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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Classification of weakly infinite-dimensional spaces
Part II: Essential mappings

by

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Abstract. We will prove several results concerning the relation between the transfinite dimension dim (introduced in Part I) and essential mappings onto D. W. Henderson's [He] cubes J^α, where α is a countable ordinal number. Counterexamples will show that the obtained results are also sharp.

Chapter IV. Henderson's problem

The purpose of this chapter is to investigate how well the value of dim and index can be determined by using essential maps to Henderson's transfinite cubes J^α $(\alpha<\omega_1)$, [He].

To begin with we recall Henderson's Theorem:

If a space X admits an essential map onto J^α then $\operatorname{Ind} X \geqslant \alpha$.

The converse of this theorem is false. In Section 5.1 we will construct a compact space X with $\operatorname{Ind} X = \omega_0+1$ which admits no essential map onto J^{ω_0+1}. Independently, R. Pol [P2] constructed spaces such that the difference between $\operatorname{Ind} X$ and the least α such that X admits no essential map onto J^α is arbitrarily big.

The situation for dim is much better. First of all we have the following theorem

(0) *If X admits an essential map onto J^α then* $\dim X \geqslant \alpha$.

Unfortunately, the converse is false, see Section 5.2 for a compact X with $\dim X \geqslant \omega_0+1$ without an essential map $f\colon X \to J^{\omega_0+1}$.

However, we can prove

(1) *If* $\dim X \geqslant \alpha+1$ *then X admits an essential map onto J^α.*

For limit ordinals α we even have

(2) $\dim X \geqslant \alpha$ *iff X admits an essential map onto J^α.*

Thus, the difference in the case of dim is at most 1.

For locally compact spaces we can prove even more:

(3) *If X is locally compact then* $\dim X \geqslant \alpha$ *iff* $X \times C$ *admits an essential map onto* J^α.

Finally for s.c.d. spaces we have

(4) *If* α *is a limit ordinal and* X *is* s.c.d. *then* $\dim X \geqslant \alpha+1$ *iff* X *admits an essential map onto* $J^{\alpha+1}$.

This last theorem is also sharp, in Section 5.3 we will construct a compact s.c.d. space X with $\dim X \geqslant \omega_0+2$ which admits no essential map onto J^{ω_0+2}.

4.1. Essential mappings I; introduction. Let I^n denote the n-dimensional unit cube, i.e. $I^n = \prod_{i=1}^{n} [0, 1]$. By ∂I^n we denote the geometrical boundary of I^n, i.e.,

$$\partial I^n = \{x \in I^n\colon\ x_i \in \{0, 1\} \text{ for some } i \in 1, \ldots, n\}.$$

In addition, for a mapping $f\colon X \to Y$ and a subset B of Y we denote

$$f_B = f \mid f^{-1}(B)\colon f^{-1}(B) \to B \subset Y.$$

In finite-dimension theory the following concept is well known.

4.1.1. Definition [A]. A continuous mapping f of a space X into I^n is called *essential*, if there does not exist a continuous mapping g of X into ∂I^n such that

$$f = g \quad \text{on the subset } f^{-1}(\partial I^n) \text{ of } X.$$

Observe that each essential mapping into I^n is surjective.

4.1.2. Theorem [A]. *A space X satisfies* $\dim X \geqslant n$ *if and only if there exists an essential mapping* $f\colon X \to I^n$.

Our goal is to extend this theorem to transfinite values.

The following lemmas from finite-dimension theory will be useful in the remaining part of this treatise. Together they actually constitute a proof of Theorem 4.1.2. Their proofs can be found in [N, Ch. III].

4.1.3. Lemma. *Let* $\{(A_i, B_i)\}_{i=1}^n$ *be an essential sequence of pairs of disjoint closed sets in a space X and let the continuous mapping* $f = \bigtimes_{i=1}^{n} f_i\colon X \to I^n$ *be such that* $f_i(A_i) = 0$ *and* $f_i(B_i) = 1$ *for* $i = 1, \ldots, n$. *Then f is essential.*

4.1.4. Lemma. *Let* $f = \bigtimes_{i=1}^{n} f_i\colon X \to I^n$ *be an essential mapping of a space X into I^n. Then* $\{(f_i^{-1}(0), f_i^{-1}(1))\}_{i=1}^n$ *is essential.*

We now describe Henderson's [He] cubes and essential maps to them.

4.1.5. Definition. We define for every $\alpha < \omega$, J^α, T^α and p_α as follows:

(0): $J^0 = \{0\}$,

(i): $J^1 = [0, 1]$, $T^1 = \{0, 1\}$ and $p_1 = 0$,

(ii): $J^{\alpha+1} = J^\alpha \times I$, $T^{\alpha+1} = (T^\alpha \times I) \cup (J^\alpha \times \{0\}) \cup (J^\alpha \times \{1\})$, and $p_{\alpha+1} = p_\alpha \times \{0\}$.

(iii): α a limit.

For $\beta < \alpha$ let A_α^β be a half-open arc with $J^\beta \cap A_\alpha^\beta = \{p_\beta\}$, let $J_\alpha = \omega(\bigoplus_{\beta<\alpha} J^\beta \cup A_\alpha^\beta)$ (one-point compactification), let $T^\alpha = J^\alpha - \bigcup_{\beta<\alpha}(J^\beta - T^\beta)$, and let p_α be the compactifying point. For each $\beta < \alpha$ by i_α^β we denote the embedding of J^β into J^α.

The concept of an essential mapping to J^α is defined as follows:

4.1.6. Definition. A continuous mapping $f\colon X \to J^\alpha$ of a space X into J^α is called *essential* if there does not exist a mapping g of X into J^α such that $g(X) \neq J^\alpha$ and $f = g$ on $f^{-1}(T)$. Otherwise f will be called *inessential.*

We need to isolate special subsets of the spaces J^α.

4.1.7. Definition. For each $\alpha < \omega_1$ we define the *collection of cells* $\mathscr{C}_\alpha$ in J^α as follows:

(i) $\mathscr{C}_1 = \{J^1\}$,

(ii) $\mathscr{C}_{\alpha+1} = \{C \times I\colon\ C \in \mathscr{C}_\alpha\}$, and

(iii) α limit: $\mathscr{C}_\alpha = \{i_\alpha^\beta(C)\colon\ C \in \mathscr{C}_\beta, \beta < \alpha\}$.

Observe that every cell in every $\mathscr{C}_\alpha$ is a homeomorph of some I^n.

From [He] we quote:

H1: (1) $\operatorname{Ind} J^\alpha = \alpha$ for each $\alpha < \omega_1$.

(2) Each space J^α is homeomorphic to a retract of the Hilbert cube I^∞.

H2: Let f be a continuous mapping of X onto J^α. Then f is essential if and only if $f_C\colon f^{-1}(C) \to C$ is essential for every cell C in $\mathscr{C}_\alpha$.

H3: If a space X admits an essential mapping onto J^α then $\operatorname{Ind} X \geqslant \alpha$.

In this section we prove that H3 also holds if we replace Ind by dim. For technical reasons we need to define the following collection in J^α.

4.1.8. Definition. For each $\alpha < \omega_1$ we define the *collection of opposite faces* $\mathscr{P}_\alpha$ in J^α as follows:

(i) $\mathscr{P}_1 = \{(\{0\}, \{1\})\}$.

(ii) $\mathscr{P}_{\alpha+1} = \{(F \times I, G \times I)\colon (F, G) \in \mathscr{P}_\alpha\} \cup \{(J^\alpha \times \{0\}, J^\alpha \times \{1\})\}$.

(iii) α limit: $\mathscr{P}_\alpha = \{(i_\alpha^\beta(F), i_\alpha^\beta(G))\colon (F, G) \in \mathscr{P}_\beta, \beta < \alpha\}$.

4.1.9. Lemma. *If* $\{(F_i, G_i)\}_{i=1}^n \in M_{P_\alpha}$ *then there is a cell C in $\mathscr{C}_\alpha$ such that* $\{(C \cap F_i, C \cap G_i)\}_{i=1}^n$ *is a collection of distinct pairs of opposite faces of C.*

Proof. By induction on the number α. If α is finite then let $C = J^\alpha$. Assume the lemma holds for ordinal numbers smaller than a given ordinal number α. First assume that α is a limit ordinal. Recall that $J^\alpha = \{p_\alpha\} \cup \bigoplus_{\beta<\alpha} (J^\beta \cup A_\alpha^\beta)$. The collection $\{(F_i, G_i)\}_{i=1}^n$ is essential, so one sees that $F_i \cap F_j \neq \emptyset$ for all $i, j \in \{1, \ldots, n\}$. It is readily seen from the definition of opposite faces that $\{(F_i, G_i)\}_{i=1}^n$ consists of opposite faces in $J^\beta \subset J^\alpha$ for some $\beta < \alpha$. Our inductive hypothesis gives a cell C in J^β with the required property. Clearly $C \in \mathscr{C}_\alpha$ by definition.

Now assume that α is a successor, say $\alpha = \beta+1$. Consider $\{(F_i, G_i)\}_{i=1}^n$. Then for at most one $i_0 \in \{1, \ldots, n\}$ we have

$$(F_{i_0}, G_{i_0}) = (J^\beta \times \{0\}, J^\beta \times \{1\}).$$

Let $\sigma = \{1, \ldots, n\} - \{i_0\}$. Then for each $i \in \sigma$ we have

$$(F_i, G_i) = (F_i' \times I, G_i' \times I) \quad \text{for some } (F_i', G_i') \in \mathscr{P}_\beta.$$

Observe that $\{(F_i, G_i)\}_{i\in\sigma}$ is essential in J^β. We can apply our inductive hypothesis and find a cell C' in J^β such that $\{(C' \cap F_i', C' \cap G_i')\}_{i\in\sigma}$ is a collection of distinct pairs of opposite faces of C'.

Put $C = C' \times I \in C_\alpha$. Observe that for each $i \in \sigma$

$$((C' \cap F_i') \times I, (C' \cap G_i') \times I) = (C \cap F_i, C \cap G_i).$$

So we conclude that

$$\{(C \cap F_i, C \cap G_i)\}_{i=1}^n \subset \{(C \cap F_i, C \cap G_i)\}_{i\in\sigma} \cup \{(C \cap J^\beta \times \{0\}, C \cap J^\beta \times \{1\})\}$$

consists of distinct pairs of opposite faces of C. ■

4.1.10. Proposition. *For each $\alpha < \omega_1$ we have* $\operatorname{Ord} M_{\mathscr{P}_\alpha} = \alpha$.

Proof (induction on α). For $\alpha = 1$ $M_{\mathscr{P}_1} = \{(\{0\}, \{1\})\} = \mathscr{P}_1$. Hence $\operatorname{Ord} M_{\mathscr{P}_1} = 1$. Assume that the proposition is true for all $\beta < \alpha$. Let us assume first that α is a limit ordinal. Since for each $\beta < \alpha$ we can consider the space J^β as a subspace of J^α and accordingly $\mathscr{P}_\beta$ as a subcollection of $\mathscr{P}_\alpha$ we then clearly have by our inductive hypothesis

$$\operatorname{Ord} M_{\mathscr{P}_\alpha} \geqslant \operatorname{Ord} M_{\mathscr{P}_\beta} = \beta.$$

Hence $\operatorname{Ord} M_{\mathscr{P}_\alpha} \geqslant \alpha$. On the other hand if $(F, G) \in \mathscr{P}_\alpha$ then $(F, G) \in \mathscr{P}_\beta$ for some $\beta < \alpha$, it then follows readily that $M_{\mathscr{P}_\alpha}^{(F,G)} = M_{\mathscr{P}_\beta}^{(F,G)}$ so that

$$\operatorname{Ord} M_{\mathscr{P}_\alpha}^{(F,G)} < \beta < \alpha.$$

Hence also $\operatorname{Ord} M_{\mathscr{P}_\alpha} \leqslant \alpha$. Next assume that α is a successor, say $\alpha = \beta+1$. Put $\mathscr{P}_\beta' = \{(F \times I, G \times I): (F, G) \in \mathscr{P}_\beta\}$ and let $a = (F_0, G_0) = (I^\beta \times \{0\}, I^\beta \times \{1\})$.

