Cells and cubes in hyperspaces

by

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Abstract. In this paper we prove that the hyperspace $C(X)$ of all subcontinua of a continuum X contains n-cells (Hilbert cubes) if and only if X contains n-odds (∞-odds).

Introduction. A *continuum* is a nonempty, connected, compact metric space. Throughout this paper X will denote a nondegenerate continuum with metric d. The *hyperspaces* of X are the spaces $2^X = \{A \subset X: A$ is closed and nonempty$\}$ and $C(X) = \{A \in 2^X: A$ is connected$\}$ both with the Hausdorff metric D. A subcontinuum M of X is an *n-odd* (∞-odd) if there exists $K \in C(M)$ such that $M-K$ has at least n components (an infinite number of components).

In studying the structure of hyperspaces, it is useful to know if they contain cells or Hilbert cubes. In this direction, S. Mazurkiewicz in [2] proved that 2^X always contains Hilbert cubes, so $\dim(2^X) = \infty$. In [3], S. B. Nadler, Jr. found a sufficient condition in order that $C(X)$ contains a Hilbert cube (see Property C in Section 2). He showed some interesting consequences from this result. Also he showed that $C(X)$ contains 2-cells if and only if X is not hereditarily indecomposable. J. T. Rogers, Jr. proved in [5] that if X contains n-odds, then $C(X)$ contains n-cells. The latter suggested the question asked by S. B. Nadler, Jr. [4, Question 1.47] of if the fact that $C(X)$ contains n-cells implies that X contains n-odds (the question for $n = 3$ was proposed by B. J. Ball).

In this paper we prove, with two different technics, that the answer to this question is positive (Thms. 1.9 and 2.8). In the first proof, we suppose that X does not contains n-odds, we calculate an upper bound for the dimension of certain subspaces ($\mathscr{C}(H, M)$) of $C(X)$ and this implies that $C(X)$ cannot contain n-cells. In the second proof n-odds are constructed explicitely from supposing that $C(X)$ contains n-cells. This proof is more involved but it allow us to prove also that if $C(X)$ contains Hilbert cubes then X contains ∞-odds.

1. First proof. We need the following conventions: A *map* is a continuous function. The unit interval in the real line is denoted by I. A nondegenerate subcontinuum $\mathscr{E}$ of $C(X)$ is an *order arc* if $A, B \in \mathscr{E}$ implies that $A \subset B$ or $B \subset A$ (then $\mathscr{E}$ is homeomorphic to I). If $A \subset H \subset M \subset X$ and $A, H, M \in C(X)$ we say that A *is* *leaving* H *within* M if there exist a map $\alpha\colon I\to C(M)$ such that $\alpha(0)=A$ and $\alpha(t)\cap(M-H)\neq\emptyset$ for each $t>0$. We define

$$\mathscr{C}(H,M)=\{B\in C(X)\colon B \text{ is leaving } H \text{ within } M\},$$

$$\mathscr{D}(H)=\{B\in C(X)\colon B\cap H\neq\emptyset\},$$

$$\mathscr{E}(H,M)=\{B\in C(X)\colon H\subset B\subset M\}$$

and, for all $R\subset X$,

$$\mathscr{F}(R)=\{A\in C(X)\colon A\subset R\}.$$

X is *locally intersectable* (LI) if for all $H\in C(X)$ and $A\in\mathscr{D}(H)$, there exists an open subset U of $\mathscr{D}(H)$ such that $A\in U$ and, for each $B\in U$, $A\cap B\neq\emptyset$ or $B\subset H$. X is *finitely intersectable* (FI) if $A\cap B$ has a finite number of components for all A, $B\in C(X)$. A continuum $\mathscr{C}\subset\mathscr{C}(H,M)$ is *upperly closed* (UC) if $A\in\mathscr{C}$, $B\in C(X)$ and $A\subset B\subset H$ implies that $B\in\mathscr{C}$. $\mathscr{C}$ *has the property* $\mathscr{S}(n)$ $(n\geqslant 1)$ if it is not possible to construct n-odds with its elements, this means that if A, $B\in\mathscr{C}$ and $A\subset B$, then $B-A$ has at most $n-1$ components (for $n=1$ this means that $B-A=\emptyset$). We will use the concept of dimension such as in the book [1]. Throughout this section H, M will denote elements of $C(X)$ such that $H\subset M$.

LEMMA 1.1. *$\mathscr{C}(H,M)$ is* UC *and connected.*

Proof. Let $A\in\mathscr{C}(H,M)$ and $B\in C(X)$ be such that $A\subset B\subset H$. Take a map $\alpha\colon I\to C(M)$ such that $\alpha(0)=A$ and $\alpha(t)\cap(M-H)\neq\emptyset$ for all $t>0$. To prove that $B\in\mathscr{C}(H,M)$, it is enough to define $\beta\colon I\to C(M)$ by

$$\beta(t)=B\cup\big(\bigcup\{\alpha(s)\colon 0\leqslant s\leqslant t\}\big).$$

LEMMA 1.2. *Suppose that $A\in C(H)$ and that $(A_n)_n$ is a sequence of elements of $C(M)$ such that $A_n\to A$ and $A_n\cap A\neq\emptyset$ and $A_n\cap(M-H)\neq\emptyset$ for all n. Then $A\in\mathscr{C}(H,M)$.*

Proof. For each n, choose a map $\alpha_n\colon I\to C(X)$ such that $\alpha_n(0)=A$, $\alpha_n(t)=A\cup A_n$ for each $t\geqslant 1/n$ and $s\leqslant t$ implies that $\alpha_n(s)\subset\alpha_n(t)$. To prove that $A\in\mathscr{C}(H,M)$, define $\alpha\colon I\to C(X)$ by $\alpha(t)=\bigcup\{\alpha_n(t)\colon n\geqslant 1\}$.

