMMA I (see [P–S]). *Let L be a complex of length d whose components are free R-modules. If for any prime ideal P of R such that depth $PR_P < d$ the complex $L\otimes R_P$ is acyclic then so is the complex L.*

LEMMA II (see [J]). *Let T be a commutative ring with a unity and let $Y = (t_{ij})$ be an $n\times n$ symmetric matrix with entries in T. Moreover, let $I_p(Y)$ denote the ideal generated by $p\times p$ minors. If $I_{n-j}(Y) = R$, then there exist an invertible matrix C over T, invertible elements $z_1, z_2 \dots z_{n-j}$ and a $j\times j$ symmetric matrix $\bar{Y}$ such that*

$$C^t YC = \left(\begin{array}{ccc|c} z_1 & & 0 & 0 \\ & \ddots & & \\ 0 & & z_{n-j} & \\ \hline & 0 & & \bar{Y} \end{array}\right).$$

Moreover, $I_{n-i}(Y) = I_i(\bar{Y})$ for $i<j$.

The polynomial ring $R = K[x_{ij}]$ can be viewed as a coordinate ring of the affine space $\mathrm{Sym}_n(K)$ of all $n\times n$ symmetric matrices with entries in K. If we identify this space with $S_2(U)$ where U is a vector space of dimension n over K, then R is identifed with the symmetric algebra $S(S_2U^*)$.

We denote by E the free R-module $R\underset{K}{\otimes} U$ of rank n and we fix a basis $\{1, 2, \ldots, n\}$ of E. The dual basis of the dual module E^* is denoted by $\{1^*, 2^*, \ldots, n^*\}$. Furthermore we write $\varphi\colon E\to E^*$ for the linear map determined by the matrix X in these two bases. With E and every partition I of a natural number one can associate the Schur module S_IE which is a free R-module with basis consisting of all standard Young tableaux of shape I (see [A–B–W] for details). The map $E\to E^*$ can be treated as a complex having E^* in degree 0 and E in degree 1. With this complex and arbitrary partition I one can associate the Schur complex $S_I(\varphi)$ (see [A–B–W]).

We will need in the sequel the following Schur complexes:

$$S_{11}\varphi\colon\ S_2E\xrightarrow{d_2} E\otimes E^*\xrightarrow{d_1}\textstyle\bigwedge^2E^*$$

where

$$d_1(i\otimes j^*) = \begin{matrix} j^* \\ \varphi(i)\end{matrix},\qquad d_2(ij) = i\otimes\varphi(j)+j\otimes\varphi(i)\,;$$

$$S_2\varphi\colon\ \textstyle\bigwedge^2E\xrightarrow{d_2} E\otimes E^*\xrightarrow{d_1} S_2E^*$$

where

$$d_1(i\otimes j^*) = \varphi(i)j^*,\qquad d_2\begin{pmatrix} j\\ i\end{pmatrix} = i\otimes\varphi(j)-j\otimes\varphi(i)\,;$$

$$S_4\varphi\colon\ \textstyle\bigwedge^4E\xrightarrow{d_4}\bigwedge^3E\otimes E^*\xrightarrow{d_3}\bigwedge^2E\otimes S_2E^*\xrightarrow{d_2} E\otimes S_3E^*\xrightarrow{d_1} S_4E^*$$

where

$$d_{i+1}(j_1\wedge\ldots\wedge j_{i+1}\otimes k_1^*\ldots k_{3-i}^*) = \sum_{s=1}^{i+1}(-1)^s j_1\wedge\ldots\wedge\hat{j}_s\wedge\ldots\wedge j_{i+1}\otimes k_1^*\ldots k_{3-i}^*\varphi(j_s);$$

$$S_{22}\varphi\colon\ S_{22}E\xrightarrow{d_4} S_{21}E\otimes E^*\xrightarrow{d_3}\begin{matrix} S_2E\otimes\bigwedge^2E\\ \oplus\\ \bigwedge^2E\otimes S_2E^*\end{matrix}\xrightarrow{d_2} E\otimes S_{21}E^*\xrightarrow{d_1} S_{22}E^*$$

where

$$d_4\begin{pmatrix} i & k\\ j & t\end{pmatrix} = \begin{matrix} i\\ jt\end{matrix}\otimes\varphi(k)-\begin{matrix} i\\ jk\end{matrix}\otimes\varphi(t)+\begin{matrix} k\\ tj\end{matrix}\otimes\varphi(i)-\begin{matrix} k\\ ti\end{matrix}\otimes\varphi(j)\,;$$