Claim. $M_{\mathscr{P}_\beta'} = M_{\mathscr{P}_\alpha}^a$.

Proof of the claim. By definition $\mathscr{P}_\beta' = \mathscr{P}_\alpha - \{(F_0, G_0)\}$ so that $M_{\mathscr{P}_\alpha}^a \subset M_{\mathscr{P}_\beta'}$ is immediate.

Let $\sigma = \{(F_i, G_i)\}_{i=1}^n \in M_{\mathscr{P}_\beta'}$. Then $\sigma \in M_{\mathscr{P}_\alpha}$ so that by Lemma 4.1.9 we can find a cell C in $\mathscr{C}_\alpha$ such that $\{(C \cap F_i, C \cap G_i)\}_{i=1}^n$ is a collection of distinct pairs of opposite faces of C.

For $i = 1, \ldots, n$ put $D_i = C \cap F_i$ and $E_i = C \cap G_i$. In addition put

$$D_0 = C \cap (J^\beta \times \{0\}) = C \cap F_0 \quad \text{and} \quad E_0 = C \cap (J^\beta \times \{1\}) = C \cap G_0.$$

Observe that $\{(D_i, E_i)\}_{i=0}^n$ consists of distinct pairs of opposite faces of a cell C. Then [E2; 1.8.1] implies that $\{(D_i, E_i)\}_{i=0}^n$ is essential in C.

Since $D_i \subset F_i$ and $E_i \subset G_i$ for each $i = 0, \ldots, n$, we conclude that $\gamma = \{(F_i, G_i)\}_{i=0}^n$ is essential in J^α. Consequently, $\gamma = \sigma \cup \{a\} \in M_{\mathscr{P}_\alpha}$ and $a \notin \sigma$ which implies $\sigma \in M_{\mathscr{P}_\alpha}^a$ and our claim is proved. Now it is easy to see that for $\sigma = \{(F_i, G_i)\}_{i=1}^n \in \mathscr{P}_\beta$ σ is essential in J^β iff $\{(F_i \times I, G_i \times I)\}_{i=1}^n$ is essential in J^α. This implies that

$$\operatorname{Ord} M_{\mathscr{P}_\alpha}^a = \operatorname{Ord} M_{\mathscr{P}_\beta'} = \operatorname{Ord} M_{\mathscr{P}_\beta} = \beta.$$

(Formally we should apply Lemma 2.1.6.) It follows that $\operatorname{Ord} M_{\mathscr{P}_\alpha} \geqslant \alpha$. Also if $b = (F \times I, G \times I) \in \mathscr{P}_\beta'$ then

$$M_{\mathscr{P}_\alpha}^{\{a,b\}} = \{\{(F_i \times I, G_i \times I)\}_{i=1}^n : \{(F_i, G_i)\}_{i=1}^n \in M_{\mathscr{P}_\beta}^b\}.$$

Hence $\operatorname{Ord} M_{\mathscr{P}_\alpha}^b = \operatorname{Ord} M_{\mathscr{P}_\beta}^b + 1 \leqslant \beta$. Together with $\operatorname{Ord} M_{\mathscr{P}_\alpha}^a = \beta$ this implies that $\operatorname{Ord} M_{\mathscr{P}_\alpha} = \alpha$. ■

4.1.11. Theorem. *For $\alpha < \omega_1$ we have* $\dim J^\alpha = \alpha$.

Proof. From Proposition 4.1.10, Theorem 3.2.4 and H1 we obtain

$$\alpha \leqslant \dim J^\alpha \leqslant \operatorname{Ind} J^\alpha = \alpha. ■$$

Let f be a continuous mapping from a space X onto J^α and consider $\mathscr{P}_\alpha$. Then we define

$$L(f) = \{(f^{-1}(F), f^{-1}(G)): (F, G) \in \mathscr{P}_\alpha\}.$$

4.1.12. Lemma. *Let f be an essential mapping from a space X onto J^α. Then*

$$\operatorname{Ord} M_{L(f)} \geqslant \alpha.$$

Proof. Let $\Phi: \mathscr{P}_\alpha \to L(f)$ be defined by $\Phi((F, G)) = (f^{-1}(F), f^{-1}(G))$. We show that Φ satisfies the conditions of Lemma 2.1.6. Let $\sigma = \{(F_i, G_i)\}_{i=1}^n \in M_{\mathscr{P}_\alpha}$. Then, by Lemma 4.1.9, $\{(C \cap F_i, C \cap G_i)\}_{i=1}^n$ is a collection of distinct pairs of opposite faces of a cell C in $\mathscr{C}_\alpha$. H2 and Lemma 4.1.4 give us that

$$\{(f^{-1}(C \cap F_i), f^{-1}(C \cap G_i))\}_{i=1}^n$$

is essential in X.

Consequently $\Phi(\sigma) = \{(f^{-1}(F_i), f^{-1}(G_i))\}_{i=1}^n$ is essential in X_1 so that

$$\Phi(\sigma) \in M_{L(f)}.$$

We conclude that Φ satisfies the conditions of Lemma 2.1.6 so that by Proposition 4.1.10

$$\operatorname{Ord} M_{L(f)} \geqslant \operatorname{Ord} M_{\mathscr{P}_\alpha} \geqslant \alpha.$$

4.1.13. Theorem. *Let f be an essential mapping from a space X onto J^α. Then*

$$\dim X \geqslant \alpha.$$

Proof. Since $M_{L(f)}$ is a subset of $M_{L(X)}$ we obtain by Lemma 4.1.12 the (in) equalities

$$\dim X = \operatorname{Ord} M_{L(X)} \geqslant \operatorname{Ord} M_{L(f)} \geqslant \alpha . \ \blacksquare$$

In Section 5.2 we will show that the converse of Theorem 4.1.13 is not true. We will construct a compact space X satisfying $\dim X \geqslant \omega_0+1$ which does not admit an essential mapping onto J^{ω_0+1}. The converse does hold when α is a limit ordinal. For this see Section 4.3. We finish this section with a lemma which is the key to the results in the next sections.

4.1.14. LEMMA. *Let X be a space and let $\{F_i\}_{i=1}^{\infty}$ be a sequence of closed sets in X such that $\{F_i\}_{i=1}^{\infty}$ is a pairwise disjoint clopen collection in $F = \overline{\bigcup_{i=1}^{\infty} F_i}$. Let α be a countable limit ordinal and let $\{\beta\colon \beta<\alpha\}$ be indexed as $\{\beta_i\colon i = 1, 3, 5, \ldots\}$ and let $\beta_i = 1$ for $i = 2, 4, 6, \ldots$ Moreover, for some $n \geqslant 0$, let $g\colon X \to I^n$ be a continuous mapping. If for each $i = 1, 2, \ldots$ there exists a mapping $f_i\colon F_i \to J^{\beta_i}$ such that*

$$(1) \qquad f_i \times (g|F_i)\colon F_i \to J^{\beta_i}\times I^n = J^{\beta_i+n}$$

is essential then we can find a mapping $f\colon X \to J^{\alpha}$ such that

$$f\times g\colon X \to J^{\alpha}\times I^n = J^{\alpha+n}$$

is essential.

Proof. For even i we consider f_i as a map from F_i to $A_{\alpha}^{\beta_i-1}\cup\{p_{\alpha}\}$. Let h be the constant function from $F-\bigcup_{i=1}^{\infty} F_i$ to p_{α}.

Since $\{F_i\}_{i=1}^{\infty}$ is a pairwise disjoint collection of clopen subsets in F one easily verifies that

$$f = h\cup\bigcup_{i=1}^{\infty} f_i\colon F \to \{p_{\alpha}\}\cup \bigcup_{i=1,3,5} (J^{\beta_i}\cup A_{\alpha}^{\beta_i}) = J^{\alpha}$$

is continuous, and $f\times(g|F)\colon F\to J^{\alpha}\times I^n$ is a continuous surjection.

Since every cell C in $J^{\alpha}\times I^n = J^{\alpha+n}$ can be considered as a cell in one of the subspaces $J^{\beta_i}\times I^n$, $i = 1, 3, 5, \ldots$ and we also have that

$$(f\times g)|F_i = f_i\times(g|F_i)$$

is essential, by H2 we obtain that $f\times(g|F)$ is essential. Now extend f over X to a map f', H1. Then also $f'\times g\colon X\to J^{\alpha+n}$ is essential. ■

4.2. Essential mapping II; a characterization. As noted before we cannot characterize the value of dim directly by means of essential mappings into the cubes J^{α}. However for locally compact spaces we can characterize the value of dim by essential mappings as follows:

4.2.1. THEOREM. *Let X be locally compact and $\alpha<\omega_1$. Then $\dim X \geqslant \alpha$ iff $X\times C$ admits an essential map onto J^{α}.*

For the proof of the necessity we need the following proposition.

4.2.2. PROPOSITION. *Let X be a space. Let $\operatorname{Ord} M_{L(X)}^{\sigma} \geqslant \alpha$ for some*

$$\sigma = \{(A_i, B_i)\}_{i=1}^{n} \in \{\emptyset\}\cup M_{L(X)}, \quad n\geqslant 0$$

and let, for each $i = 1, \ldots, n$, $g_i'\colon X\to I$ be a continuous map such that $g_i'(A_i) = 0$ and $g_i'(B_i) = 1$. For each $i = 1, \ldots, n$ define $g_i\colon X\times C\to I$ by $g_i(x, c) = g_i'(x)$ and let

$$g = \bigtimes_{i=1}^{n} g_i\colon X\times C\to I^n .$$

Then we can find a mapping $f\colon X\times C\to J^{\alpha}$ such that the mapping

$$f\times g\colon X\times C\to J^{\alpha}\times I^n = J^{\alpha+n}$$

is essential.

Proof. By transfinite induction on the number α. If $\alpha = 0$ then $J^0 = \{p_0\}$ is a single point. Apply Lemma 4.1.3 Assume that the proposition is true for all ordinal numbers $\beta<\alpha$. Assume first that α is a successor, say $\alpha = \beta+1$. There is some $a = (A_{n+1}, B_{n+1}) \in L(X)$ such that $a\notin\sigma$, $\sigma\cup\{a\}\in M_{L(X)}$ and $\operatorname{Ord} M_{L(X)}^{\sigma\cup\{a\}} \geqslant \beta$. Let $g_{n+1}'\colon X\to I$ be a continuous mapping such that $g_{n+1}'(A_{n+1}) = 0$ and $g_{n+1}'(B_{n+1}) = 1$. Let $g_{n+1}\colon X\times C\to I$ be defined by

$$g_{n+1}(x, c) = g_{n+1}'(x) .$$

Then by our inductive hypothesis there is a mapping $f'\colon X\times C\to J^{\beta}$ such that

$$f'\times(g_{n+1}\times g)\colon X\times C\to J^{\beta}\times I\times I^n = J^{\alpha+n}$$

is essential. Clearly,

$$f = f'\times g_{n+1}\colon X\times C\to J^{\beta}\times I = J^{\alpha}$$

is the required map. Assume now that α is a limit ordinal. Let us first consider the Cantor set C. Put $C_i = [n_i, m_i]\cap C$ for each $i = 1, 2, 3, \ldots$ where the points n_i and m_i are defined inductively as follows:

$$n_1 = 0, \quad m_i = n_i+1/3^i \quad \text{and} \quad n_{i+1} = m_i+1/3^i .$$

Observe that

(a) $\{C_i\}_{i=1}^{\infty}$ is a disjoint clopen collection in C,

(b) $C-\bigcup_{i=1}^{\infty} C_i = \{1\}$,

(c) C_i and C are homeomorphic for each i.