LEMMA 1.3. *If X is* LI, *then $\mathscr{C}(H,M)$ is compact.*

Proof. It follows from Lemma 1.2 and the fact that for each $B\in\mathscr{C}(H,M)$, there exists a sequence $(B_n)_n$ of elements of $C(M)$ such that $B\subset B_n$, $B_n\cap(M-H)\neq\emptyset$ for all n and $B_n\to B$.

LEMMA 1.4. *Let G be a closed subset of H and $\{G_j\colon j\in J\}$ the set of components of G. Let $\mathscr{C}\subset\mathscr{C}(H,M)$ be a* UC *continuum. Let $\mathscr{E}=\{B\in\mathscr{C}\colon B\subset G\}$. Suppose that X is* LI. *Then the set of components of $\operatorname{Fr}_{\mathscr{C}}(\mathscr{E})$ is*

$$\mathscr{F}=\{\mathscr{C}\cap\mathscr{C}(G_j,H)\colon j\in J \text{ and } \mathscr{C}\cap\mathscr{C}(G_j,H) \text{ is nonempty}\}.$$

Proof. We only proof that $\operatorname{Fr}_{\mathscr{C}}(\mathscr{E})=\bigcup\{E\colon E\in\mathscr{F}\}$. Let $B\in\operatorname{Fr}_{\mathscr{C}}(\mathscr{E})$, then there exists $j\in J$ such that $B\subset G_j$. Since X is LI, there exists an open subset U of $\mathscr{C}(H,M)$ such that $B\in U$ and $B\cap F\neq\emptyset$ for each $F\in U$. Let $(B_n)_n$ a sequence of elements of $C(H)$ such that $B_n\to B$, and $B_n\in U$ and $B_n\not\subset G_j$ for each n. Applying Lemma 1.2, we obtain that $B\in\mathscr{C}(G_j,H)$. Hence $\operatorname{Fr}_{\mathscr{C}}(\mathscr{E})\subset\bigcup\{E\colon E\in\mathscr{F}\}$ the other inclusion follows from the fact that $\mathscr{C}\subset\mathscr{C}(H,M)$ is UC.

LEMMA 1.5. *Suppose that X is* FI. *Let A, $B\in\mathscr{C}(H,M)$ be such that $A\subset B$ and $B-A$ has at least n components, then there exist A_1, $B_1\in C(M)$ such that $A\subset A_1\subset B_1\cap H$ and B_1-A_1 has at least $n+1$ components.*

The above lemmas imply:

LEMMA 1.6. *Suppose that X is* LI *and* FI. *If $\mathscr{C}\subset\mathscr{C}(H,M)$ is* UC *and has property $\mathscr{S}(n)$ $(n\geqslant 2)$, then for all $K\in C(H)$, $\mathscr{C}\cap\mathscr{C}(K,H)\subset\mathscr{C}(K,H)$ is* UC *and has property $\mathscr{S}(n-1)$.*

THEOREM 1.7. *Suppose that X is* LI *and* FI. *If $\mathscr{C}\subset\mathscr{C}(H,M)$ is* UC *and has property $\mathscr{S}(n)$, then $\dim C\leqslant n-1$ $(n\geqslant 1)$.*

Proof. (1) For $n=1$: Let $A\in\mathscr{C}$. Since X is LI, here exists an open subset U of $\mathscr{C}(H,M)$ such that $A\in U$ and $A\cap B\neq\emptyset$ for each $B\in U$. If $B\in U\cap C$, then $A\cup B\in C$, so $A=B$ ($\mathscr{C}$ has property $\mathscr{S}(1)$). Hence $\mathscr{C}=\{A\}$ and $\dim\mathscr{C}=0$.

(2) Suppose the theorem holds for $n-1$ and $n\geqslant 2$. Suppose also that $\mathscr{C}\subset\mathscr{C}(H,M)$ is UC and has property $\mathscr{S}(n)$. Let $A\in\mathscr{C}$ and let U be an open subset of $C(X)$ such that $A\in U\cap\mathscr{C}$. Since X is LI, we can suppose that $A\cap B\neq\emptyset$ for each $B\in U\cap\mathscr{C}(H,M)$. Let $U_1,\dots,U_m$ be open subsets of X such that $A\subset\langle U_1,\dots,U_m\rangle\subset U$, where $\langle U_1,\dots,U_m\rangle=\{B\in C(X)\colon B\subset U_1\cup\ldots\cup U_m$ and $B\cap U_i\neq\emptyset$ for each $i\}$. Take an open subset U_0 of X such that

$$A\subset U_0\subset \operatorname{Cl}_X(U_0)\subset U_1\cup\ldots\cup U_m.$$

For $i=1,\dots,m$, define $U_i^*=\{B\in\mathscr{C}\colon B\cap U_i\neq\emptyset\}$ and $G_i=H-U_i$. Suppose that $\{G_j^i\colon j\in J_i\}$ are the components of G_i. By Lemma 1.4, the components of $\operatorname{Fr}_{\mathscr{C}}(U_i^*)$ are $\{\mathscr{C}\cap\mathscr{C}(G_j^i,H)\colon \mathscr{C}\cap\mathscr{C}(G_j^i,H)\neq\emptyset$ and $j\in J_i\}$. Lemma 1.6 implies that each continuum $\mathscr{C}\cap\mathscr{C}(G_j^i,H)\subset\mathscr{C}(G_j^i,H)$ is UC and has property $\mathscr{S}(n-1)$. Then (see [1, Theorem VI.7]) $\dim\operatorname{Fr}_{\mathscr{C}}(U_i^*)\leqslant n-2$.