$$d_3\left(\begin{matrix} i\\ jk\end{matrix}\otimes t^*\right) = \begin{matrix} i\\ j\end{matrix}\otimes\varphi(k)t^*+\begin{matrix} i\\ k\end{matrix}\otimes\varphi(j)t^*+jk\otimes\begin{matrix} t^*\\ \varphi(i)\end{matrix}-ik\otimes\begin{matrix} t^*\\ \varphi(j)\end{matrix}\,;$$

$$d_2\left(ij\otimes\begin{matrix} k^*\\ t^*\end{matrix}\right) = i\otimes\begin{matrix} k^*\\ t^*\varphi(j)\end{matrix}+j\otimes\begin{matrix} k^*\\ t^*\varphi(i)\end{matrix}\,;$$

$$d_2\left(\begin{matrix} i\\ j\end{matrix}\otimes k^*t^*\right) = i\otimes\begin{matrix}\varphi(j)\\ k^*t^*\end{matrix}-j\otimes\begin{matrix}\varphi(i)\\ k^*t^*\end{matrix}\,;$$

$$d_1\left(i\otimes\begin{matrix} k^*\\ t^*j^*\end{matrix}\right) = \begin{matrix} k^*\varphi(i)\\ t^*\ \ j^*\end{matrix}.$$

Now we define two maps of complexes. If X is a complex we write $X[p]$ for a shifted complex, i.e. $X[p]_k = X_{p-k}$.

Let $\mathrm{Tr}\colon R[1]\to S_{11}\varphi$ be the map defined by

$$\begin{array}{rccc} & 0 & \longrightarrow & S_2E\\ & \downarrow & & \downarrow\\ \mathrm{Tr}\colon & R & \xrightarrow{\mathrm{tr}} & E\otimes E^*\\ & \downarrow & & \downarrow\\ & 0 & \longrightarrow & \bigwedge^2E^*\end{array}\qquad \mathrm{tr}(1) = \sum_{i=1}^{n} i\otimes i^*.$$

Let $\mathrm{Ev}\colon S_2\varphi\to R[1]$ be the map defined by

$$\begin{array}{rccc} & \bigwedge^2E & \longrightarrow & 0\\ & \downarrow & & \downarrow\\ \mathrm{Ev}\colon & E\otimes E^* & \xrightarrow{\mathrm{ev}} & R\\ & \downarrow & & \downarrow\\ & S_2E^* & \longrightarrow & 0\end{array}\qquad \mathrm{ev}(i\otimes j^*) = j^*(i).$$

Both maps are maps of complexes since X is symmetric. Moreover Ev and Tr are $\mathrm{Gl}(E)$-invariant, $\mathrm{Ker}\,\mathrm{Tr} = 0$ and Ev is nonzero.

3. Construction of the complex $W(\varphi)$. Now we construct the double complex $S_{\cdot\cdot}$.

$$S_{\cdot\cdot}\colon\ S_{4*}\xrightarrow{\partial_{4*}} S_{3*}\xrightarrow{\partial_{3*}} S_{2*}\xrightarrow{\partial_{2*}} S_{1*}\xrightarrow{\partial_{1*}} S_{0*}$$

where

$$S_{4*} := R[2],\quad S_{3*} := S_{11}\varphi[1],\quad S_{2*} := S_{22}\varphi\oplus S_4\varphi,\quad S_{1*} := S_2\varphi,$$

$$S_{0*} := R[2],\quad \partial_{4*} := \mathrm{Tr},\quad \partial_{1*} := \mathrm{Ev}.$$

The differential ∂_{3*} is defined as the following composition:

$$S_{11}\varphi[1]\xrightarrow{1\otimes\mathrm{Tr}} S_{11}\varphi\otimes S_{11}\varphi\xrightarrow{\pi} S_{22}\varphi\oplus S_4\varphi$$

where π is a projection

$$S_{11}\varphi\otimes S_{11}\varphi\cong S_4\varphi\oplus S_{211}\varphi\oplus S_{22}\varphi\xrightarrow{\pi} S_4\varphi\oplus S_{22}\varphi\,.$$

The differential ∂_{2*} is defined as

$$S_{22}\varphi\oplus S_4\varphi\hookrightarrow S_2\varphi\otimes S_2\varphi\xrightarrow{1\otimes\mathrm{Ev}} S_2\varphi[1]\,.$$

The injection is from the equality

$$S_2\varphi\otimes S_2\varphi = S_{22}\varphi\oplus S_{211}\varphi\oplus S_4\varphi\,.$$