We want to apply Lemma 4.1.14 for the construction of f. Because α is a countable ordinal number, the collection $\{\beta\colon \beta<\alpha\}$ can be indexed as $\{\beta_i\colon i = 1, 3, 5, \ldots\}$. Let $\beta_i = 1$ for $i = 2, 4, 6, \ldots$ Since $\operatorname{Ord} M_{L(X)}^{\sigma} \geqslant \beta_i$ for $i = 1, 2, \ldots$, by our inductive hypothesis and (c), we can find mappings

$$f_i\colon F_i = X\times C_i\to J^{\beta_i} \quad \text{for } i = 1, 2, \ldots$$

such that

$$f_i \times (g|F_i): F_i \to J^{\beta_i} \times I^n$$

is essential. Then we may apply Lemma 4.1.14, which gives the required mapping

$$f: X \times C \to J^{\alpha}. \ ■$$

Taking $\sigma = \emptyset$ in Proposition 4.2.2 we get:

4.2.3. THEOREM. *Let X be a space such that $\dim \geqslant \alpha$ for some countable ordinal number α. Then we can find an essential mapping $f: X \times C \to J^{\alpha}$.* ■

For the sufficiency we use the following argumentation. The existence of an essential mapping from $X \times C$ onto J^{α} implies according to Theorem 4.1.10 that $\dim(X \times C) \geqslant \alpha$. But by Theorem 3.5.7 $\dim X = \dim(X \times C)$ whenever X is locally compact, so we are done. This completes the proof of Theorem 4.2.1. ■

Together with the results of Section 3.3 we obtain the following:

4.2.4. THEOREM. *Let X be a compact space and let $\alpha < \omega_1$. Then the following statements are equivalent*:

(1) $\operatorname{index} X \geqslant \omega_0^{\alpha}$,
(2) $\dim X \geqslant \alpha$, *and*
(3) *there exists an essential mapping $f: X \times C \to J^{\alpha}$.* ■

4.3. Essential mappings III; the difference; limit ordinal numbers. To be able to state the results of this section in an economic way we define (informally) for a space X

$$\operatorname{Ess} X = \sup\{\alpha < \omega_1: X \text{ admits an essential map onto } J^{\alpha}\}.$$

R. Pol, [P2], has shown that the difference between $\operatorname{Ind} X$ and $\operatorname{Ess} X$ can be arbitrarily large. We show that the difference between $\dim X$ and $\operatorname{Ess} X$ is at most 1.

4.3.1. THEOREM. *If $\dim X \geqslant \alpha+1$ then X admits an essential map onto J^{α}.*

Combining this result and Theorem 4.1.13 we may conclude

$$\operatorname{Ess} X \leqslant \dim X \leqslant \operatorname{Ess} X + 1.$$

The second result shows among other things that if $\operatorname{Ess} X = \alpha < \omega_1$ then X actually admits an essential map onto J^{α}.

4.3.2. THEOREM. *If X a space and $\alpha < \omega_1$ is a limit ordinal then $\dim X \geqslant \alpha$ iff X admits an essential map onto J^{α}.*

Clearly if $\alpha = \operatorname{Ess} X$ is a successor then it is a maximum so X admits an essential map onto J^{α}. If α is a limit then by Theorem 4.1.13 $\dim X \geqslant \beta$ for every $\beta < \alpha$. Hence $\dim X \geqslant \alpha$, but then we can apply Theorem 4.3.2.

4.3.3. PROPOSITION. *Let X be a space. In addition, let $\operatorname{Ord} M^{\sigma}_{L(X)} \geqslant \alpha$ for some $\sigma = \{(A_i, B_i)\}_{i=0}^{n} \in M_{L(X)}$, $n \geqslant 0$ and let for every $i = 1, \ldots, n$, $g_i: X \to I$ be a continuous mapping such that $g_i(A_i) = 0$ and $g_i(B_i) = 1$ and let*

$$g = \bigtimes_{i=1}^{n} g_i: X \to I^n.$$

Then we can find a mapping $f: X \to J^{\alpha}$ such that the mapping

$$f \times g: X \to J^{\alpha} \times I^n = J^{\alpha+n}$$

is essential.

Proof. By transfinite induction on the number α. If $\alpha = 0$ then $J^0 = \{p_0\}$ is a single point. Apply Lemma 4.1.3. Assume that the proposition holds for all ordinal numbers $\beta < \alpha$. Assume first that α is a successor, say $\alpha = \beta+1$. There exists an $a = (A_{n+1}, B_{n+1}) \in L(X)$ such that $a \notin \sigma$, $\sigma \cup \{a\} \in M_{L(X)}$ and

$$\operatorname{Ord} M^{\sigma\cup\{a\}}_{L(X)} \geqslant \beta.$$

Take a continuous mapping $g_{n+1}: X \to I$ with $g_{n+1}(A_{n+1}) = 0$ and $g_{n+1}(B_{n+1}) = 1$. Then by our inductive hypothesis there is a mapping $f': X \to J^{\beta}$ such that

$$f' \times (g_{n+1} \times g): X \to J^{\beta} \times I \times I^n = J^{\beta+1} \times I^n$$

is essential. Clearly $f = f' \times g_{n+1}$ is as required.

Now let us assume that α is a limit ordinal. We want to apply Lemma 4.1.14. Put $b = (A_0, B_0)$ and $\gamma = \sigma - \{b\}$. Using the normality of X we can find a sequence $O_1, O_2, \ldots$ of open sets in X such that

$$A_0 \subset O_1 \subset \bar{O}_1 \subset O_2 \subset \bar{O}_2 \subset \ldots \subset X - B_0.$$

Putting $F_i = \operatorname{Fr} O_i$ we see by virtue of Proposition 3.2.1 (1) that $M^{b}_{L(X)} \subset \tilde{M}_{L(X)|F_i}$ for $i = 1, 2, \ldots$ Consequently, for $i = 1, 2, \ldots$

$$M^{\sigma}_{L(X)} = M^{\{b\}\cup\gamma}_{L(X)} = (M^{b}_{L(X)})^{\gamma} \subset \tilde{M}^{\gamma}_{L(X)|F_i}$$

so that

$$\operatorname{Ord} \tilde{M}^{\gamma}_{L(X)|F_i} \geqslant \alpha.$$

Because α is a countable ordinal number the set $\{\alpha: \beta < \alpha\}$ can be indexed as $\{\beta_i: i = 1, 3, 5, \ldots\}$. For $i = 2, 4, 6, \ldots$ let $\beta_i = 1$. Since $\beta_i < \alpha$ there exist $b_i \in L(X)$ such that $\operatorname{Ord} \tilde{M}^{\gamma\cup\{b_i\}}_{L(X)|F_i} \geqslant \beta_i$ for $i = 1, 2, \ldots$ Then by our inductive hypothesis we can find mappings $f_i: F_i \to J^{\beta_i}$ for $i = 1, 2, \ldots$ such that

$$f_i \times g|F_i: F_i \to J^{\beta_i} \times I^n = J^{\beta_i+n}$$

is essential. An application of Lemma 4.1.14 gives us the required mapping $f: X \to J^{\alpha}$. ■

Theorem 4.3.1 now follows by taking $n = 0$ in Proposition 4.3.3. In the following lemma we shall apply Theorem 3.4.4.

4.3.4. LEMMA. *Let X be a space and let α be a countable limit ordinal such that* $\dim X \geqslant \alpha$. *Moreover, let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of ordinal numbers such that $\alpha_i < \alpha$ for every $i = 1, 2, \ldots$ There exist sequences $\{F_i\}_{i=1}^{\infty}$ and $\{G_i\}_{i=1}^{\infty}$ of closed sets of X such that for $i = 1, 2, \ldots$*

(1) $G_{i+1} \subset G_i$,

(2) $F_i \subset G_i - G_{i+1}$,

(3) $\dim F_i \geqslant \alpha_i$, *and*

(4) $\dim G_i \geqslant \alpha$.

Proof. Put $G_1 = X$ and assume that G_i has been constructed. Since $\alpha_i < \alpha$ we can find some $(A, B) \in L(X)$ such that

$$\operatorname{Ord} \tilde{M}^{(A,B)}_{L(X)|G_i} \geqslant \alpha_i .$$

Let W be an open set in G_i such that $A \cap G_i \subset W \subset \overline{W}^{G_i} \subset G_i - B$. Put $C = \overline{W}^{G_i}$ and $D = G_i - W$. Then $G_i = C \cup D$. Because α is a limit ordinal, by virtue of Theorem 3.4.4 we have

(i) $\dim C \geqslant \alpha$ or

(ii) $\dim D \geqslant \alpha$.

Without loss of generality (i) holds. Let $G_{i+1} = C$ and O be an open set in G_i such that

$$A \cap G_i \subset C \subset O \subset \overline{O}^{G_i} \subset G_i - B .$$

Then by Corollary 3.2.2 for $F_i = \operatorname{Fr}_{G_i} O$ we have by our choice of (A, B) that $\dim F_i \geqslant \alpha_i$. Consequently, F_i and G_{i+1} are as required. ■

4.3.5. Proof of Theorem 4.3.2. "⇒": Theorem 4.1.13.

"⇒": Write $\{\beta: \beta < \alpha\}$ as $\{\beta_i: i = 1, 3, 5, \ldots\}$ and for $i = 2, 4, 6, \ldots$ let $\beta_i = 1$. Put $\alpha_i = \beta_i + 1$ for $i = 1, 2, \ldots$ Then we can find a sequence $\{F_i\}_{i=1}^{\infty}$ such as in Lemma 4.3.4. By Theorem 4.3.1 we can find mappings f_i, $i = 1, 2, \ldots$, such that

$$f_i\colon F_i \to J^{\beta_i}$$

is essential. Consequently by using Lemma 4.1.14 (the special case that $n = 0$), we obtain the required map. ■

4.4. Essential mappings IV; strongly countable dimensional spaces. As mentioned before, in Section 5.2 we shall construct a compact space X with $\dim X = \omega_0 + 1$ for which there does not exist an essential map $f\colon X \to J^{\omega_0+1}$. The space X contains no finite-dimensional open subsets and is c.d. but not s.c.d., Lemma 1.2.4.

It is natural to ask whether for s.c.d. spaces it is possible to improve Theorem 4.2.1. The answer to this question is rather surprising. We shall prove that if X is s.c.d. and if α is the successor of a limit ordinal then $\dim X \geqslant \alpha$ iff there is an map $f\colon X \to J^{\alpha}$. In Section 5.3 we shall construct a compact s.c.d. space X such that $\dim X \geqslant \omega_0 + 2$ which does not admit an essential map $f\colon X \to J^{\omega_0+2}$. Consequently, our result is best possible.

We shall need the following lemma.