Let L_0 be the component of $U_0\cap H$ which contains A and let $L=\operatorname{Cl}_X(L_0)$. Put $\mathscr{D}=\{B\in\mathscr{C}\colon B\subset L\}$. Then $A\in\{B\in\mathscr{C}\colon B\subset U_0$ and $B\cap U_i\cap U_0\neq\emptyset$ for each $i\}\subset\mathscr{D}$, so $A\in\operatorname{Int}_{\mathscr{C}}(\mathscr{D})$. Define $V=U_1^*\cap\ldots\cap U_n^*\operatorname{Int}_{\mathscr{C}}(\mathscr{D})$. Since

$$\dim\operatorname{Fr}_{\mathscr{C}}(\operatorname{Int}_{\mathscr{C}}(\mathscr{D}))\leqslant n-2,$$

we have that $\dim(\operatorname{Fr}_{\mathscr{C}}(V))\leqslant n-2$. Notice that $A\in V\subset U\cap\mathscr{C}$. This proves that $\dim\mathscr{C}\leqslant n-1$ and completes the proof of the theorem.

LEMMA 1.8. *If X does not contain n-odds, then X is* LI *and* FI $(n\geqslant 1)$.

THEOREM 1.9. *If X does not contain n-odds, then $C(X)$ does not contain n-cells* ($n \geqslant 2$).

Proof. Suppose $\mathscr{F}$ is an n-cell contained in $C(X)$. Since $n \geqslant 2$, there exist A, $B \in \mathscr{F}$ such that $B \not\subset A$, Let $\mathscr{G}$ be an n-cell contained in $\mathscr{F}$ such that $A \in \mathscr{G}$ and $B \not\subset \bigcup \{G: G \in \mathscr{G}\}$. Let $H_0 = \bigcup \{G: G \in \mathscr{G}\}$. Lemmas 1.5 and 1.7 imply $\dim \mathscr{C}(H_0, X) \leqslant n-2$. Then $\mathscr{G} \not\subset \mathscr{C}(H_0, X)$ and $\mathscr{C}(H_0, X) \cap \mathscr{F}$ does not separate $\mathscr{F}$ (see [1, Corollary to Theorem IV.4]). Let $C \in \mathscr{G} - \mathscr{C}(H_0, X)$ and $\sigma\colon I \to \mathscr{F}$ a map such that $\sigma(0) = C$, $\sigma(1) = B$ and $\operatorname{Im}\sigma \cap \mathscr{C}(H_0, X) = \emptyset$. Let

$$t_0 = \sup\{t \in I\colon \sigma(t) \subset H\}.$$

Then $0 \leqslant t_0 < 1$ and $\sigma(t_0) \in \mathscr{C}(H_0, X)$. This contradiction proves the theorem.

2. Second proof. We say X has *property* A if there exists a sequence $A, A_1, \ldots$ of subcontinua of X such that

(a) $A \cap A_n \neq \emptyset$ and $A_n - A \neq \emptyset$ for all n.

(b) $A_1 - A$, $A_2 - A, \ldots$ are pairwise disjoint.

X has *Property* B (resp. *property* C) if there exists a sequence $A, A_1, \ldots$ of subcontinua of X which satisfy (a), (b) and

(c) $A_n \to A$

(resp. (c′) $\operatorname{diam} A_n \to 0$).

In [3], S. B. Nadler, Jr. proved that if X has property C, then $C(X)$ contains a Hilbert cube. He asked in [4, Question 1.148] if the converse is true. In Section 3 we will show, with an example, that the answer is no. However, in this section we will prove that are equivalent:

(1) X has Property A;

(2) X has Property B;

(3) X contains ∞-odds;

(4) $C(X)$ contains Hilbert cubes.

The equivalences between (1), (2) and (3) are easy to prove, so we will only show that they are equivalent to (4). A little modification in the proof of Theorem 6 in [3] gives:

PROPOSITION 2.1. *If X has property* B, *then $C(X)$ contains Hilbert cubes.*

From now on, we suppose that X does not contains ∞-odds and, in consequence, X is FI.