It is easy to see that these maps define a double complex $S..$, i.e. that in the following diagram $d_{i-1j}\partial_{ij}+\partial_{ij-1}d_{ij} = 0$, for $1\leqslant i,j\leqslant 4$, and $\partial_{i-1j}\partial_{ij} = 0$, $d_{ij-1}d_{ij} = 0$

$$\begin{array}{ccccccccc}
0 & \xrightarrow{\partial_{44}} & 0 & \xrightarrow{\partial_{34}} & S_{22}E\oplus\bigwedge^4E & \xrightarrow{\partial_{24}} & 0 & \xrightarrow{\partial_{14}} & 0\\
\downarrow{\scriptstyle d_{44}} & & \downarrow{\scriptstyle d_{34}} & & \downarrow{\scriptstyle d_{24}} & & \downarrow{\scriptstyle d_{14}} & & \downarrow{\scriptstyle d_{04}}\\
0 & \xrightarrow{\partial_{43}} & S_2E & \xrightarrow{\partial_{33}} & \begin{matrix} S_{21}E\otimes E^*\\ \oplus\\ \bigwedge^3E\otimes E^*\end{matrix} & \xrightarrow{\partial_{23}} & \bigwedge^2E & \xrightarrow{\partial_{13}} & 0\\
\downarrow{\scriptstyle d_{43}} & & \downarrow{\scriptstyle d_{33}} & & \downarrow{\scriptstyle d_{23}} & & \downarrow{\scriptstyle d_{13}} & & \downarrow{\scriptstyle d_{03}}\\
R & \xrightarrow{\partial_{42}} & E\otimes E^* & \xrightarrow{\partial_{32}} & \begin{matrix} \bigwedge^2E\otimes S_2E^*\\ \oplus\\ S_2E\otimes\bigwedge^2E^*\\ \oplus\\ \bigwedge^2E\otimes S_2E^*\end{matrix} & \xrightarrow{\partial_{22}} & E\otimes E^* & \xrightarrow{\partial_{12}} & R\\
\downarrow{\scriptstyle d_{42}} & & \downarrow{\scriptstyle d_{32}} & & \downarrow{\scriptstyle d_{22}} & & \downarrow{\scriptstyle d_{12}} & & \downarrow{\scriptstyle d_{02}}\\
0 & \xrightarrow{\partial_{41}} & \bigwedge^2E^* & \xrightarrow{\partial_{31}} & \begin{matrix} E\otimes S_{21}E^*\\ \oplus\\ E\otimes S_3E^*\end{matrix} & \xrightarrow{\partial_{21}} & S_2E^* & \xrightarrow{\partial_{11}} & 0\\
\downarrow{\scriptstyle d_{41}} & & \downarrow{\scriptstyle d_{31}} & & \downarrow{\scriptstyle d_{21}} & & \downarrow{\scriptstyle d_{11}} & & \downarrow{\scriptstyle d_{01}}\\
0 & \xrightarrow{\partial_{40}} & 0 & \xrightarrow{\partial_{30}} & S_{22}E^*\oplus S_4E^* & \xrightarrow{\partial_{20}} & 0 & \xrightarrow{\partial_{10}} & 0
\end{array}$$

Let $q\colon R\to S_{22}E\oplus\bigwedge^4E$ be the map defined by

$$q(1) = \sum(-1)^{i+j+t+k}\begin{matrix} i & t\\ j & k\end{matrix}\,M\begin{pmatrix} i, & t\\ j, & k\end{pmatrix}$$

where $M\begin{pmatrix} i, & t\\ j, & k\end{pmatrix}$ denotes $(n-2)\times(n-2)$ minor of the matrix X obtained by omitting rows i, j and columns k, t.

Let $r\colon S_{22}E^*\oplus S_4E^*\to R$ be the map defined by

$$r\begin{pmatrix} i^* & t^*\\ j^* & k^*\end{pmatrix} = (-1)^{i+t+j+k}\,M\begin{pmatrix} i, & t\\ j, & k\end{pmatrix}\det X\,,$$

$$r(i^*j^*k^*t^*) = (-1)^{i+j+t+k}M(i,j)M(t,k)+M(i,t)M(j,k)+M(i,k)M(j,t)$$

where $M(i,j)$ is $(n-1)\times(n-1)$ minor of the matrix X obtained by omitting row i and column j. Notice that $\operatorname{Ker}q = 0$. It is easy to show that $d_{24}q = 0$ and $rd_{21} = 0$.

