Dualizing cubic graph theory

by

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Abstract. While the concept of cubic graph cannot itself be dualized, various characterizations of cubic can be, producing a family of reasonable possibilities for the dual of cubic. The resulting dual characterizations of cubic provide insight into both cubic graphs and the meaning of duality within graph theory.

1. Introduction. Graph-theoretic duality exploits the relationships between circuits and cutsets (as well as between spanning trees and cotrees, edge contraction and deletion, etc.; see [6] for all unexplained terminology) for planar graphs (especially three-connected planar graphs), and extends nicely to general matroids. But duality is most commonly used within traditional graph theory: finite graphs without loops or multiple edges. The artificiality of duality in this setting was explored in the companion, but nonprerequisite, paper [3]. That paper made the somewhat negative point that graph-theoretic concepts (such as eulerian or cubic) cannot themselves be dualized, much as arbitrary graphs do not have dual graphs unless they can be embedded in the plane. Concepts can only be said to have duals in terms of a characterization (or embedding) in a language. In other words, concepts cannot be dualized; only specific characterizations can be. The present paper provides a natural example lacking in [3] and takes a more positive approach to the same issue, considering to what extent a concept such as being cubic can be dualized within graph theory. (An alternative description is to study how cubic could be defined within matroid theory.)

As discussed in [3], it is natural to restrict attention to graphs having no cutsets of size less than three, corresponding to excluding graphs with circuits of size less than three. Thus, by graphs we mean three-edge-connected graphs.

The notions of edge deletion and edge contraction are central to duality. We allow edge deletion only when the resulting graph is still three-edge-connected, and so no edge which is in a size three cutset can be deleted. Dually, we do not allow edge contraction which produces loops or multiple edges, and so no edge in a triangle can be contracted. (Many writers allow contraction within a triangle, identifying the two multiple edges formed; we would demand this be done by a deletion, followed by a contraction.)

In our context of three-edge-connected graphs, there are natural operations of vertex joining and vertex splitting, inverse to edge deleting and edge contracting. Vertex joining means inserting an edge between nonadjacent vertices; multiple edges cannot be formed. Vertex splitting (as in [1, Theorem 5.7]) introduces one new edge joining two new vertices, with the other incident edges divided arbitrarily between the new vertices, making sure the minimal degree remains at least three for three-edge-connectedness.

In this context, cubic graphs are precisely those which are maximal with respect to vertex splitting, just as complete graphs are maximal with respect to vertex joining. This naturalness (and usefulness) is part of the reason that we have chosen cubic as a natural notion to try to dualize. Another part is that many cubic graphs are planar and so have actual dual graphs. These duals of planar cubics — precisely the *maximal* (or triangulated) *planar graphs* — need to be included in any reasonable possible dual for cubic.

Unless stated otherwise, all graphs will be assumed to be three-connected. For cubic graphs, this is equivalent to three-edge-connectedness. For planar graphs, it insures that the dual graph is unique. Moreover, three-connected graphs have a nice self-dual characterization in Tutte's Wheel Theorem [5] or [1, Theorem 5.7]: all can be obtained from wheels by a sequence of vertex joinings and splittings.

2. Dual characterizations of cubic. Call a statement *admissible* if it can is expressed in a form which can be dualized; that is, in terms of edges and set theoretic concepts, plus circuits and cutsets, spanning trees and cotrees, edge deletion and contraction, vertex joining and spliting, and rank ($r = q-p+1$, where q and p are the number of edges and vertices, respectively) and corank ($c = p-1$). (Although we are being purposely vague about the formal language involved, some sort of higher-order version of predicate logic will suffice). For any admissible statement, the dual is defined by interchanging words (for instance "circuits" with "cutsets") within the pairings listed above. These pairings are justified by the natural geometric duality of plane graphs.

Consider, for example, the simple characterization of cubic by $3p = 2q$. The difficulty with this statement is that vertices are not susceptible to dualization. But the number p of vertices is, since $p = c+1$ where c is again the corank (= the number of edges in a spanning tree; see "cutset rank" in [6] or "cocycle rank" in [1]). Hence $3(c+1) = 2q$ is an admissible characterization of cubic, dualizing into $3(r+1) = 2q$ where r is the rank (= $q-p+1$ = the number of edges in a cotree; see "circuit rank" or "cyclomatic number" in [6] or "cycle rank" in [1]). This is then equivalent to $3(q-p+2) = 2q$ which reduces to $q = 3p-6$. This is then a reasonable dual characterization of cubic which describes the maximal planar graphs plus many nonplanar ones. The $q = 3p-6$ graphs of orders through six are shown in Figure 1.