LEMMA 2.2. *If $A, B \in C(X)$, $A \subset B$ and $A \neq B$, then there exist $A_1, B_1 \in C(X)$ such that $A \subset A_1$, $B_1 \subset B$ and $\mathscr{E}(A_1, B_1)$ is an order arc.*

Proof. Suppose that this assertion is not true. We will obtain a contradiction by proving that X has Property A. For this, we will construct, inductively, sequences $F_1, F_2, \ldots$ and $E_1, E_2, \ldots$ of subcontinua of B such that:

(i) $F_1 - E_1$, $F_2 - E_2, \ldots$ are nonempty and pairwise disjoint;

(ii) $A \subset E_n \cap F_n \in C(B)$ and $E_n - F_n \neq \emptyset$ for each n;

(iii) $E_1 \cap F_1 \subset E_2 \cap F_2 \subset \ldots$ and $E_1 \supset E_2 \supset \ldots$

(1) For $n = 1$. Since $\mathscr{E}(A, B)$ is a subcontinuum of $C(X)$ which is not an order arc, there exist $E_1, F_1 \in C(B)$ such that $E_1 \not\subset F_1$, $F_1 \not\subset E_1$ and $A \subset E_1 \cap F_1$. Since X is FI, we can suppose that $E_1 \cap F_1$ is connected.

(2) Suppose that $E_1, \ldots, E_n$ and $F_1, \ldots, F_n$ have been constructed. Since $\mathscr{E}(E_n \cap F_n, E_n)$ is not an order arc, there exist F_{n+1} $E_{n+1} \in C(E_n)$ such that $E_n \cap F_n \subset E_{n+1} \cap F_{n+1}$, $E_{n+1} \cap F_{n+1}$ is connected $E_{n+1} \not\subset F_{n+1}$ and $F_{n+1} \not\subset E_{n+1}$. This completes the induction.

Now we define $H = \operatorname{Cl}_x(\bigcup \{E_n \cap F_n\colon n \geqslant 1\})$. Then the sets $F_1 - H$, $F_2 - H, \ldots$ are nonempty and pairwise disjoint and, for each n, $F_n \cap H \neq \emptyset$. Thus X has Property A. This contradiction completes the proof.

LEMMA 2.3. *Let $A \in C(X) - \{X\}$ and $\varepsilon > 0$. Then there exist $\delta > 0$, $n \geqslant 1$ and H, $K_1, \ldots, K_n \in C(X)$ such that $A \subset H \subset K_1 \cap \ldots \cap K_n$; $K_1 - H, \ldots, K_n - H$ are pairwise disjoint; $D(A, K_r) < \varepsilon$ (D is the Hausdorff metric for $C(X)$); $\mathscr{E}(H, K_r)$ is an order arc for each r and if $H \subset K$, $K \in C(X)$ and $D(H, K) < \delta$, then $K \subset K_1 \cup \ldots \cup K_n$.*

Proof. Suppose that this assertion is not true. We will prove, inductively, that there exist sequences $K_1, K_2, \ldots$ and $H_1, H_2, \ldots$ of subcontinua of X and a sequence $p_1, p_2, \ldots$ of points of X such that $A \subset H_1 \subset H_2 \subset \ldots$; $K_1 - H_1$, $K_2 - H_2, \ldots$ are pairwise disjoint; for all n and m, $D(A, K_n) < \varepsilon$, $p_n \in K_n - H_n$, $H_n \subset K_n$, $\mathscr{E}(H_n, K_n)$ is an order arc and $H_n \cap K_m$ is connected; and if $m > n$, $D(H_n, H_m) < \frac{1}{2} d(p_n, H_n)$.

(1) By Lemma 2.2, there exist K_1, $H_1 \in C(X)$ such that $A \subset H_1 \subset K_1$, $D(A_1, K_1) < \varepsilon$ and $\mathscr{E}(K_1, H_1)$ is an order arc. Choose a point $p_1 \in K_1 - H_1$.

(2) Suppose that $K_1, \ldots, K_n$; $H_1, \ldots, H_n$ and $p_1, \ldots, p_n$ have been constructed. Let $\delta = \operatorname{minimum}\big(\{(\frac{1}{2})(d(p_r, H_r) - D(H_r, H_m))\colon 1 \leqslant r < m \leqslant n\} \cup \{(\frac{1}{2}) d(p_r, H_r)\colon 1 \leqslant r \leqslant n\} \cup \{\varepsilon - D(A, H_n)\}\big) > 0$. Notice that, for $1 \leqslant r \leqslant n$, $H_n \neq H_n \cup K_r$, so $\mathscr{E}(H_n, H_n \cup K_r)$ is an order arc. Since we are supposing the lemma is not true, there exist $K \in C(X)$ such that $H_n \subset K$, $D(H_n, K) < \delta$ and $K \not\subset (H_n \cup K_1) \cup \ldots \cup (H_n \cup K_n)$. We can suppose that $E = K \cap (K_1 \cup \ldots \cup K_n)$ is connected (X is FI). By Lemma 2.2, there exist H_{n+1}, $K_{n+1} \in C(X)$ such that $E \subset H_{n+1} \subset K_{n+1} \subset K$ and $\mathscr{E}(H_{n+1}, K_{n+1})$ is an order arc. Choose a point $p_{n+1} \in K_{n+1} - H_{n+1}$. Since $K_1 - H_n, \ldots, K_n - H_n$ are pairwise separated and $H_n \subset E$, we have that $E \cap (K_r \cup H_n)$ is connected for each $r = 1, \ldots, n$. Then $E \cap K_r$ is connected and $H_{n+1} \cap K_r$ is connected for each $r = 1, \ldots, n$. The remaining properties for K_{n+1}, H_{n+1} and p_{n+1} are easy to check. This completes the induction.