4.4.1. LEMMA. *Let X be a space and $\sigma = \{(A_i, B_i)\}_{i=1}^{n} \in M_{L(X)}$. If $E = \bigcap_{i=1}^{n} \overline{X - (A_i \cup B_i)}$ then the following holds:*

$$M^{\sigma}_{L(X)} = \tilde{M}^{\sigma}_{L(X)|E} .$$

Proof. Clearly $\tilde{M}^{\sigma}_{L(X)|E} \subset M^{\sigma}_{L(X)}$. Also, $M^{\sigma}_{L(X)} \subset \tilde{M}^{\sigma}_{L(X)|E}$ for let

$$\gamma = \{(A_i, B_i)\}_{i=n+1}^{m} \in M^{\sigma}_{L(X)}$$

and fix open sets O_i, $i = 1, \ldots, m$, such that $A_i \subset O_i \subset \overline{O}_i \subset X - B_i$. Then $F = \bigcap_{i=1}^{n} \operatorname{Fr} O_i \subset E$ and $\sigma \cup \gamma = \{(A_i, B_i)\}_{i=1}^{m}$ is essential, so that

$$\emptyset \neq \bigcap_{i=1}^{m} \operatorname{Fr} O_i = \bigcap_{i=1}^{m} \operatorname{Fr} O_i \cap F \subset \bigcap_{i=1}^{m} \operatorname{Fr} O_i \cap E .$$

Consequently, $\sigma \cup \gamma \in \tilde{M}_{L(X)|E}$ so that $\gamma \in \tilde{M}^{\sigma}_{L(X)|E}$. ■

We now turn to s.c.d. spaces. We refer to 1.2.3 for the definition of the sets A_η and P_η.

4.4.2. THEOREM. *Let X be* s.c.d. *and* S-w.i.d. *Then*

(1) $A_\alpha = \emptyset$ *for some* α.

Put $O_\xi = X - A_\xi$ for $\xi \leqslant \alpha$. Then

(2) *for every closed $F \subset X$, $\eta = \min\{\xi\colon F \subset O_\xi\}$ is a successor ordinal.*

Proof. The first part follows from the fact that by Theorem 1.2.6. A_{ω_0} is compact so we can apply Theorem 1.2.5 on A_{ω_0}.

For the second part observe that the case $F \cap A_{\omega_0} = \emptyset$ is already covered by (3) of Theorem 1.2.6. For $F \cap A_{\omega_0} \neq \emptyset$, observe that $F \cap A_{\omega_0}$ is compact; each O_ξ is open in X and for every limit γ we have $O_\gamma = \bigcup_{\xi<\gamma} O_\gamma$ is an increasing union. Consequently, if $F \subset O_\gamma$ for some limit ordinal γ then

$$F = (F \cap O_{\omega_0}) \cup (F \cap A_{\omega_0}) \subset O_{\omega_0} \cup O_\xi = O_\xi$$

for some $\omega_0 < \xi < \gamma$. ■

Using this theorem we can prove:

4.4.3. LEMMA. *Let X be a* s.c.d. *space such that for some countable limit ordinal α we have $\dim X = \alpha + 1$. Moreover, let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of ordinal numbers such that $\alpha_i < \alpha$, for every $i = 1, 2, \ldots$ Then there exist sequences $\{F_i\}_{i=1}^{\infty}$ and $\{G_i\}_{i=1}^{\infty}$ of closed sets in X and an $a \in L(X)$ such that for $i = 1, 2, \ldots$*

(1) $G_{i+1} \subset G_i$,

(2) $F_i \subset G_i - G_{i+1}$,

(3) $\operatorname{Ord} \tilde{M}^{a}_{L(X)|F_i} \geqslant \alpha_i$, *and*

(4) $\operatorname{Ord} \tilde{M}^{a}_{L(X)|G_i} \geqslant \alpha$.

Proof. Since $\dim X$ exists, X is S-w.i.d. by Theorem 3.1.3. Let $\eta = \min\{\eta'$: there is a closed set F in X with $F \subset O_{\eta'}$ and $\dim F = \alpha+1\}$. Observe that by the second part of Theorem 4.4.2 η is a successor. Put $\gamma = \lambda(\eta)$ and $n = n(\eta) > 0$. Let G_1 be a closed subset of X with $G_1 \subset O_\eta$ and $\dim G_1 = \alpha+1$. Then let $a = (A, B) \in L(X)$ be such that

$$\operatorname{Ord} \tilde{M}^{a}_{L(X)|G_1} = \alpha .$$

Assume that G_i has been constructed. By Lemma 2.1.5, since $\alpha_i+n+1 < \alpha$, we can find some $\sigma = \{(A_j, B_j)\}_{j=1}^{n+1} \in \operatorname{Fin} L(X)$ such that $a \notin \sigma$ and

$$\operatorname{Ord} \tilde{M}^{\sigma\cup\{a\}}_{L(X)|G_i} \geqslant \alpha_i .$$

We have

$$G_i \subset O_\eta = O_\gamma \cup P_n(A_\gamma)$$

so that

$$G = G_i - O_\gamma \subset P_n(A_\gamma) .$$

Observe that G is closed and since $\dim P_n(A_\gamma) \leqslant n$, $\dim G \leqslant n$.

By Lemma 1.1.4 we can find open sets O_j in X for $j = 1, \dots, n+1$ such that

$$A_j \subset O_j \subset \bar{O}_j \subset X - B_j \quad \text{and} \quad \bigcap_{j=1}^{n+1} \operatorname{Fr} O_j \cap G = \emptyset .$$

Put $F_i = \bigcap_{j=1}^{n+1} \operatorname{Fr} O_j \cap G_i \subset G_i - G = G_i \cap O_j$. By virtue of Proposition 3.2.1(1) we see that $\tilde{M}^{\sigma}_{L(X)|G_i} \subset \tilde{M}_{L(X)|F_i}$.

Consequently,

$$\tilde{M}^{\sigma\cup\{a\}}_{L(X)|G_i} = (\tilde{M}^{\sigma}_{L(X)|G_i})^a \subset \tilde{M}^{a}_{L(X)|F_i}$$

so that

$$\operatorname{Ord} \tilde{M}^{a}_{L(X)|F_i} \geqslant \alpha_i .$$

Now let U and V be open subsets of G_i such that

$$F_i \subset U \subset \bar{U} \subset O_\gamma, \quad A_\gamma \cap G_i = G_i - O_\gamma \subset V \subset \bar{V} \subset G_i - F_i$$

and

$$U \cup V = G_i .$$

Now apply Proposition 3.4.5: since α is a limit ordinal we must have

(i) $\operatorname{Ord} \tilde{M}^{a}_{L(X)|\bar{U}} \geqslant \alpha$ or

(ii) $\operatorname{Ord} \tilde{M}^{a}_{L(X)|\bar{V}} \geqslant \alpha$ or

(iii) $\operatorname{Ord} \tilde{M}^{b}_{L(X)|G_i} \geqslant \alpha$ where $b = (G_i - U, G_i - V)$.

If (i) is true then $\dim \bar{U} = \alpha+1$ and $\bar{U} \subset O_\gamma$ which is impossible because $\gamma < \eta$ and by our choice of η.

If (iii) is true then by Lemma 4.4.1 for $E = \bar{U} \cap \bar{V} \subset O_\gamma$ we have $\dim E = \alpha+1$ which is again impossible.

We conclude that (ii) must hold. Clearly $G_{i+1} = \bar{V}$ is as required. ■

We can now prove the main result in this section.

4.4.4. THEOREM. *Let X be a s.c.d. space and let α be a countable limit ordinal number. Then $\dim X \geqslant \alpha+1$ if and only if there exists an essential mapping f: $X \to J^{\alpha+1}$.*

Proof. "$\Leftarrow$": Theorem 4.1.13.

"$\Rightarrow$": If $\dim X > \alpha+1$ then we are done by Theorem 4.3.1. For the case $\dim X = \alpha+1$, write $\{\beta: \beta < \alpha\}$ as $\{\beta_i: i = 1, 3, 5, \dots\}$ and for $i = 2, 4, 6, \dots$ let $\beta_i = 1$. Putting $\alpha_i = \beta_i+1$ for $i = 1, 2, \dots$ we can find a sequence $\{F_i\}_{i=1}^{\infty}$ and $a \in L(X)$ as in Lemma 4.4.3. Let g: $X \to I$ be such that $g(A) = 0$ and $g(B) = 1$. Then since $\operatorname{Ord} \tilde{M}^{\{a,b_i\}}_{L(X)|F_i} \geqslant \beta_i$ for some $b_i \in L(X)$, $b_i \neq a$ for $i = 1, 2, \dots$ we can find by Proposition 4.3.3 a mapping f_i: $X \to J^{\beta_i}$ for $i = 1, 2, \dots$ such that

$$f_i \times (g|F_i)\colon F_i \to J^{\beta_i} \times I$$

is essential.

Then by Lemma 4.1.14 we can find an essential mapping from X onto $J^{\alpha+1}$. ■

Chapter V. Counterexamples

In this chapter we will construct some counterexamples to show that our results on dim are best possible. The spaces will be compact and metrizable. The first space X satisfies $\dim X = \omega_0$ and $\operatorname{Ind} X = \omega_0+1$ showing that in general dim and Ind are different. Thus Theorem 3.2.4 is best possible. The second space Y satisfies $\dim Y = \omega_0+1$, yet Y admits no essential map onto J^{ω_0+1}. This concerns Theorems 4.2.3 and 4.3.1. Finally, the third space Z shows that Theorem 4.4.4 is best possible: Z is s.c.d., $\dim Z = \omega_0+2$ yet Z admits no essential map onto J^{ω_0+2}.

5.1. A space X with $\dim X = \omega_0$ and $\operatorname{Ind} X = \omega_0+1$. In this section we first prove that the space X constructed in [B+D] satisfies $\dim X = \omega_0$ and $\operatorname{Ind} X = \omega_0+1$.

5.1.1. EXAMPLE. According to [B+D] there exists a compact metric space X such that

(1) $\operatorname{Ind} X = \omega_0+1$,

(2) X admits no essential mapping onto J^{ω_0+1}, and

(3) X is s.c.d.

We prove that also

(4) $\dim X = \omega_0$.

Proof. $\dim X \geqslant \omega_0$ since X contains copies of I^n for every $n = 1, 2, \dots$ The inequality $\dim X \leqslant \omega_0$ follows readily applying (2), (3) and Theorem 4.4.4. ■

In [P1] R. Pol constructed a w.i.d. compact metric space which is not c.d. From this fact he was able to derive in [P2] the existence of a class $\mathscr{T}$ of s.c.d. compact metric spaces satisfying

$$\sup\{\operatorname{index} T\colon T \in \mathscr{T}\} < \omega_1 \quad \text{and} \quad \sup\{\operatorname{Ind} T\colon T \in \mathscr{T}\} = \omega_1 .$$

Clearly by Theorem 3.3.8, $\sup\{\dim T\colon T\in\mathcal{T}\}<\omega_1$. Consequently, we see that the gap between dim and Ind can be arbitrarily large.

R. Pol [P2; Remark 3.3(a)] put forth the problem of obtaining such spaces in a more explicit way. We restate Pol's problem as follows.