We define $W(\varphi)$ as the following complex

$$0\to R\xrightarrow{q} H_2(S..)\xrightarrow{r} R$$

where $H_2(S..)$ is the homology of $S..$ with respect to the horizontal differential ∂.

Lemma 1. $\operatorname{Im}r = I_{n-1}^2(X)$.

Proof. From the equality $M\begin{pmatrix} i, & t\\ j, & k\end{pmatrix}\cdot\det X = M(j,k)M(i,t)-M(j,t)M(i,k)$ it follows that $\operatorname{Im}r\subset I_{n-1}^2(X)$. Moreover, using elementary linear algebra, every generator $M(i,j)M(p,q)$ of $I_{n-1}^2(X)$ can be expressed as a linear combination of elements from $\operatorname{Im}r$. In other terms, $H_0(W(\varphi)) = R/I_{n-1}^2(X)$.

Theorem. *The complex $W(\varphi)$ is a minimal free resolution of the ideal $I_{n-1}^2(X)$.*

4. Acyclicity of the complex $W(\varphi)$. Because the complex $W(\varphi)$ has length 6 it follows from Lemma I that the acyclicity of the complex $W(\varphi)$ is equivalent to the acyclicity of the complexes $W(\varphi)_P$ where $P\subset R$ are prime ideals such that depth $PR_P<6$. We know that depth $I_{n-2}(X) = 6$ (see [K]). Hence $I_{n-2}(X)\not\subset P$. Therefore in $I_{n-2}(X)_P$ there exists an invertible element. Hence $R_P = I_{n-2}(X)_P$. In this situation we can use Lemma II. This means that in E there exists a basis $f_1\ldots f_n$ such that the matrix of φ with respect to this basis has the form

$$\left(\begin{array}{ccc|c} z_1 & & 0 & 0\\ & \ddots & & \\ 0 & & z_{n-2} & \\ \hline & 0 & & \overline{X}\end{array}\right)$$

where z_i are invertible elements of R_P and $\overline{X}$ is 2×2 symmetric matrix. Moreover, $I_{n-1}(X)_P = I_1(\overline{X})_P$. In the sequel we will write R instead of R_P.

Let N be the R-module generated by $f_1\ldots f_{n-2}$ and let F be the R-module generated by f_{n-1}, f_n. Let $\psi\colon N\to N^*$ be the map defined by the matrix

$$\begin{pmatrix} z_1 & & 0\\ & \ddots & \\ 0 & & z_{n-2}\end{pmatrix}$$

and let $\theta\colon F\to F^*$ be the map defined by the matrix $\overline{X}$. Hence $\varphi = \psi\oplus\theta$. Because ψ is an isomorphism and a Schur complex of an isomorphism is exact (see [A–B–W]) and moreover $S_I\varphi = S_I(\psi\oplus\theta) = \sum_{J\subset I}S_{I/J}\psi\otimes S_J\theta$ we infer that $H.(S_I\varphi) = H.(S_I\theta)$.

In order to prove that $W(\varphi)$ is acyclic we utilize the theory of the spectral sequences associated with the double complex $S..$. It will be shown that ${}^{\mathrm{II}}E_{2,q}^{\infty}$ is equal to $H_q(H_2(S))$. Furthermore we compute ${}^{\mathrm{I}}E_{p,q}^{\infty}$ and comparison with ${}^{\mathrm{II}}E_{p,q}^{\infty}$ gives us the desired description of the homology of $W(\varphi)$.

LEMMA 2. *The only nonzero elements of the sequence* ${}^{\mathrm{I}}E^{\infty}_{p,q}$ *are*:

$$ {}^{\mathrm{I}}E^{\infty}_{2,0} \cong \frac{I_1^2(\overline{X})}{I_1(\overline{X})I_2(\overline{X})}, \quad {}^{\mathrm{I}}E^{\infty}_{1,1} \cong \frac{I_1(\overline{X})I_2(\overline{X})}{I_2(\overline{X})^2}, \quad {}^{\mathrm{I}}E^{\infty}_{0,2} \cong I_2^2(\overline{X}), \quad {}^{\mathrm{I}}E^{\infty}_{4,2} \cong R. $$

Proof. First we have to calculate the homology ${}^{\mathrm{II}}H_{p,q}(S)$ of the columns of the complex $S_{..}$. The columns of $S_{..}$ are Schur complexes and we can replace φ by θ. It is obvious that

$$ {}^{\mathrm{II}}H_{0,q}(S) = \begin{cases} R & q = 2, \\ 0 & q \neq 2, \end{cases} \qquad {}^{\mathrm{II}}H_{4,q}(S) = \begin{cases} R & q = 2, \\ 0 & q \neq 2. \end{cases} $$