Define $A_0 = \operatorname{Cl}_x(\bigcup \{H_n\colon n \geqslant 1\})$. Then $A_0 \in C(X)$, $A_0 \cap K_n \neq \emptyset$, $p_n \in K_n - A_0$; and $K_1 - A_0$, $K_2 - A_0, \ldots$ are pairwise disjoint. This contradicts the fact that X does not contains ∞-odds and ends the proof.

LEMMA 2.4. *Let H, $K_1, \ldots, K_r \in C(X)$ be such that $H \subset K_1 \cap \ldots \cap K_r$, $K_1 - H, \ldots, K_r - H$ are pairwise disjoint and $\mathscr{E}(H, K_1), \ldots, \mathscr{E}(H, K_r)$ are order arcs. Suppose that $D \in C(K)$, where $K = K_1 \cup \ldots \cup K_r$, then:*

(a) *$(D \cap K_s) - H$, $D \cap K_s$ and $D \cap H$ are connected for each s.*

(b) *For each* $s = 1, \ldots, r$, *the function* $\psi\colon \mathscr{D}(H)\cap C(K) \to \mathscr{E}(H, K_s)$ *given by* $\psi(A) = (A\cap K_s)\cup H$ *is continuous.*

Proof. We only prove (b). Let $\varepsilon > 0$. Put

$$\delta = \operatorname{minimum}\big(\{\varepsilon\}\cup\{\operatorname{distance}(K_s - N(\varepsilon, H), K_t - N(\varepsilon, H))\colon s \neq t\}\big),$$

where $\operatorname{distance}(L, M) = \inf\{d(x, y)\colon x\in L \text{ and } y\in M\}$ and $N(\varepsilon, H) = \{x\in X\colon d(x, H) < \varepsilon\}$. Let $A, B\in \mathscr{D}(H)\cap C(K)$ be such that $D(A, B) < \delta$. Given $x\in (A\cap K_s) - N(\varepsilon, H)$, there exists $b\in B$ such that $d(x, b) < \varepsilon$; then $b\in N(\varepsilon, H)\cup \cup(B\cap K_s)$. This implies that $\psi(A)\subset N(2\varepsilon, \psi(B))$. Similarly, $\psi(B)\subset N(2\varepsilon, \psi(A))$. Hence ψ is continuous.

Lemma 2.5. *Let* $H, K\in C(X)$ *be such that* $H\subset K$ *and* $\mathscr{E}(H, K)$ *is an order arc. If* $\mathscr{A}$ *is a locally connected closed subset of* $C(K)$, *then* $\mathscr{A}\cap C(H)\cap \operatorname{Cl}_{\mathscr{A}}(\mathscr{A}\cap\mathscr{F}(K-H))$ *has at most one element.*

Proof. Suppose that $A, B\in \mathscr{A}\cap C(H)\cap \operatorname{Cl}_{\mathscr{A}}(\mathscr{A}\cap \mathscr{F}(K-H))$ with $B\not\subset A$. Let $\mathscr{G}$ be a closed connected subset of A such that $A\in \operatorname{Int}_{\mathscr{A}}(\mathscr{G})$ and $B\not\subset \bigcup\{E\colon E\in\mathscr{G}\}$. Define $G = \bigcup\{E\colon E\in\mathscr{G}\}$. Then $G\in C(K)$, $G\cap H\neq\emptyset$, $B\not\subset G$ and $G\cap(K-H)\neq\emptyset$. Let $\mathscr{F}$ be a closed connected subset of $\mathscr{A}$ such that $B\in \operatorname{Int}_{\mathscr{A}}(\mathscr{F})$ and $G-H\not\subset \bigcup\{E\colon E\in\mathscr{F}\}$. Define $L = \bigcup\{E\colon E\in\mathscr{F}\}\in C(K)$. Since $\mathscr{E}(H, K)$ is an order arc and $H\cup G\not\subset H\cup L$, we have that $H\cup L\subset H\cup G$. This is not possible since $B\in \operatorname{Cl}_{\mathscr{A}}(\mathscr{A}\cap\mathscr{F}(K-H))$. This contradiction proves the lemma.

Lemma 2.6. *Let* $H, K_1, \ldots, K_r\in C(X)$ *be such that* $H\subset K_1\cap\ldots\cap K_r$, $\mathscr{E}(H, K_1), \ldots, \mathscr{E}(H, K_r)$ *are order arcs and* $K_1 - H, \ldots, K_r - H$ *are pairwise disjoint. Let* $K = K_1\cup\ldots\cup K_r$. *Suppose that* $A\subset C(K)$ *is such that* $A\cap H\neq\emptyset$. *Then there exists* $\varepsilon > 0$ *such that if* $0 < \delta < \varepsilon$ *and* $F\in C(K)$ *is such that* $F\cap H\neq\emptyset$, $F\subset N(\varepsilon, A)$ *and* $(A\cap K_s)\cup H = (F\cap K_s)\cup H$ *for each* s, *then there exists* $\lambda > 0$ *such that* $D(B, F) < \lambda$, $B\in C(K)$ *and* $B\cap H\neq\emptyset$ *implies that* $B\subset N(\delta, A)\cup H$.