As another possibility, consider having at least p 3-*cuts* (that is, cutsets of size three). This does not quite characterize cubic, however, because of the noncubic counterexample with $p = 10$ 3-cuts in Figure 2. One natural strengthening is having

a set of p 3-cuts such that each edge is in exactly two. But this is too strong since it would dualize into having $q-p+2$ triangles such that each edge is in exactly two. All but one of these would constitute a circuit basis and so by MacLane's Planarity

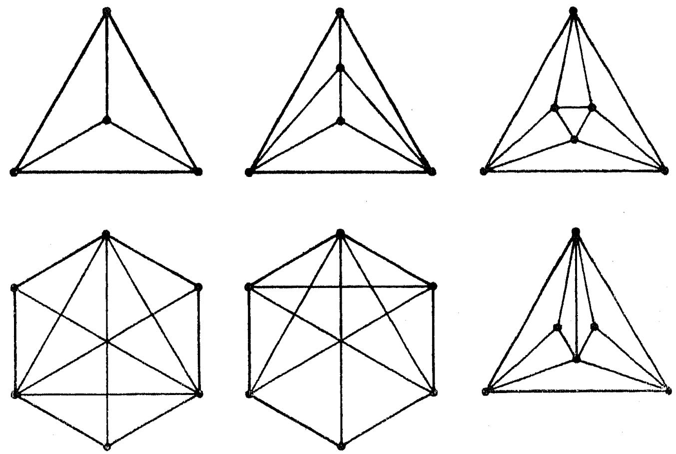

Fig. 1

Theorem ([2] or [1, Theorem 11.16]) would force planarity. This would lead to a dual characterization of cubic describing precisely the maximal planar graphs. But this would be the dual of the family of planar cubics, rather than of all cubics.

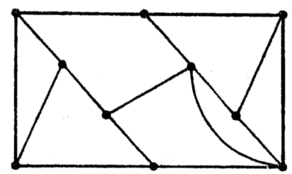

Fig. 2

Call an admissible characterization of cubic *graphic* if its dual implies planarity, and *nongraphic* otherwise. Graphic characterizations can be thought of as those which would imply the corresponding matroid is graphic. The dual would then imply being cographic and so, within our chosen context of graphs, being planar. A graphic characterization is too strong in that it would characterize being a cubic graph, not just being cubic. So having p 3-cuts such that each edge is in exactly two is a graphic characterization, while having $q = 3p-6$ is nongraphic. Our goal is to find nongraphic characterizations of cubic. This requires that the dual characterization describe precisely the maximal planar graphs from among planar graphs, plus at least one nonplanar graph.

PROPOSITION 1. *"$2q = 3p$" is a nongraphic characterization of cubic. The corresponding dual characterization is*

$$(1) \qquad q = 3p-6$$

PROPOSITION 2. *"Having a set of p 3-cuts with each edge in an average of two" is a nongraphic characterization of cubic. The corresponding dual characterization is*

(2) *having a set of $q-p+2$ triangles with each edge in an average of two.*

Proof. Each cubic graph has p such vertex-cutsets, and having such a set of 3-cuts immediately implies $2 = 3p/q$. Using all but any one triangle in the bottom-center graph of Figure 1 shows nongraphicness by satisfying (2).

Being a maximal graph with respect to vertex splitting is also graphic; its dual would involve being maximal cographic (and so planar) with respect to vertex joining. Proposition 1 can be viewed as a nongraphic "softening" of this, as Proposition 2 is of our first graphic example.

Another characterization of cubic involves being constructible from the complete graph K_4 by repeated use of the following construction: place a new vertex in the interior of any two edges and then join them with an edge [1, Exercise 5.15]. We call this construction an *illegal joining* since it can be though of as two illegal (because of introducing vertices of degree two) vertex splittings, followed by a vertex joining. Notice that each illegal joining introduces three new edges and two new vertices. By looking at plane graphs, the dual construction is seen to be an *illegal splitting*, obtained by illegally joining a vertex to two adjacent vertices (introducing multiple edges) and then splitting that vertex so as to remove the multiple edges; three new edges and one new vertex are introduced.

PROPOSITION 3. *"Being constructable from K_4 by illegal joining" is a nongraphic characterization of cubic. The corresponding dual characterization is*

(3) *being constructable from K_4 by illegal splitting.*

The three duals we have seen of nongraphic characterizations of cubic are related by the following theorem.