5.1.2. PROBLEM. Construct compact metric (s.c.d.) spaces T such that the family $\mathcal{T}$ consisting of these spaces satisfies

$$\sup\{\dim T\colon T\in\mathcal{T}\}<\omega_1 \quad\text{and}\quad \sup\{\operatorname{Ind} T\colon T\in\mathcal{T}\}=\omega_1 .$$

Since we have constructed a space X with $\dim X=\omega_0$ and $\operatorname{Ind} X=\omega_0+1$ the following more specific question arises:

5.1.3. PROBLEM. Can we construct for every ordinal α, $\omega_0<\alpha<\omega_1$, a compact metric (s.c.d.) space T_α such that

$$\dim T_\alpha=\omega_0 \quad\text{and}\quad \operatorname{Ind} T_\alpha=\alpha?$$

As was mentioned by R. Pol, it should be noted that if we have constructed such spaces T_α, an application of Theorem 3.3.9 gives us a w.i.d. compact space X that contains every T_α topologically.

This space X would have no Ind so that by [E3; Th. 4.2.] X would not be c.d. Consequently, we would then have constructed another, more explicit, example of a w.i.d. compact space X which is not c.d.

5.2. The failure of Henderson's characterization for dim. In this and the next section we will use upper semicontinuous decompositions to construct our counter-examples. We fix some notation on decompositions and prove two results that we will need in all constructions.

Let E be a decomposition (partition) of a space X. For $x\in X$, $E[x]$ denotes the unique element of E containing x. Also for $A\subset X$ we put $E[A]=\bigcup_{x\in A}E[x]$. Moreover, if d is a metric on X then

$$\mu(E)=\sup\{\operatorname{diam}_d(E[x])\colon x\in X\}.$$

5.2.1. DEFINITION. A decomposition E of a space X is *upper semicontinuous* (abbreviated u.s.c.) iff for every closed set $A\subset X$, $E[A]$ is closed. It is easily seen that E is u.s.c. iff the corresponding quotient map is closed.

5.2.2. PROPOSITION. *Let X be a space and d a metric on X. Let $\{C_n\colon n\in N\}$ be a pairwise disjoint collection of closed subsets of X; let for each n E_n be an* u.s.c. *decomposition of C_n such that*

$$\lim_{n\to\infty}\mu(E_n)=0 .$$

Then $E=\bigcup_{n=1}^{\infty}E_n\cup\{\{x\}\colon x\in X-\bigcup_{n=1}^{\infty}C_n\}$ *is* u.s.c.

Proof. Let $A\subset X$ be closed. Note that $E[A]=A\cup\bigcup_{n=1}^{\infty}E_n[A\cap C_n]$. Let $x\in X-E[A]$. First pick $\varepsilon>0$ such that $B(x,3\varepsilon)\cap A=\emptyset$. Second find $n_0\in N$ such that $\mu(E_n)<\varepsilon$ for $n\geqslant n_0$. Third $\bigcup_{n=1}^{n_0}E_n[A\cap C_n]$ is closed so pick $\delta\in(0,\varepsilon)$ such that

$$B(x,\delta)\cap\bigcup_{n=1}^{n_0}E_n[A\cap C_n]=\emptyset .$$

Now also $B(x,\delta)\cap E[A]=\emptyset$. For if $n>n_0$ and $y\in A\cap C_n$ then $d(x,y)>3\varepsilon$ and so

$$B(x,\delta)\cap E(y)\subset B(x,\varepsilon)\cap B(y,\varepsilon)=\emptyset .$$

It follows that $E[A]$ is closed. ■

Without proof we state

5.2.3. THEOREM. *If X is compact metric and E is an* u.s.c. *decomposition of X then the quotient space X/E is compact and metrizable as well.*

For a proof use [E1; 3.2.11+4.2.13].

We will construct a compact metric space X with $\dim X\geqslant\omega_0+1$, but which cannot be mapped onto J^{ω_0+1} by an essential mapping. By this we see that the converse of Theorem 4.1.13 is not true. We also see that the characterization of dim by essential mappings to Henderson's cubes stated in Theorem 4.2.1 is indeed one of the nicest we can obtain.

We construct the space as a quotient space of the unit interval I by taking a proper semicontinuous decomposition.

5.2.4. EXAMPLE. A compact space X such that $\dim X=\omega_0+1$ and which does not admit an essential map onto J^{ω_0+1}.

Construction. To begin, we put for $n\geqslant 3$ and $i\in\{1,\dots,2^n\}$

$$U_{ni}=((i-1)\cdot 2^{-n},\, i\cdot 2^{-n}) \quad\text{and}\quad \mathcal{U}_n=\{U_{ni}\colon i=1,\dots,2^n\}.$$

Next we choose, by induction on n, a copy C_{ni} of the Cantor set in U_{ni} for each i as follows. For $n=3$ choose $C_{31},\dots,C_{38}$ arbitrarily. Let $n>3$ and assume the C_{mi} are chosen for $1\leqslant i\leqslant 2^m$, $m<n$. Let $D_n=\bigcup_{m<n}\bigcup_{i\leqslant 2^m}C_{mi}$. Then D_n is closed and nowhere dense in I, so for each $i\leqslant 2^n$ we can find a nonempty open interval $O_{ni}\subset U_{ni}-D_n$. Choose C_{ni} inside O_{ni}.

We list the properties of the C_{ni}:

C1: $\{C_{ni}\colon 1\leqslant i\leqslant 2^n,\ n\geqslant 3\}$ is disjoint,
C2: $C_{ni}\subset U_{ni}$,
C3: $\operatorname{diam}(C_{ni})<2^{-n}$,
C4: $\operatorname{diam}(C_{ni}\cup C_{ni+1})<2^{-n+1}$.

Next let for each n, $C_n = \bigcup_{i=1}^{2^n} C_{ni}$ and let $q_n\colon C_n \to I^n$ be a continuous surjection such that for every i

$$q_n(C_{ni}) = \{x \in I^n\colon (i-1)\cdot 2^{-n} \leqslant x_1 \leqslant i\cdot 2^{-n}\}\,.$$

Note that this implies that

C5: for each $y \in I^n$, $q_n^{-1}(y) \subset C_{ni} \cup C_{ni+1}$ for a (unique) $i \in \{1, \ldots, 2^n-1\}$.

Let $E = \{\{x\}\colon x \notin \bigcup_{n=3}^{\infty} C_n\} \cup \bigcup_{n=3}^{\infty} \{q_n^{-1}(y)\colon y \in I^n\}$. Then E is upper semicontinuous. We let $X = I/E$ and $q\colon I \to X$ will be the quotient map. We shall now show that X is as required.

The following claim will be very useful in what follows.

5.2.5. Claim. *For $n \geqslant 3$: if $i, j \in \{1, \ldots, 2^n\}$ and $|i-j| > 1$ then*

$$q(\overline{U}_{ni}) \cap q(\overline{U}_{nj}) \subset q(D_n)$$

(*for* $n = 3$, $D_n = \emptyset$).

Proof. Let $x \in \overline{U}_{ni}$ and $y \in \overline{U}_{nj}$ be such that $q(x) = q(y)$. Take the unique m for which $x, y \in C_m$. We must show $m < n$.

Set $r = q(x)$. Then for some $k < 2^m$

$$q^{-1}(r) = q_m^{-1}(r) \subset U_{mk} \cup U_{mk+1}\,.$$

The assumption $m \geqslant n$ now readily gives us an $l < 2^n$ with

$$q^{-1}(r) \subset U_{nl} \cup U_{nl+1}$$

contradicting the assumption on x and y. ■

5.2.6. Claim. $\dim X \geqslant \omega_0 + 1$.

Proof. Let $F = q([0, 1/8])$ and $G = q([7/8, 1])$. Now by C4 and C5 for every $x \in X$ we have

$$\operatorname{diam} q^{-1}(x) < 2^{-2} = 1/4$$

so that $q^{-1}(F) \subset [0, 3/8]$ and $q^{-1}(G) \subset [5/8, 1]$. It follows that F and G are closed and disjoint in X. We show that

$$\operatorname{Ord} M_{L(X)}^{(F,G)} \geqslant \omega_0\,.$$

Let $n \geqslant 3$, and let σ_n be the collection of pairs of opposite faces in the n-cube $I^n = q(C_n) \subset X$. Then σ_n is essential in X. Let $A = \{0\} \times I^{n-1}$ and $B = \{1\} \times I^{n-1}$. Then

$$A \subset q(C_{n1}) \subset F \quad\text{and}\quad B \subset q(C_{n2^n}) \subset G$$

and

$$(A, B) \in \sigma_n\,.$$

It follows that $\tau_n = \{(F, G)\} \cup (\sigma_n - \{(A, B)\})$ is essential in X, so that by Lemma 2.1.4.

$$\operatorname{Ord} M_{L(X)}^{(F,G)} \geqslant n-1\,.$$

Since n was arbitrary we conclude that $\operatorname{Ord} M_{L(X)}^{(F,G)} \geqslant \omega_0$. ■

5.2.7. Claim. *X admits no essential mapping onto J^{ω_0+1}.*

Proof. Let $f\colon X \to J^{\omega_0+1}$ be continuous. We find a cell K in J^{ω_0+1} such that $f_K\colon f^{-1}(K) \to K$ is not essential. This is sufficient of course by H2. Write

$$J^{\omega_0+1} = J^{\omega_0} \times I = \{p_{\omega_0}\} \times I \cup \bigoplus_{i=1}^{\infty} (J^i \times I \cup A^i \times I)\,.$$

Set for $i \in N$, $K_i = J^i \times I$ and $P_i = f^{-1}(K_i)$. Note that $\{K_i\colon i \in N\}$ is the collection of cells of J^{ω_0+1}. Finally put

$$F = f^{-1}(J^{\omega_0} \times \{0\})\,, \qquad A = q^{-1}(F)\,,$$
$$G = f^{-1}(J^{\omega_0} \times \{1\})\,, \qquad B = q^{-1}(G)\,.$$

Let $0 = d_0 < c_1 < d_1 < \ldots < d_p < c_{p+1} = 1$ be a sequence of points in I such that for $i = 1, \ldots, p$

$$(c_i, d_i) \cap (A \cup B) = \emptyset$$

and for $i = 1, \ldots, p+1$

$$(d_{i-1}, c_i) \cap A = \emptyset \quad\text{or}\quad (d_{i-1}, c_i) \cap B = \emptyset\,.$$

Let $d_0' = 0$ and $c_{p+1}' = 1$ and find for $i = 1, \ldots, p$ c_i', d_i' and k_i as follows:

1. If $(c_i, d_i) \cap q^{-1}(P_k) = \emptyset$ for every k then let $(c_i', d_i') = (c_i, d_i)$ and $k_i = 1$.
2. If $(c_i, d_i) \cap q^{-1}(P_k) \neq \emptyset$ for some k then let k_i be the least such k.