Proof. First, lemma is proved for $r = 1$. Let $\omega\colon C(X)\to \mathbf{R}$ be a Withney map. Let $a = \omega(H)$, $b = \omega(K)$, $\omega_1 = \omega|\mathscr{E}(H, K)$ and $a_0 = \omega_1(A\cup H)$. If $a_0 = b$, then $K\subset N(\delta, A)\cup H$. Suppose then that $a_0 < b$. We analize two cases:

(a) There exists $\varepsilon > 0$ such that $\operatorname{Cl}_x(\omega_1^{-1}(t) - H)\not\subset N(\varepsilon, A)$ for all $t\in (a_0, b]$. Take $\delta\in (0, \varepsilon)$ and $F\in C(K)$ such that $F\subset N(\varepsilon, A)$. Choose $\lambda > 0$ such that $D(B, F) < \lambda$ implies that $B\subset N(\varepsilon, A)$. If $B\in C(K)$, $D(B, F) < \lambda$ and $B\cap H\neq\emptyset$, then $\omega_1(B\cup H)\leqslant a_0$, so $B\cup H\subset A\cup H$.

(b) For all $\varepsilon > 0$, there exists $t\in (a_0, b]$ such that $\operatorname{Cl}_x(\omega_1^{-1}(t) - H)\subset N(\varepsilon, A)$. Take any $\delta > 0$ and let $F\in C(K)$ be such that $F\cap H\neq\emptyset$, $F\subset N(\varepsilon, A)$ and $(A\cap K)\cup H = (F\cap K)\cup H$. Let $t_0\in (a_0, b]$ be such that $\operatorname{Cl}_x(\omega_1^{-1}(t_0) - H)\subset N(\delta, A)$. Take $\lambda > 0$ such that $B\in C(K)$, $D(B, F) < \lambda$ and $B\cap H\neq\emptyset$ implies that $\omega_1(B\cup H) < t_0$. Then $B - H\subset N(\delta, A)$. This completes the proof for $r = 1$.

Suppose now that $r > 1$. For $s = 1, \ldots, r$, put $H_s = \bigcup\{K_l\colon l\neq s\}$. Applying the first part of this proof to the continua H_s, K and A, we obtain $\varepsilon_s > 0$ with the mentioned properties. Finally we define $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_r\}$.

Theorem 2.7. *Let* $n\geqslant 1$. *Suppose that* $C(X)$ *contains a path connected subcontinuum* $\mathscr{A}$ *which has a basis of open sets* U *such that* $\operatorname{Cl}_{\mathscr{A}}(U)$ *is connected, locally connected; no one of the points of* U *separates it; and* $\dim U\geqslant n$. *Then there exist:*

(a) *Subcontinua* $K_1^1, K_1^2, K_2^2, \ldots, K_1^n, K_2^n, \ldots, K_n^n$ *of* X;

(b) *Subcontinua* $\mathscr{A}_1, \ldots, \mathscr{A}_n$ *of* $\mathscr{A}$ *with nonempty interior (in* $\mathscr{A}$);

(c) *Points* $p_1\in K_1^1, \ldots, p_n\in K_n^n$; *and*

(d) *Subcontinua* $H_1, \ldots, H_n$ *of* X

such that, for each $r\in\{1, \ldots, n\}$:

(1) $H_r\subset K_1^r\cap\ldots\cap K_r^r$; $K_1^r - H^r, \ldots, K_r^r - H_r$ *are pairwise disjoint and*

$$\mathscr{E}(H_r, K_1^r), \ldots, \mathscr{E}(H_r, K_r^r)$$

are order arcs.

(2) *If* $r > 1$, *then* $H_r\subset H_{r-1}$ *and* $K_1^r\subset K_1^{r-1}, \ldots, K_{r-1}^r\subset K_{r-1}^{r-1}$, $K_r^r\subset H_{r-1}$.

(3) *If* $A\in\mathscr{A}_r$, *then* $A\subset K_1^r\cup\ldots\cup K_r^r$, $A\cap H_r\neq\emptyset$ *and, for each* $s = 1, \ldots, r$, $p_s\in A\cap K_s^r - H_r$.

(4) $\mathscr{A}_r\subset\ldots\subset\mathscr{A}_1\subset\mathscr{A}$.

Furthermore, if $\dim U = \infty$, *infinite sequences with this properties can be constructed.*

Proof. We will prove this theorem inductively.

(i) For $n = 1$. Let $A_0, B\in A$ be such that $B\not\subset A_0$. Let $A\in C(X)$ be such that $B\not\subset A$ and $A_0\in \operatorname{Int}_{\mathscr{A}}(C(A)\cap\mathscr{A})$ and let $\varepsilon > 0$ be such that $B\not\subset N(\varepsilon, A)$. Take $\delta > 0$ and $H, K_1, \ldots, K_m$ as in Lemma 2.3. Then $B\not\subset K_1\cup\ldots\cup K_m$. Let $\sigma\colon I\to\mathscr{A}$ be an injective map such that $\sigma(0) = A_0$ and $\sigma(1) = B$. Put $t_0 = \sup\{t\in I\colon \sigma[0, t]\subset C(H)\}$ and $S = \sigma(t_0)$. Then $0 < t_0 < 1$, $S\subset H$ and there exists $i\in\{1, \ldots, m\}$ such that $S\in \operatorname{Cl}_{\mathscr{A}}(\mathscr{D}(K_i - H)\cap\mathscr{A})$.