Now we compute ${}^{\mathrm{II}}H_{1,q}(S)$, $S_{1*} = S_{22}\theta[\theta]$. From Lemma I and Lemma II it follows that $H_{1,2}(S.) = H_1(S_2\theta)$ and $H_{1,3}(S.) = H_2(S_2\theta)$ are zero. We must compute $H_{1,1}(S) = H_0(S_2\theta)$. We compare the complex $F\otimes F^* \xrightarrow{d_{12}} S_2F^*$ with the Koszul complex $T.$ on a regular sequence $\bar{x}_{11}$, $\bar{x}_{12}$, $\bar{x}_{22}$. Notice that $\operatorname{depth} I_1(\overline{X}) = \operatorname{depth} I_{n-1}(X) = 3$.

$$ T. \qquad 0 \to = \textstyle\bigwedge^3(S_2F^*) \xrightarrow{\pi_2} = \bigwedge^2(S_2F^*) \xrightarrow{\pi_1} S_2F^* \xrightarrow{r'} I_1(\overline{X})I_2(\overline{X}) \to 0 $$

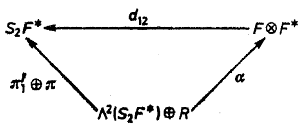
where $r'(i^*j^*) = (-1)^{i+j}M(i,j)\det X$.

The complex $T.$ is exact; hence $\operatorname{Coker}\pi_1 \cong I_1(\overline{X})I_2(\overline{X})$.

Let us consider the map $\alpha\colon \bigwedge^2(S_2F^*)\oplus R \to F\otimes F^*$ defined by:.

$$ \alpha((1^*1^*)\wedge(1^*2^*)) = -2\otimes 1^*, \quad \alpha((1^*1^*)\wedge(2^*2^*)) = 2\otimes 2^* - 1\otimes 1^*, $$
$$ \alpha((1^*2^*)\wedge(2^*2^*)) = 1\otimes 2^*, \quad \alpha(1) = 1\otimes 1^*. $$

The following diagram is commutative:



where $\pi_1'(1) = \bar{x}_{11}(1^*1^*)+\bar{x}_{12}(1^*2^*)$.

Since α is an isomorphism $\operatorname{Im}(\pi_1\oplus\pi_1') = \operatorname{Im}\pi_1 + \operatorname{Im}\pi_1' = \operatorname{Im} d_{12}$,

$$ {}^{\mathrm{II}}H_{1,1}(S..) = \operatorname{Coker} d_{12} \cong \frac{S_2F^*}{\operatorname{Im} d_{12}} = \frac{S_2F^*}{\operatorname{Im}\pi_1 \oplus \operatorname{Im}\pi_1'} = \frac{S_2F^*/\operatorname{Im}\pi_1^*}{\operatorname{Im}\pi_1 + \operatorname{Im}\pi_1'/\operatorname{Im}\pi_1} = \frac{I_1(\overline{X})I_2(\overline{X})}{I_2^2(\overline{X})}. $$

Now we compute ${}^{\mathrm{II}}H_{2,q}(S..)$. From Lemma I and Lemma II it follows that the complex $S_4\theta \subset S_{2*}$ is acyclic and $S_{22}\theta \subset S_{2*}$ has nonzero homology in first and zero place only.

Let us compute ${}^{\mathrm{II}}H_{2,1}(S..)$. We handle the complex $S_{22}\theta$ only because $H_1(S_4\theta) = 0$. Let $T.$ be the Koszul complex as above. Let us consider the map of complexes $\xi\colon T. \to S_{22}\theta$.

$$ \begin{array}{llllllllll} S_{22}\theta & S_{22}F & \longrightarrow & S_{21}F\otimes F^* & \longrightarrow & \begin{array}{c}\bigwedge^2F\otimes S_2F^* \\ \oplus \\ S_2F\otimes\bigwedge^2F^*\end{array} & \longrightarrow & F\otimes S_{21}F^* & \longrightarrow & S_{22}F^* \\ & \uparrow{\scriptstyle\xi_3} & & \uparrow{\scriptstyle\xi_2} & & \uparrow{\scriptstyle\xi_1} & & \uparrow{\scriptstyle\xi_0} & & \uparrow \\ T. & \bigwedge^3(S_2F) & \longrightarrow & \bigwedge^2(S_2F) & \longrightarrow & S_2F & \longrightarrow & R & \longrightarrow & 0 \end{array} $$