Next set $O = q^{-1}(f^{-1}(K_k \cup A^k \times I))$; then O is open and $q^{-1}(P_k) \subset O$. Let (c_i', d_i') be a nonempty open interval in $(c_i, d_i) \cap O$. Next set

$$G_A = \bigcup \{[d_{i-1}', c_i']\colon [d_{i-1}, c_i] \cap A \neq \emptyset\}\,,$$
$$G_B = \bigcup \{[d_{i-1}', c_i']\colon [d_{i-1}, c_i] \cap A = \emptyset\}\,.$$

Then $A \subset G_A$, $B \subset G_B$ and G_A and G_B are closed and disjoint. Find $n_0 \in N$ so big that

$$2^{-n_0} < \tfrac{1}{2}\min\{d_i' - c_i'\colon 1 \leqslant i \leqslant p\}\,, \qquad k_i < n_0 \text{ for } i = 1, \ldots, p.$$

Now put

$$U_A = \bigcup \{U \in \mathscr{U}_{n_0}\colon U \cap G_A \neq \emptyset\}\,, \qquad U_B = \bigcup \{U \in \mathscr{U}_{n_0}\colon U \cap G_B \neq \emptyset\}\,.$$

Note that if $U_{n_0 i} \cap G_A \neq \emptyset$ and $U_{n_0 j} \cap G_B \neq \emptyset$ then $|i-j| > 1$, so that by Claim 5.2.5

$$q(\overline{U}_A) \cap q(\overline{U}_B) \subset q(D_{n_0})\,.$$

Now consider $q^{-1}(P_{n_0})$. Note that

$$q^{-1}(P_{n_0}) \cap (c_i', d_i') = \emptyset$$

for $i = 1, \ldots, p$. Hence
$$q^{-1}(P_{n_0}) \subset G_A \cup G_B \subset \bar{U}_A \cup \bar{U}_B .$$
Now $q^{-1}(F) \cap \bar{U}_B = \emptyset$ so that $F \cap q(\bar{U}_B) = \emptyset$. Likewise $G \cap q(\bar{U}_A) = \emptyset$. Also
$$P_{n_0} \subset q(\bar{U}_A) \cup q(\bar{U}_B) .$$
Set $O = P_{n_0} - q(\bar{U}_B)$. The set O is open in P_{n_0} and $\mathrm{Fr}_{P_{n_0}} O \subset q(\bar{U}_B)$. Also $O \subset q(\bar{U}_A)$ so $\mathrm{Fr}_{P_{n_0}} O \subset q(\bar{U}_A)$. Now
$$F \cap P_{n_0} \subset O \subset \bar{O} \subset P_{n_0} - G ,$$
$$\mathrm{Fr}_{P_{n_0}} O \subset q(\bar{U}_A) \cap q(\bar{U}_B) \subset q(D_{n_0}) ,$$
we see that $\dim \mathrm{Fr}_{P_{n_0}} O \leqslant n_0 - 1$. But
$$(F \cap P_{n_0}, G \cap P_{n_0}) \in \{(f^{-1}(A_i), f^{-1}(B_i)) \colon i = 1, \ldots, n_0+1\} = \sigma$$
where $\{(A_i, B_i) \colon i = 1, \ldots, n_0+1\}$ are the pairs of opposite faces in K_{n_0}. It follows from Corollary 3.2.2 that σ is not essential in P_{n_0}. Consequently by Lemma 4.1.4, $f|P_{n_0}$ is not essential. ■

By virtue of Theorem 4.3.1 we have

5.2.8. Corollary. $\dim X \leqslant \omega_0 + 1$. ■

5.3. A strongly countable dimensional example. In this section we show that Theorem 4.4.4 is best possible by constructing a compact s.c.d. space X with $\dim X = \omega_0 + 2$ but which does not admit an essential map onto J^{ω_0+2}.

Before we start the construction we prove two lemmas which are interesting in their own right and which we need when verifying the properties of our space X.

5.3.1. Lemma. *Let X be a space, let $\sigma = \{(A_i, B_i)\}_{i=1}^m$ and $\gamma = \{(C_i, D_i)\}_{i=1}^m$ be elements of* $\mathrm{Fin}L(X)$ *and set* $G = \bigcup_{i=1}^m ((A_i - C_i) \cup (B_i - D_i))$. *Finally assume that*
$$n = \dim \bar{G} < \omega_0 .$$
Then
$$\mathrm{Ord}\, M^{\sigma}_{L(X)} \leqslant \mathrm{Ord}\, M^{\gamma}_{L(X)} + n + 1 .$$

Proof. We show that $\mathrm{Ord}\, M^{\sigma}_{L(X)} \geqslant \alpha + n + 1$ implies $\mathrm{Ord}\, M^{\gamma}_{L(X)} \geqslant \alpha$ for every α. Find $\tau = \{(F_i, G_i)\}_{i=1}^{n+1} \in M^{\sigma}_{L(X)}$ such that $\mathrm{Ord}\, M^{\sigma\cup\tau}_{L(X)} \geqslant \alpha$ and, as $\dim \bar{G} = n$, find open sets O_i, $i = 1, \ldots, n+1$ such that $F_i \subset O_i \subset \bar{O}_i \subset X - G_i$ and $F = \bigcap_{i=1}^{n+1} \mathrm{Fr}\, O_i$ is disjoint from $\bar{G}$.

By Proposition 3.2.1 we have $\mathrm{Ord}\, \tilde{M}^{\sigma}_{L(X)|F} \geqslant \mathrm{Ord}\, M^{\sigma\cup\tau}_{L(X)} \geqslant \alpha$. As $F \cap \bar{G} = \emptyset$ we have
$$A_i \cap F \subset C_i \cap F \quad \text{and} \quad B_i \cap F \subset D_i \cap F$$
for $i = 1, \ldots, m$. Hence by Lemma 3.3.6
$$\mathrm{Ord}\, M^{\gamma}_{L(X)} \geqslant \mathrm{Ord}\, \tilde{M}^{\gamma}_{L(X)|F} \geqslant \mathrm{Ord}\, \tilde{M}^{\sigma}_{L(X)|F} \geqslant \alpha . ■$$

Loosely speaking this lemma says that if σ and τ differ only by sets of dimension $\leqslant n$ then the difference between $\mathrm{Ord}\, M^{\sigma}_{L(X)}$ and $\mathrm{Ord}\, M^{\gamma}_{L(X)}$ is at most $n+1$.

5.3.2. Lemma. *Let X and Y be spaces. Let $f\colon X \to Y$ be closed and continuous. Let $F \subset X$ be closed such that*

— *$\dim F$ and $\dim f(F)$ are finite and*

— *for every closed set G in X disjoint from F $\dim G$ is finite and $f|G$ is a homeomorphism.*

Then $\dim Y \leqslant \omega_0 + \dim F$ and in particular, taking $f = \mathrm{id}$, $\dim X \leqslant \omega_0 + \dim F$.

Proof. Set $n = \dim F$. We must show that $\mathrm{Ord}\, M^{\sigma}_{L(Y)} < \omega_0$ whenever $\sigma \in \mathrm{Fin}L(Y)$ and $|\sigma| = n+1$. Let $\sigma = \{(A_i, B_i)\}_{i=1}^{n+1} \in \mathrm{Fin}L(Y)$. In X find open sets O_i for $i = 1, \ldots, n+1$, such that
$$f^{-1}(A_i) \subset O_i \subset \bar{O}_i \subset X - f^{-1}(B_i)$$
and $G = \bigcap_{i=1}^{n+1} \mathrm{Fr}\, O_i$ is disjoint from F. Let $U_i = Y - f(X - O_i)$; then
$$A_i \subset U_i \subset \bar{U}_i \subset Y - B_i \quad \text{for} \quad i = 1, \ldots, n+1 .$$

Claim. $\bigcap_{i=1}^{n+1} \mathrm{Fr}\, U_i \subset f(F) \cup f(G)$.

Proof of the claim. Let $y \in \bigcap_{i=1}^{n+1} \mathrm{Fr}\, U_i$ and suppose $y \notin f(F)$, then for some $x \in X$ have $f^{-1}(y) = \{x\}$. For every i, $U_i \subset f(O_i)$, hence $\bar{U}_i \subset \overline{f(O_i)} \subset f(\bar{O}_i)$ so that $x \in \bigcap_{i=1}^{n+1} \bar{O}_i$. On the other hand if for some i we have $x \in O_i$ then $f^{-1}(y) \subset O_i$ and $y \in U_i$ which is impossible. Hence $x \in \bigcap_{i=1}^{n+1} \mathrm{Fr}\, O_i$, i.e. $y \in f(G)$. So now by Proposition 3.2.1(2)
$$\mathrm{Ord}\, M^{\sigma}_{L(Y)} \leqslant \dim(f(F) \cup f(G)) < \omega_0 . ■$$

Before we start the construction of our space X, we first define some auxiliary notions. To begin, let $M \subset I^2$ be Sierpiński's universal curve [E2; p. 122, Fig. 12]. Let $\mathcal{K}$ be the collection of open squares removed from I^2, in the construction of M. For $i \in N$ let
$$\mathcal{K}_i = \{K \in \mathcal{K} \colon K \text{ has edge-length } 3^{-i}\} .$$
We shall use the following properties of M and $\mathcal{K}$.

K1: $M = I^2 - (\bigcup \mathcal{K})$; $\dim M = 1$.

K2: $\bar{\mathcal{K}} = \{\bar{K} \colon K \in \mathcal{K}\}$ is pairwise disjoint.

K3: $\mathcal{K} = \bigcup_{i=1}^{\infty} \mathcal{K}_i$.

For $K \in \mathcal{K}$ and $\varepsilon \geqslant 0$ we set
$$D(K, \varepsilon) = \{x \in I^2 \colon d(x, I^2 - K) = \varepsilon\}$$
(here d is the euclidean metric on I^2).

Next we need a space defined by Smirnov [Sm]. For every $\alpha<\omega_1$ Smirnov defines a space S_α as follows:

— $S_0 = \{0\}$,
— $S_{\alpha+1} = S_\alpha \times I$,
— if α is a limit then $S_\alpha = \omega(\bigoplus_{\beta<\alpha} S_\beta)$ (one-point compactification).

We need S_{ω_0+2}. We write

$$S = S_{\omega_0+2} = S_{\omega_0}\times I^2 = \{p\}\times I^2 \cup (\bigoplus_{n<\omega_0} S_n\times I^2).$$

We simply identify $\{p\}\times I^2$ with I^2 and we write T_n for $S_n\times I^2$. We define our space X in two steps.

Step 1. For each $K\in\mathscr{K}$ define $f_K\colon \overline{K}\to I$ by $f_K(x) = d(x, I^2-K\}$. Note that $f_K(\overline{K}) = [0, \frac{1}{2}\cdot 3^{-i}]$, if $K\in\mathscr{K}_i$.

The f_K's induce by Proposition 5.2.1 an u.s.c decomposition of X:

$$E' = \{\{x\}\colon x\notin \bigcup\mathscr{K}\}\cup \bigcup_{K\in\mathscr{K}}\{f_K^{-1}f_K(x)\colon x\in\overline{K}\}.$$

Our first space is $X_1 = S/E'$, the quotient map is denoted by q. By x_K we denote the (unique) point in $q(\overline{K})\cap q(I^2-K)$.

Step 2. Let $i\in N$ and $K\in\mathscr{K}_i$. For $n\geqslant i$ we let

$$I_{K,n} = q(f_K^{-1}([3^{-(n+1)}, \tfrac{1}{2}\cdot 3^{-n}])) = \{q(x)\colon x\in K, 3^{-(n+1)}\leqslant f_K(x)\leqslant \tfrac{1}{2}\cdot 3^{-n}\}.$$

Of course, basically, $I_{K,n} = [3^{-(n+1)}, \frac{1}{2}\cdot 3^{-n}]$. Note that $\{I_{K,n}\colon n\geqslant i\}$ converges to x_K. Next let

$$Q_{K,n} = S_n\times D(K, 3^{-n})\subset T_n$$

for $n\geqslant i$. Then $q|Q_{K,n}$ is a homeomorphism and in X_1, $\{q(Q_{K,n})\colon n\geqslant i\}$ converges to x_K. Let $f_{K,n}\colon I_{K,n}\cup q(Q_{K,n})\to q(Q_{K,n})$ be continuous such that

$$f_{K,n}(I_{K,n}) = q(Q_{K,n}) \quad\text{and}$$
$$f_{K,n}|q(Q_{K,n}) \text{ is the identity.}$$

Again by Proposition 5.2.1 the decomposition E'' determined by

$$\{f_{K,n}\colon K\in\mathscr{K}_i, n\geqslant i, i\in N\}$$

is u.s.c. We let $X = X_1/E''$ and $r\colon X_1\to X$ the quotient map.