Define $K_1^1 = K_1\cup\ldots\cup K_m$ and $H_1 = H\cup(\bigcup\{K_j\colon j\neq i\})$. Take an open subset V of $\mathscr{A}$ such that $S\in V\subset \operatorname{Cl}_{\mathscr{A}}(V)$ is connected, locally connected; $D(E, S) < \delta/2$ for all $E\in V$ and V is not separated by any of its points. Then $\operatorname{Cl}_{\mathscr{A}}(V)\subset C(K_1^1)$. By Lemma 2.5, $\operatorname{Cl}_{\mathscr{A}}(V)\cap C(H_1)\cap \operatorname{Cl}_{C(X)}(\operatorname{Cl}_{\mathscr{A}}(V)\cap\mathscr{F}(K_1^1 - H))$ has at most one element. Notice that $V - \{S\}$ is connected and it intersects $C(H_1)$ and $C(K_1^1) - C(H_1)$. Hence there exists $T\in V - \{S\}$ such that $T\in \operatorname{Fr}_{\mathscr{A}}(\mathscr{A}\cap C(H_1))$. Thus $S, T\in V\cap \cap C(H_1)\cap \operatorname{Fr}_{\mathscr{A}}(\mathscr{A}\cap C(H_1))$ and $S\neq T$. Then one of them is not in

$$\operatorname{Cl}_{\mathscr{A}}(\operatorname{Cl}_{\mathscr{A}}(V)\cap\mathscr{F}(K_1^1 - H)).$$

So that there exists a nonempty, connected open subset W of $\mathscr{A}$ such that

$$\operatorname{Cl}_{\mathscr{A}}(W)\subset V - \big(C(H_1)\cup \operatorname{Cl}_{\mathscr{A}}(\operatorname{Cl}_{\mathscr{A}}(V)\cap\mathscr{F}(K_1^1 - H_1))\big).$$

Define $\mathscr{A}_1 = \operatorname{Cl}_{\mathscr{A}}(W)$. Since the function from $\mathscr{A}_1$ in $\mathscr{E}(H_1, K_1^1)$ given by $E\to E\cup H_1$ is continuous, there exists $E_0\in\mathscr{A}$, such that $E_0\cup H_1\subset E\cup H_1$ for each $E\in\mathscr{A}_1$. Choose a point $p_1\in E_0 - H_1$.

(ii) Suppose that $K_1^1; K_1^2, K_2^2; \ldots; K_1^r, K_2^r, \ldots, K_r^r; \mathscr{A}_1, \ldots, \mathscr{A}_r; p_1, \ldots, p_r$ and $H_1, \ldots, H_r$ have been constructed and that $r < n$. We will construct $K_{r+1}^{r+1}, \mathscr{A}_{r+1}$, p_{r+1} and H_{r+1}.

Let $\omega\colon C(X)\to \boldsymbol{R}$ be a Whitney map. By Lemma 2.4, the function $\psi\colon \mathscr{A}_r\to \boldsymbol{R}^r$ given by $\psi(A) = \big(\omega((A\cap K_1^r)\cup A_r), \ldots, \omega((A\cap K_r^r)\cup H_r)\big)$ is continuous. By Theorem 7, Chap. VI in [1], there exists a nondegenerate continuum $\mathscr{F}\subset \operatorname{Int}_{\mathscr{A}}(\mathscr{A}_r)$ such that $\psi|\mathscr{F}$ is constant. Take $A, B\in\mathscr{F}$ such that $B\not\subset A$. Let $\varepsilon>0$ be as in Lemma 2.6 applied to $H_r, H_1^r, \ldots, H_r^r$ and A. We can suppose that $B\not\subset \operatorname{Cl}_x(N(2\varepsilon, A))$ and that $\{E\in\mathscr{A}\colon D(A,E)<2\varepsilon\}\subset\mathscr{A}_r$. Let $K = K_1^r\cup\ldots\cup K_r^r$ and $L = \operatorname{Cl}_x$ (component of $N(\varepsilon/2, A)\cap K$ which contains A). Since $\psi(A)=\psi(B)$, we have that $B-A\subset H_r$. So that L is a proper subcontinuum of $H_r\cup L$. Let $\delta>0, m\geqslant 1$ and $H, K_1, \ldots, K_m \in C(H_r\cup L)$ be as in Lemma 2.3 applied to $L\in C(L\cup H_r)-\{L\cup H_r\}$ and $\varepsilon/2$.

Since $A\notin C(H)$, $B\in C(H)$ and $\mathscr{F}$ is connected, we have that there exists $F\in\mathscr{F}-C(H)$ such that $F\subset N(\varepsilon/2, L)\cap N(\delta/2, H)\subset N(\varepsilon, A)$. Choose a point $q\in F-H$. Let $\varepsilon_1 = d\big(q, ((\bigcup\{K_s\colon q\notin K_s\})\cup H)>0\big)$. From the choice of ε, there exists an open connected subset W of $\mathscr{A}$ such that $F\in W\subset \operatorname{Cl}_{\mathscr{A}}(W)\subset\mathscr{A}_r$ and, for each $E\in\operatorname{Cl}_{\mathscr{A}}(W)$, $E\subset N(\varepsilon/2, A)\cup H_r$ and $D(E,F)<\varepsilon_1, \delta/2$. Define $\mathscr{A}_{r+1} = \operatorname{Cl}_{\mathscr{A}}(W)$.