THEOREM. *Condition* (2) *implies condition* (3) *which in turn implies condition* (1), *but not conversely.*

Proof. Suppose graph G satisfies condition (2). By (2), $2 = 3(q-p+2)/q$ and so $q = 3p-6$. Suppose also, contrary to (3), that G is a minimal graph which cannot be obtained from K_4 by illegal splitting. By this minimality, no two of the $q-p+2$ triangles can even share an edge, so $q \geqslant 3(q-p+2)$, but this together with $q = 3p-6$ would force $q \geqslant 2q$. This contradiction proves condition (3). The graph of Figure 3 satisfies condition (3) (with the two starred vertices resulting from the last illegal split), but the total number of triangles is $11 < 12 = q-p+2$, thus providing a counterexample to the converse.

Finally, suppose condition (3). Since each illegal splitting increases q by 3 and p by 1, the $q = 3p-6$ relation is injerited from K_4, thus proving condition (1). Removing one edge from the complete bipartite graph $K_{5,5}$ leaves a graph which satisfies (1) but not (3), since it contains no triangles at all. This completes the proof of the Theorem.

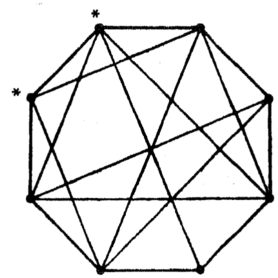

Fig. 3

Our final example does not begin with a characterization of cubic, but rather with the family of maximal planar graphs enlarged by adding certain nonplanar graphs. Proposition 4 below shows that the dual formulation is another nongraphic characterization of cubic. Suppose a, b, c, d are the vertices, in order, of a length-4 circuit in graph G, and suppose a is adjacent to c, but b is not adjacent to d. By a *diagonal transformation*, we mean deleting the edge ac and then joining the vertices b and d. As in [4, Theorem 1.3.4], all the maximal planar graphs of a given order can be obtained from any one by a sequence of diagonal transformations. If we also allow nonplanar graphs to result, we can define a reasonable possibility for the dual of cubic as

(4) *being constructable from a bipyramid via diagonal transformations.* (A *bipyramid* is a circuit graph augmented by two vertices, each joined to every vertex of the circuit. Bipyramids are planar, and their geometric duals are cubic graphs called *prisms*.) By looking at plane graphs, the dual transformation is seen to be contracting an edge and then resplitting the resulting vertex.

PROPOSITION 4. *"Obtainable from a prism via contracting and then resplitting" is a nongraphic characterization of cubic. The corresponding dual characterization is given in* (4)

Proof. Since the family of cubic graphs is clearly closed under contraction followed by resplitting, we merely need to show that every cubic graph G of (even) order p can be so obtained from the *Mobius ladder* of order p (i.e., an order p circuit graph plus the $p/2$ diagonals joining opposite vertices).

Let C be any circuit of G. Suppose first that C is not hamiltonian. Then there must be an edge e with endvertices u and v where $u \notin C$ and $v \in C$. Say v^- and v^+ are adjacent to v in C. Three-connectedness implies that u is not adjacent to both v^- and v^+. If say u is adjacent to v^-, then we can enlarge C to include $v^- uvv^+$. If u is adjacent to neither v^- nor v^+, then e can be contracted and the resulting vertex resplit to produce an enlarged circuit C. Continuing in this way will eventually produce a hamiltonian circuit C with $p/2$ diagonals in a cubic graph G. Repeated contraction and resplitting among the edges of C will permute these diagonals to produce a Moebius ladder. This completes the proof of Proposition 4.

The Theorem and Propositions above show that while various nongraphic characterizations of cubic can appear to be indistinguishable (in that they are equivalent within graph theory), their dual characterizations can vary greatly. This allows a graph-theoretic (as opposed to matroidal) means of describing their relative "strength", as partially illustrated in the distinction between graphic and nongraphic characterizations.

Each dual characterization of cubic corresponds to a family of graphs which includes the maximal planar graphs (but no other planar ones). Other than that, any assortment of nonplanar graphs can be accommodated. Each nonplanar graph G can be characterized up to isomorphism by a sentence $\sigma(G)$ which specifies the existence of the proper number of edges, circuits and cutsets, plus their incidences. This description can be dualized to a sentence $\sigma^*(G)$. Suppose α is any nongraphic characterization of cubic. Since $\sigma^*(G)$ will never hold in our graph setting (because it would describe the dual of the nonplanar graph G), the disjunction "α or $\sigma^*(G)$" will be another (nongraphic) characterization of cubic. The corresponding dual characterization corresponds to all the graphs of the dual characterization of α, together with G. In this way, any assortment of nonplanar graphs could be subsumed by a dual characterization of cubic.

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