We introduce a few more pieces of notation.

— $y_K = r(x_K)$ for each $K\in\mathscr{K}$.
— $s = r\circ q\colon S\to X$ is the quotient map from S onto X.
— $Z = s(I^2)$.
— $s_n = s|T_n$; note that, basically, s_n is the identity ($n\in N$).
— For $K\in\mathscr{K}_i$ and $n\geqslant i+1$ set

$$J_{K,n} = q(\{x\in K\colon \tfrac{1}{2}\cdot 3^{-n}\leqslant f_K(x)\leqslant 3^{-n}\}).$$

Note that for each K and n $r|J_{K,n}$ is a homeomorphism.

We pause to describe a picture of X_1 and X. The space X_1 is obtained from S by replacing each $K\in\mathscr{K}$ by a piece of string, of length $\frac{1}{2}\cdot 3^{-i}$ if $K\in\mathscr{K}_i$. In each square $\{x_K\} = q(\overline{K}-K)$ and x_K is the place where the piece of string belonging to K is attached to $q(M)$.

Then X is obtained from X_1 as follows: whenever $K\in\mathscr{K}_i$ and $n\geqslant i$ the piece $I_{K,n}$ is wound around the cylinder $Q_{K,n}$. The piece $J_{K,n+1}$ stays as it is connecting $Q_{K,n}$ to $Q_{K,n+1}$.

Between T_1 and T_2 there is one piece; between T_2 and T_3 there are 9 pieces; in general T_n is connected to T_{n+1} by 9^{n-1} pieces of string.

The first few properties of X are easily established.

5.3.3. Claim. X *is* s.c.d.

Proof. For each $n\in N$ set $T_n^+ = \bigcup_{i\leqslant n} s(T_n)\cup\bigcup\{r(J_{K,i})\colon K\in\mathscr{K}_i \text{ and } i\leqslant n\}$. Then T_n^+ is closed and $X = s(M)\cup\bigcup_{n\in N} T_n^+$.

— For each n, $\dim T_n^+ = n+2$.
— $s(M)$ is obtained from M by identifying each set of the form $\overline{K}-K$ ($K\in\mathscr{K}$) to the point y_K. So $s(M) = \{y_K\colon K\in\mathscr{K}\}\cup s(I^2-\bigcup\mathscr{K})$ with $\dim\{y_K\colon K\in\mathscr{K}\}^K = 0$, $\dim s(I^2-\bigcup\mathscr{K}) = 1$ so that $\dim s(M)\leqslant 2$.

In fact $s(M)$ is homeomorphic to I^2 so $\dim s(M) = 2$.

We conclude that X is s.c.d. ■

5.3.4. Claim. $\dim X\geqslant \omega_0+2$.

Proof. Let

$$A_1 = s(S_{\omega_0}\times\{0\}\times I),\quad B_1 = s(S_{\omega_0}\times\{1\}\times I),$$
$$A_2 = s(S_{\omega_0}\times I\times\{0\})\quad\text{and}\quad B_2 = s(S_{\omega_0}\times I\times\{1\}).$$

Note that (A_1, B_1) and (A_2, B_2) are in $L(X)$. We show that $\operatorname{Ord} M^\sigma_{L(X)}\geqslant\omega_0$, where $\sigma = \{(A_1, B_1), (A_2, B_2)\}$. To this end note that for each n $\sigma|s(T_n)$ consists of the last two pairs of $(n+1)$-dimensional opposite faces of $s(T_n)$. This shows immediately that for every n

$$\operatorname{Ord}\tilde{M}^\sigma_{L(X)|s(T_n)} = n.$$

This proves our claim. ■

The inequality $\dim X\leqslant\omega_0+2$ follows from Theorem 4.3.1 and the fact that X admits no essential mapping onto J^{ω_0+2}, which we prove in the remaining part of this section.

To be able to formulate our next two lemmas we introduce for each $n\in N$

$$M_n = I^2-\bigcup\{K\in\mathscr{K}_i\colon i\leqslant n\}.$$

Let us call a pair (A, B) of disjoint closed subsets of X *o.k.* iff for every $K \in \mathcal{K}$ we have

$$A \cap s(\bar{K}) = \emptyset \quad \text{or} \quad B \cap s(\bar{K}) = \emptyset .$$

5.3.5. LEMMA. *Let (A, B) be a pair of disjoint closed subsets of Z. Then for some $n \in N$ $(A \cap s(M_n), B \cap s(M_n))$ is* o.k.

Proof. Let $\mathcal{K}' = \{K \in \mathcal{K}: A \cap s(\bar{K}) \neq \emptyset \neq B \cap s(\bar{K})\}$. Then $\mathcal{K}'$ is finite. This follows readily using property K3 of $\mathcal{K}$ and the compactness of S_{ω_0+2}. For some $n \in N$ $\mathcal{K}' \subset \bigcup_{i \leqslant n} \mathcal{K}_i$. Clearly, this n works. ■

5.3.6. LEMMA. *Let $n \in N$ and let (A, B) be an* o.k. *pair of disjoint closed subsets of Z. Then there are open sets U and V in I^2 such that*

(i) $A \subset U' = s(S_{\omega_0} \times U)$, $B \subset V' = s(S_{\omega_0} \times V)$, $\bar{U} \cap \bar{V} = \emptyset$,

(ii) *if $i \leqslant n$ and $K \in \mathcal{K}_i$ then $\bar{K} \subset U$ or $\bar{K} \subset V$;*

if $i > n$ and $K \in \mathcal{K}_i$ then $\bar{K} \subset U$ or $\bar{K} \subset V$ or $\bar{K} \cap (\bar{U} \cup \bar{V}) = \emptyset$.

Note that by (ii) $S_{\omega_0} \times U = s^{-1}s(S_{\omega_0} \times U)$ *and likewise for $\bar{U}$, V and $\bar{V}$ so that $\bar{U}' \cap \bar{V}' = \emptyset$.*

Proof. Let

$$A_M = (A \cap s(M)) \cup \{y_K: A \cap s(\bar{K}) \neq \emptyset\} \quad \text{and}$$

$$B_M = (B \cap s(M)) \cup \{y_K: B \cap s(\bar{K}) \neq \emptyset\} .$$

In $s(M)$ find disjoint open sets U_0 and V_0 such that $\bar{U}_0 \cap \bar{V}_0 = \emptyset$, $A_M \subset U_0$, $B_M \subset V_0$, $\{y_K: K \in \mathcal{K}_i, i \leqslant n\} \subset U_0 \cup V_0$ and $\{y_K: K \in \mathcal{K}\} \cap \mathrm{Fr}(U_0 \cup V_0) = \emptyset$. In I^2 let

$$U = s^{-1}(U_0) \cup \bigcup \{K: y_K \in U_0\} \quad \text{and} \quad V = s^{-1}(V_0) \cup \bigcup \{K: y_K \in V_0\} .$$

It is not hard to verify that U and V are as required. ■

5.3.6.a. Remark. Because $s_n = s|T_n$ is essentially the identity it is easy to see that

$$s_n^{-1}(U' \cap s(T_n)) = S_n \times U , \quad s_n^{-1}(\bar{U}' \cap s(T_n)) = S_n \times \bar{U}$$

and likewise for V' and $\bar{V}'$. ■

5.3.7. CLAIM. *Every closed set G in X disjoint from Z can be covered by finitely many sets $s(T_n)$, $n \in N$. In addition,* $\dim G$ *is finite.*

Proof. Since $s^{-1}(G) \cap I^2 = \emptyset$, we have $s^{-1}(G) \subset \bigcup_{n=1}^{\infty} T_n$. Because each T_n is open in S and $s^{-1}(G)$ is compact $s^{-1}(G)$ can be covered by finitely many T_n where $n \in N$. The fact that $s|T_n$: $T_n \to s(T_n)$ is a homeomorphism proves the claim. ■

5.3.8. CLAIM. *Let $\sigma = \{(A_i, B_i)\}_{i=1,2} \in L(X)$ and let F be a closed subset of X such that*

(1) $F \cap Z \subset \bigcup_{n=1}^{\infty} s(T_n)$ *for some $k \in N$,*

(2) *σ is inessential on $F \cap s(T_n)$ for every $n \in N$.*

Then σ is inessential on F.

Proof. From the compactness of $F \cap Z$ and the fact that each $s(T_n) \cap Z$ is open in $\bigcup_{n=1}^{\infty} s(T_n) \cap Z$ follows

$$F \cap Z \subset \bigcup_{n=1}^{k} s(T_n)$$

for some $k \in N$.

We may put

$$F_1 = \bigoplus_{n=1}^{k} F \cap s(T_n) .$$

$F_2 = F - F_1$ is disjoint from Z and closed in X. Hence we have by Claim 5.3.7

$$F_2 = \bigoplus_{n=k+1}^{l} F \cap s(T_n)$$

for some l.

Consequently, $F = \bigoplus_{n=1}^{l} F \cap s(T_n)$ so that by (2) σ is essential on F. ■

5.3.9. CLAIM. *X admits no essential map onto J^{ω_0+2}.*

Proof. Assume that f: $X \to J^{\omega_0+2}$ is essential. We will derive a contradiction. We write

$$J^{\omega_0+2} = J^{\omega_0} \times I^2 = \{p_\omega\} \times I^2 \cup \bigoplus_{i=1}^{\infty} (J^i \times I^2 \cup A^i \times I^2) .$$

We let $K_i = J^i \times I^2$ and $P_i = f^{-1}(K_i)$ for $i \in N$. Note that by H2 $f|P_i$ is essential for each i. We set

$$A_1 = f^{-1}(J^{\omega_0} \times \{0\} \times I) , \qquad B_1 = f^{-1}(J^{\omega_0} \times \{1\} \times I) ,$$

$$A_2 = f^{-1}(J^{\omega_0} \times I \times \{0\}) \quad \text{and} \quad B_2 = f^{-1}(J^{\omega_0} \times I \times \{1\}) .$$

Let $\sigma = \{(A_1, B_1), (A_2, B_2)\}$; it follows readily that

$$\mathrm{Ord}\, \tilde{M}^{\sigma}_{L(X)|P_i} \geqslant i$$

for each $i \in N$: $\sigma|P_i$ consists of two pairs of inverse images of opposite faces of K_i through an essential map.