Let $D\in\mathscr{A}_{r+1}$. Given $s\in\{1,\ldots,r\}$, $p_s\in A\cap\big((D\cap K_s^r)-H_r\big)$ and $(D\cap K_s^r)-H_r$ is a connected subset of $\operatorname{Cl}_x(N(\varepsilon/2, A)\cap K)$. It follows that $D\subset L\cup H_r$. Furthermore $D\subset N(\delta, H)$, so that $D\subset K_1\cup\ldots\cup K_m$. In particular, there exists $s_0\in\{1,\ldots,m\}$ such that $q\in K_{s_0}-H$. And then $D\cap(K_{s_0}-H)\neq\emptyset$. Define

$$N = (\bigcup\{K_s\colon s\neq s_0\})\cup H;\quad K_1^{r+1} = K_1^r\cap N, \ldots, K_r^{r+1} = K_r^r\cap N;$$
$$K_{r+1}^{r+1} = (N\cup K_{s_0})\cap H_r$$

and $H_{r+1} = N\cap H_r$. The point p_{r+1} can be choose in a similar way that p_1 was choosen. This completes the induction and the proof of the theorem.

THEOREM 2.8. *If $C(X)$ contains n-cells, then X contains n-odds, $(n\geqslant 2)$.*

Proof. Let $\mathscr{A}$ be an n-cell in $C(X)$. Suppose that X has not ∞-odds. Let $K_1^n, \ldots, K_n^n; \mathscr{A}_n; p_1, \ldots, p_n$ and H_n be as in Theorem 2.7. Then every element of $\mathscr{A}_n$ is an n-odd.

THEOREM 2.9. *$C(X)$ contains Hilbert cubes if and only of X contains ∞-odds.*

Proof. Let $\mathscr{A}$ be a Hilbert cube in $C(X)$. Suppose that X has not ∞-odds. Let $K_1^1; K_1^2, K_2^2; \ldots; A_1, A_2, \ldots; p_1, p_2, \ldots;$ and $H_1, H_2, \ldots$ as in Theorem 2.7. Let $A\in\bigcap\{\mathscr{A}_n\colon n\geqslant 1\}$. We will prove that A is an ∞-odd. For each n, put $L_n = (K_n^n - H_n)\cup(K_n^{n+1}-H_{n+1})\cup\ldots$ Let $H = \bigcap\{H_n\colon n\geqslant 1\}$. Then

$$A-H = ((A-H)\cup L_1)\cup((A-H)\cap L_2)\cup\ldots; L_1, L_2, \ldots$$

are pairwise disjoint and $p_n\in(A-H)\cap L_n$. Given $x\in(A-H)\cap L_s$, there exists $n\geqslant s$ such that $x\in K_s^n - H_n$. Then $x\in(A-H)\cap\big(X-(H_n\cup(\bigcup\{K_t^n\colon t\neq s\}))\big)\subset L_s$. This proves that $(A-H)\cap L_s$ is open in $A-H$. Hence A is an ∞-odd.

3. An example. In this section we show a continuum X for which $C(X)$ contains Hilbert cubes but it has not Property C.

Consider a nondegenerate, hereditarily indecomposable continuum Y. Let B be a proper, nondegenerate subcontinuum of Y an let $f\colon Y\to I$ be a map such that $B = f^{-1}(0)$. Define $A_0 = Y\times\{0\}\subset Y\times I$ and, for each n, define

$$A_n = \{(y, (1/n)f(y))\in Y\times I\colon y\in Y\}.$$

Finally, put $X = A_0\cup A_1\cup\ldots\subset Y\times I$ (X is a book where the sheets are copies of Y and the spine is B).

Notice that, for each $n\neq m$ $(n, m\geqslant 0)$, A_n is homeomorphic to Y and $A_n\cap A_m = B\times\{0\}$. Furthermore $X-(B\times\{0\})$ has an infinite number of components. Therefore $C(X)$ contains Hilbert cubes.

Now, suppose that there exist subcontinua $E, E_1, E_2, \ldots$ of X such that $E_0 - E, E_2 - E, \ldots$ are pairwise disjoint; $\operatorname{diam} E_n\to 0$ and $E_n - E$, $E_n\cap E$ are nonempty for all n. Choose N such that $\operatorname{diam} E_n < \operatorname{diam}(B\times\{0\})$ for each $n\geqslant N$. We analize two cases:

(a) There exists $n\geqslant N$ such that $E_n\cap(B\times\{0\})\neq\emptyset$. Then, for each $m\geqslant 0$, all the components of $E_n\cap A_m$ intersects $B\times\{0\}$. Hence $E_n\subset B\times\{0\}$. In particular, $E\cap(B\times\{0\})\neq\emptyset$ and $B\times\{0\}\not\subset E$. It follows that $E\subset B\times\{0\}$. Thus $E_{n+1}\cap\cap(B\times\{0\})\neq\emptyset$, so that $E_{n+1}\subset B\times\{0\}$. Then $E_n\cup E\cup E_{n+1}\subset B\times\{0\}$. This is a contradiction since B is hereditarily indecomposable.

(b) $E_n\cap(B\times\{0\}) = \emptyset$ for each $n\geqslant N$. Let $m\geqslant 0$ be such that $E_N\subset A_m$. This implies that $E\subset A_m$ and then $E_{N+1}\subset A_m$. This contradicts the fact that A_m is hereditarily indecomposable.

Hence X has not Property C.

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