$$ \xi_3((11)\wedge(12)\wedge(22)) = \frac{1}{4}\binom{22}{11}, \qquad \xi_1(11) = \frac{1}{2}\left(11\otimes\frac{2^*}{1^*} + \frac{2}{1}\otimes 2^*2^*\right), $$
$$ \xi_2((11)\wedge(12)) = \frac{1}{2}\left(\frac{2}{11}\otimes 2^*\right), \qquad \xi_1(12) = \frac{1}{2}\left(12\otimes\frac{2^*}{1^*} - \frac{2}{1}\otimes 1^*2^*\right), $$
$$ \xi_2((11)\wedge(22)) = \frac{1}{2}\left(\frac{2}{12}\otimes 2^* - \frac{2}{11}\otimes 1^*\right), \qquad \xi_1(22) = \frac{1}{2}\left(22\otimes\frac{2^*}{1^*} + \frac{2}{1}\otimes 1^*1^*\right), $$
$$ \xi_2((12)\wedge(22)) = -\frac{1}{2}\left(\frac{2}{12}\otimes 1^*\right), \qquad \xi_0(1) = 1\otimes\frac{2^*}{1^*1^*} + 2\otimes\frac{2^*}{1^*2^*}. $$

Since ξ_3 is an isomorphism, ξ_2 is a monomorphism and

$$ \operatorname{Im}\xi_2 \oplus R\left(\frac{2}{12}\otimes 2^*\right) = S_{21}F\otimes F^*, $$

the cone on ξ)topically equivalent to the cone on the following map.

$$ \begin{array}{llllllll} \overline{S_{22}\theta}. & R\left(\frac{2}{12}\otimes 2^*\right) & \longrightarrow & \begin{array}{c}\bigwedge^2F\otimes S_2F^* \\ \oplus \\ S_2F\otimes\bigwedge^2F^*\end{array} & \longrightarrow & F\otimes S_{21}F^* & \longrightarrow & S_{22}F^* \\ \uparrow{\scriptstyle\bar{\xi}} & & & \uparrow{\scriptstyle\xi_1} & & \uparrow{\scriptstyle\xi_0} & & \\ T. & & & S_2F & \longrightarrow & R & & \end{array} $$

This cone $M.$ has length 3. Using Lemmas I and II and the long sequence of homology associated with a cone, we obtain that $M.$ is acyclic. Hence we have the exact sequence:

$$ \to H_2(M.) \to H_1(T.) \to H_1(\overline{S_{22}\theta}) \to H_1(M.) $$

and consequently

$$ H_1(S_{22}\theta) = H_1(\overline{S_{22}\theta}) \cong H_1(\overline{T}) = H_1(T.) = \frac{R}{I_1(\overline{X})}. $$

Now we compute ${}^{\mathrm{II}}H_{2,0}(S..)$. Let us consider the map of complexes $\beta\colon T. \to N.$ where $T.$ as above and $N.$ is the Eagon–Northcott complex of a matrix

$$\begin{pmatrix} \bar{x}_{11} & \bar{x}_{12} & \bar{x}_{22} & 0 \\ 0 & \bar{x}_{11} & \bar{x}_{12} & \bar{x}_{22} \end{pmatrix}.$$

$$\begin{array}{lccccccccc} T. & 0 \longrightarrow & \bigwedge^3(S_2F^*) & \xrightarrow{\pi_2} & \bigwedge^2(S_2F^*) & \xrightarrow{\pi_1} & S_2F^* & \xrightarrow{r'} & I_1(\bar{X})I_2(\bar{X}) \\ & & \downarrow{\scriptstyle\beta_3} & & \downarrow{\scriptstyle\beta_2} & & \downarrow{\scriptstyle\beta_1} & & \cap\!\!\downarrow \\ N. & 0 \longrightarrow & S_{31}(S_2F^*) & \xrightarrow{\varrho_2} & S_{21}(S_2F^*) & \xrightarrow{\varrho_1} & S_2(S_2F^*) & \xrightarrow{r} & I_1^2(\bar{X}) \end{array}$$

where

$$\beta_1(i^*j^*) = (-1)^{i+j}M(i,j)\big((1^*1^*)(2^*2^*)-(1^*2^*)(1^*2^*)\big),$$