Now we replace (A_1, B_1) and (A_2, B_2) by pairs of disjoint closed sets which will be much easier to handle; here we use Lemma's 5.3.5 and 5.3.6. Consider $(A_j \cap Z, B_j \cap Z)$ for $j = 1, 2$. By Lemma 5.3.5 we can find $n_0 \in N$ such that $(A_j \cap s(M_{n_0}), B_j \cap s(M_{n_0}))$ is o.k. for $j = 1, 2$. Next we find open sets U_j and V_j

in I^2 satisfying the conditions from Lemma 5.3.6 again for $j = 1, 2$ and for $n = n_0$. We let $C_j = \overline{U}'_j$ and $D_j = \overline{V}'_j$ for $j = 1, 2$. In addition, put $\gamma = \{C_j, D_j\}_{j=1}^2$. We will show that for γ, which will replace σ in the remaining part of the argument, we have the following:

A: For some m, $i_0 \in N$ we have for each $i \geq i_0$

$$\operatorname{Ord} \tilde{M}^{\gamma}_{L(X)|P_i} \geq i-(m+1) .$$

Put $Y_{n_0} = s(S_{\omega_0} \times M_{n_0})$ and $Y_K = s(S_{\omega_0} \times \overline{K})$ for each $K \in \mathscr{K}$. Clearly

$$X = Y_{n_0} \cup \bigcup \{Y_K\colon K \in \mathscr{K}_i, i \leq n_0\}.$$

Now for $j = 1, 2$ pick open sets U''_j and V''_j in X such that

$$U'_j \cap Y_{n_0} = U''_j \cap Y_{n_0}, \quad V'_j \cap Y_{n_0} = V''_j \cap Y_{n_0}, \quad \overline{U}''_j \cap \overline{V}''_j = \emptyset,$$

$$A_j \cap Z \subset U''_j, \quad B_j \cap Z \subset V''_j \quad \text{and} \quad \{y_K\colon K \in \mathscr{K}_i, i \leq n_0\} \subset U''_j \cup V''_j .$$

We let $C''_j = \overline{U}''_j$ and $D''_j = \overline{V}''_j$ for $j = 1, 2$. It follows that

$$H = \bigcup_{j=1}^{2} \overline{(A_j - C''_j) \cup (B_j - D''_j)}$$

is closed and disjoint from Z. Hence by Claim 5.3.7 $\dim H = m \leq \omega_0$. According to Lemma 5.3.1 for $\gamma'' = \{(C''_j, D''_j)\}_{j=1}^2$ we have

$$\operatorname{Ord} \tilde{M}^{\gamma''}_{L(X)|P_i} \geq i-(m+1)$$

for every $i = 1, 2, \ldots$ An application of Lemma 4.4.1 gives us, when we put $E'' = \bigcap_{j=1}^{2} \overline{X-(C''_j \cup D''_j)}$, that $\operatorname{Ord} \tilde{M}^{\gamma''}_{L(X)|E'' \cap P_i} \geq i-(m+1)$ for $i = 1, 2, \ldots$ Observe that $E'' \subset \bigcap_{j=1}^{2} (X-(U''_j \cup V''_j))$. Put $E = \bigcap_{j=1}^{2} (Y_{n_0}-(U''_j \cup V''_j))$ and for $i \leq n_0$, $K \in \mathscr{K}_i$

$$E_k = \bigcap_{j=1}^{2} (Y_K - (U''_j \cup V''_j)) .$$

Since for each $i \leq n_0$, $K \in \mathscr{K}_i$ we have $Y_K \subset U'_j$ or $Y_K \subset V'_j$, we obtain

$$\bigcap_{j=1}^{2} (X-(U'_j \cup V'_j)) = \bigcap_{j=1}^{2} (Y_{n_0}-(U'_j \cup V'_j)) = E$$

so that

$$E \subset \operatorname{Int} Y_{n_0} .$$

Consequently, we may write $E'' \subset E \oplus (\oplus \{E_K\colon K \in \mathscr{K}_i, i \leq n_0\})$. We will prove that for each $K \in \mathscr{K}_i$, $i \leq n_0$

$$n_K = \operatorname{Ord} M^{\gamma''}_{L(X)|E_K} < \omega_0 .$$

For this let us consider the mapping r defined in step 2 of our construction. Especially we consider $r|r^{-1}(E_K)\colon r^{-1}(E_K) \to E_K$. Put $F = r^{-1}(E_K) \cap q(K)$. Then $x_K \notin F$,

since $y_K \notin E_K$. We have $\dim F \leq 1$ and the construction of r shows us that $r(F)$ is finite-dimensional.

Clearly, using the same argument as in Claim 5.3.7 we see that each closed G in $r^{-1}(E_K)$ which is disjoint from $q(K)$ is finite dimensional and $r|G\colon G \to r(G)$ is a homeomorphism.

Now we may apply Proposition 5.3.2 which gives us $\dim E_K \leq \omega_0+1$. Consequently by definition, we can find our n_K as desired.

By Proposition 3.4.1 we obtain that, if we take

$$i_0 > \max\{n_K\colon K \in \mathscr{K}_i, i \leq n_0\}+m+1, \quad \operatorname{Ord} \tilde{M}^{\gamma''}_{L(X)|E \cap P_i} \geq i-(m+1)$$

whenever $i \geq i_0$.

Then the equality $\gamma''|E = \gamma|E$ gives us A.

Now we establish the following:

B: If $K \in \mathscr{K}$ (say $K \in \mathscr{K}_i$) and $\delta > 0$ then there are j_K and n_K in N with $n_K \geq i$ such that whenever $n, j \in N$ satisfy $n > n_K$ and $j \neq j_K$ there is an $\varepsilon < \delta$ with

$$s^{-1}(P_j) \cap (S_n \times D(K, \varepsilon)) = \emptyset .$$

Case 1. For every $j \in N$ and every $n \in N$ if $n \geq i$ and $3^{-n} < \delta$ then $s^{-1}(P_j) \cap Q_{K,n} = \emptyset$.

In this case let $j_K = 1$ and pick n_K such that $3^{-n_K} < \delta$. Then given j and n take $\varepsilon = 3^{-n}$.

Case 2. There exist $j_K, m \in N$ with $m \geq i$, $3^{-m} < \delta$ and $s^{-1}(P_{jK}) \cap Q_{K,m} \neq \emptyset$. In this case we can find $y \in K$ with $s(y) \in P_{j_K}$ and $q(y) \in I_{K,m}$. Let $\varepsilon = d(y, I^2-K)$. Then $\varepsilon \leq 3^{-m} < \delta$. It follows that

$$D(K, \varepsilon) = q^{-1}q(y) \subset s^{-1}(P_{j_K}) .$$

But then we can find an $n_K \geq i$ such that

$$q^{-1}q(y) \cup \bigcup_{n \geq n_K} (S_n \times D(K, \varepsilon)) \subset s^{-1}f^{-1}(J^{j_K} \times I^2 \cup A^{j_K} \times I^2)$$

because this last set is open and contains $q^{-1}q(y)$. This last set is also disjoint from $s^{-1}(P_j)$ for $j \neq j_K$ so that we have found our j_K and n_K.

Since $\dim M = 1$, we can find open sets O_1 and O_2 in I^2 such that

$$\overline{U}_j \subset O_j \subset \overline{O}_j \subset I^2 - \overline{V}_j$$

for $j = 1, 2$ and

$$M \cap (\operatorname{Fr} O_1 \cap \operatorname{Fr} O_2) = \emptyset .$$

Let $D = \operatorname{Fr} O_1 \cap \operatorname{Fr} O_2$. D is compact, $D \subset \bigcup \mathscr{K}$ and $\mathscr{K}$ is pairwise disjoint. Hence

$$\mathscr{K}_D = \{K \in \mathscr{K}\colon K \cap D \neq \emptyset\}$$

is finite. Now fix n_1 and i_1 in N such that

$$n_1 > \max(\{n_K\colon K \in \mathscr{K}_D\} \cup \{n_0\}), \quad \text{and}$$

$$i_1 > n_1+4+m+1+\max\{j_K\colon K \in \mathscr{K}_D\} .$$

Let $F_1 = (Z - \bigcup_{n=1}^{\infty} s(T_n)) \cup \bigcup_{n<n_1} s(T_n)$. Then

$$\dim F_1 \leqslant 2+(n_1+1) = n_1+3 .$$

Find $\tau = \{(A_i, B_i)\}_{i=3}^{n_1+6} \in \operatorname{Fin} L(X)$ such that according to A

$$\operatorname{Ord} \tilde{M}^{\gamma\cup\tau}_{L(X)|P_{i_1}} \geqslant i_1-(n_1+4)-(m+1) .$$

Find open sets O_i for $i = 3, \ldots, n_1+6$ such that for every i

$$A_i \subset O_i \subset \bar{O}_i \subset X-B_i \quad \text{and} \quad F_1 \cap \bigcap_{i=3}^{n_1+6} \operatorname{Fr} O_i = \emptyset .$$

Now set $F = P_{i_1} \cap \bigcap_{i=3}^{n_1+6} \operatorname{Fr} O_i$. Then by Proposition 3.2.1

$$\operatorname{Ord} \tilde{M}^{\gamma}_{L(X)|F} \geqslant i_1-(n_1+4)-(m+1) \geqslant 1 .$$

In particular, $\gamma|F$ is essential on F. We obtain a contradiction by showing that $\gamma|F$ is not essential on F.

To this end note that $F \cap Z \subset \bigcup_{n=1}^{\infty} s(T_n)$; so that by Claim 5.3.8 it suffices to show that for each n

$$\gamma|F \text{ is not essential on } F \cap s(T_n) .$$

For $n < n_1$ this is no problem as $F \cap s(T_n) = \emptyset$ in this case. So let $n \geqslant n_1$, and let $\delta = d(D, I^2 - (\bigcup \mathscr{K}_D))$. For each $K \in \mathscr{K}_D$ choose $\varepsilon_K < \delta$ as guaranteed by B; this is possible because $n_1 > n_K$ and $i_1 > j_K$ for each $K \in \mathscr{K}_D$. Set

$$O = \bigcup_{K \in \mathscr{K}_D} \{x \in K\colon d(x, I^2-K) > \varepsilon_K\} .$$

Let $G_1 = S_n \times \bar{O}$ and $G_2 = S_n \times (I^2 - O)$. Now $G_1 \cap G_2 = \bigcup_{K \in \mathscr{K}_D} S_n \times D(K, \varepsilon_K)$ so that by B

$$s_n^{-1}(P_{i_1}) \cap G_1 \cap G_2 = \emptyset ;$$

whence

$$s_n^{-1}(P_{i_1}) = (s_n^{-1}(P_{i_1}) \cap G_1) \oplus (s_n^{-1}(P_{i_1}) \cap G_2) .$$

Now let $\gamma' = \{(S_n \times \bar{U}_1, S_n \times \bar{V}_1), (S_n \times \bar{U}_2, S_n \times \bar{V}_2)\}$, then

$$s_n(\gamma') = \gamma|s(T_n)$$

for recall (5.3.6.a) that $s_n(S_n \times \bar{U}_1) = C_1 \cap s(T_n)$ etc.

(i) For $K \in \mathscr{K}_D$: $D \cap K \neq \emptyset$ and $D \cap \bar{U}_1 = \emptyset$, hence, by 5.3.6 (ii), $\bar{K} \cap \bar{U}_1 = \emptyset$. This implies that $S_n \times \bar{U}_1 \cap (s_n^{-1}(P_{i_1}) \cap G_1) = \emptyset$, whence γ' is not essential on $s^{-1}(P_{i_1}) \cap G_1$.

(ii) By our choice of O_1 and O_2

$$S_n \times \bar{U}_j \subset S_n \times O_j \subset S_n \times \bar{O}_j \subset T_n - (S_n \times \bar{V}_j)$$

for $j = 1, 2$ and

$$\operatorname{Fr}(S_n \times O_1) \cap \operatorname{Fr}(S_n \times O_2) = S_n \times D .$$

But $S_n \times D \subset S_n \times O = T_n - G_2$ so that γ' is not essential on $s_n^{-1}(P_{i_0}) \cap G_2$ either. Consequently, γ' is not essentiation $s_n^{-1}(P_{i_0})$. Translating everything to $s(T_n)$ through s_n we obtain that γ is not essential on $P_{i_0} \cap s(T_n)$, hence certainly not on $F \cap s(T_n)$. We conclude that γ is not essential on F and we obtain the desired contradiction. ■

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