$$\beta_2\begin{pmatrix}(1^*2^*)\\(1^*1^*)\end{pmatrix} = x_{11}\genfrac{}{}{0pt}{}{(1^*2^*)}{(1^*1^*)\;(1^*1^*)} + x_{12}\genfrac{}{}{0pt}{}{(2^*2^*)}{(1^*1^*)(1^*1^*)}$$

$$+x_{22}\left(\genfrac{}{}{0pt}{}{(2^*2^*)}{(1^*1^*)\;(1^*2^*)} - \genfrac{}{}{0pt}{}{(1^*2^*)}{(1^*1^*)\;(2^*2^*)}\right),$$

$$\beta_2\begin{pmatrix}(2^*2^*)\\(1^*1^*)\end{pmatrix} = x_{11}\genfrac{}{}{0pt}{}{(1^*2^*)}{(1^*1^*)\;(1^*2^*)} + x_{22}\genfrac{}{}{0pt}{}{(2^*2^*)}{(1^*2^*)\;(1^*2^*)}$$

$$+x_{12}\genfrac{}{}{0pt}{}{(2^*2^*)}{(1^*1^*)\;(1^*2^*)},$$

$$\beta_2\begin{pmatrix}(2^*2^*)\\(1^*2^*)\end{pmatrix} = x_{22}\genfrac{}{}{0pt}{}{(2^*2^*)}{(1^*2^*)\;(2^*2^*)} + x_{12}\genfrac{}{}{0pt}{}{(2^*2^*)}{(1^*1^*)\;(2^*2^*)}$$

$$+x_{11}\genfrac{}{}{0pt}{}{(1^*2^*)}{(1^*1^*)\;(2^*2^*)},$$

$$\beta_3\big((1^*1^*)\wedge(1^*2^*)\wedge(2^*2^*)\big) = x_{11}\genfrac{}{}{0pt}{}{(1^*2^*)}{(1^*1^*)\;(1^*1^*)(2^*2^*)}$$

$$+x_{12}\genfrac{}{}{0pt}{}{(1^*2^*)}{(1^*1^*)\;(1^*2^*)(2^*2^*)}$$

$$+x_{22}\genfrac{}{}{0pt}{}{(2^*2^*)}{(1^*1^*)\;(1^*2^*)(2^*2^*)}.$$

Hence $\operatorname{Coker}(\beta_1 \oplus \varrho_1) \cong \dfrac{I_1^2(\bar{X})}{I_1(\bar{X})I_2(\bar{X})}$. Since $S_2(S_2F^*) = S_{22}F^* \oplus S_4F^*$, we have $\operatorname{Coker} d_{12} \cong \dfrac{I_1^2(\bar{X})}{I_1(\bar{X})I_2(\bar{X})}$.

Now we compute ${}^{\mathrm{II}}H_{3,q}(S..)$. It is easy to see that ${}^{\mathrm{II}}H_{3,1}(S..) \cong \dfrac{R}{I_1(\bar{X})}$. To

calculate ${}^{\mathrm{II}}H_{3,2}(S..)$ let us consider the following map:

$$\begin{array}{lccccc} D. & R & \xrightarrow{D} & R & \longrightarrow & 0 \\ & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \downarrow \\ \bigwedge^2\theta. & S_2F & \xrightarrow{d_{33}} & F\otimes F^* & \xrightarrow{d_{32}} & \bigwedge^2F^* \end{array}$$

where

$$D(1) = \det(\bar{X}), \quad g_0(1) = 1\otimes 1^* + 2\otimes 2^*,$$
$$g_1(1) = \tfrac{1}{2}\big(x_{11}(22)+x_{22}(11)-x_{12}(12)\big)$$

It is easy to see that the cone on this map $M(g)$ is acyclic. Using the long exact sequence of homology, we have:

$$\to H_3(M(g)) \to H_2(\textstyle\bigwedge^2\theta) \to H_2(D.) \to H_2(M(g)) \to H_1(\bigwedge^2\theta) \to H_1(D) \to H_1(M(g))$$

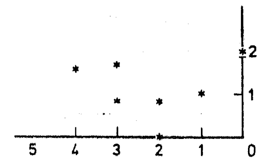
Hence

$$H_1(D.) = \frac{R}{I_2(\bar{X})}, \quad H_2(D.) = 0$$

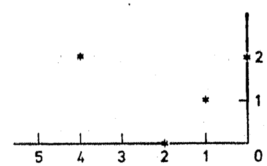
and finally

$$H_2(\textstyle\bigwedge^2\theta) = 0, \quad H_1(\bigwedge^2\theta) = \dfrac{R}{I_2(\bar{X})}.$$

We obtain that the sequence $H_{p,q}(S..)$ has the following diagram:



Simple analysis of horizontal differentials gives the terms $H_pH_q(S..)$.



From this we have that

$${}^{\mathrm{I}}E^2_{2,0} \cong \frac{I_1^2(\bar{X})}{I_1(\bar{X})I_2(\bar{X})}, \quad {}^{\mathrm{I}}E^2_{1,1} \cong \frac{I_1(\bar{X})I_2(\bar{X})}{I_2^2(\bar{X})}, \quad E^2_{0,2} \cong I_2^2(\bar{X}) \text{ and } {}^{\mathrm{I}}E^2_{4,2} \cong I_2(\bar{X}).$$

Observe that ${}^{\mathrm{I}}E^2_{p,q} = {}^{\mathrm{I}}E^{\infty}_{p,q}$ because all higher differentials are zero.

LEMMA 3.

$$ {}^{\mathrm{II}}E^{\infty}_{p,q} \cong \begin{cases} H_q(H_2(S..)), & p = 2, \\ 0, & p \neq 2. \end{cases} $$

Proof. We have to compute the homology of the rows of the complex $S..$. We can express the modules $S_{i,j}$ as the sums of Schur modules on E. It is known that $\mathrm{Hom}_{GL(E)}(S_I E, S_J E) = 0$ if $I \neq J$. Therefore it is easy to check that only $H_2(S..)$ is nonzero.

Proof of the theorem. Let $B.$ be the total complex of the double complex $S..$. By Lemma 3 it follows that $H_p(B) \cong H_{p-2}(H_2(S..))$. Moreover the geometry of ${}^{\mathrm{I}}E$ shows that $H_i(B) = 0$, for $i = 3, 4, 5$, i.e. $H_i(H_2(S..)) = 0$ for $i = 1, 2, 3$ and in turn $H_i(W(\varphi)) = 0$ for $i = 2, 3, 4$. Furthermore,

$$ H_4(H_2(S..)) = H_6(B) = \mathrm{Ker}\, d_{24} \cong R.. $$

It can easily be shown that the image of q kills this homology, i.e. $H_5(W(\varphi)) = 0$.

Finally we must prove that $H_2(B) = I^2_{n-1}(X)$. To this end let us consider a sequence of maps $B_3 \xrightarrow{h} B_2 \xrightarrow{f} I^2_{n-1}(X)$ where

$$ B_2 = S_{22}E^* \oplus S_4 E^* \oplus S_2 E^* \oplus R, \quad B_3 = E \otimes S_{21}E^* \oplus E \otimes S_3 E^* \oplus E \otimes E^*, $$

$$ h = d_{21} + \partial_{21} + d_{12} + \partial_{12}, \quad f = r + r' + d_2, \quad d_2(1) = \det X^2, $$

One can easily check that $fh = 0$, i.e. that we have a map $H_2(B) \to I^2_{n-1}(X)$ induced by f. From the analysis of the spectral sequence ${}^{\mathrm{I}}E$ and an explicit form of f we know that there exists a commutative diagram:

$$ \begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & H_2(B) & \longrightarrow & \dfrac{H_2(B)}{T} & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow f & & \downarrow \bar{f} & & \\ 0 & \longrightarrow & I_1(\bar{X})I_2(\bar{X}) & \longrightarrow & I_1^2(\bar{X}) & \longrightarrow & \dfrac{I_1^2(\bar{X})}{I_1(\bar{X})I_2(\bar{X})} & \longrightarrow & 0 \end{array} $$

where T is the image of $H_2(\mathrm{Tot}(\sum_{i \leqslant 1} S_{i,j}))$ in $H_2(B)$. Since

$$ \mathrm{Tot}(\sum_{i \leqslant 1} S_{i,j}) \colon \textstyle\bigwedge^2 E \to E \otimes E^* \to S_2 E^* + R $$

and this is (up to a splitting factor $R \xrightarrow{\sim} R$) the resolution of $I_{n-1}(X)$ described in [J], we infer that the mapping g is an isomorphism. Moreover the map $\bar{f}$ is also an isomorphism since $H_2(B)/T = {}^{\mathrm{I}}E^2_{2,0}$, this implies that we have an isomorphism of $H_2(B)$ and $I^2_{n-1}(X)$ induced by f and, in fact, by $r\colon S_{22}E^* \oplus S_4 E^* \to I^2_{n-1}(X)$.

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