Some propositions equivalent to the Sikorski Extension Theorem for Boolean algebras

by

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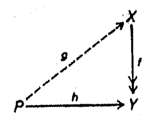
Abstract. It is known that the Sikorski Extension Theorem (every complete Boolean algebra is injective) is not a consequence of the Boolean Prime Ideal Theorem. In this paper we formulate several propositions equivalent to the Sikorski Extension Theorem and, as a consequence, solve in the negative a problem of Rubin and Rubin as to whether a certain proposition concerning Boolean algebras is implied by the Boolean Prime Ideal Theorem.

In [1], it is shown that the Sikorski Extension Theorem for Boolean algebras is not a consequence of the Boolean Prime Ideal Theorem. (It is still an open problem as to whether the Sikorski Extension Theorem is equivalent to the axiom of choice.) It is, accordingly, of some interest to discover propositions equivalent to the Sikorski Extension Theorem. In this paper we formulate several such propositions, and, as a consequence, solve in the negative a problem of Rubin and Rubin [4] as to whether a certain proposition concerning Boolean algebras is implied by the Boolean Prime Ideal Theorem.

§ 1. Preliminaries. Several of our propositions will be topological in nature; accordingly we begin with some topological definitions and results.

Let $f\colon X \to Y$ be a continuous surjection from a topological space X to a topological space Y. The map f is said to be *irreducible* if the image of a proper closed subset of X is a proper subset of Y.

Let $\mathbf{C}$ be a category of topological spaces. A space P in $\mathbf{C}$ is said to be *projective*



in $\mathbf{C}$ if for any epic arrow $f\colon X \to Y$ in $\mathbf{C}$ and any arrow $P \xrightarrow{h} Y$ in $\mathbf{C}$ there is an arrow $P \xrightarrow{g} X$ in $\mathbf{C}$ such that $f \circ g = h$.

A space X is (a) *Boolean* if it is compact Hausdorff and has a base of clopen

(= open-and-closed) subsets, (b) *extermally disconnected* if the closure of any open subset is open.

Let X be compact Hausdorff. A *Gleason cover* of X is a pair (E, π) consisting of an extremally disconnected compact Hausdorff space E and an irreducible continuous surjection $\pi\colon E \to X$.

We write **BooSp**, **CompHaus** for the categories of Boolean spaces, compact Hausdorff spaces (and continuous mappings) respectively.

Let **Bool** be the category of Boolean algebras and Boolean homomorphisms. The Boolean Prime Ideal Theorem (**BPI**), which asserts that every Boolean algebra contains a prime ideal, or, equivalently, an ultrafilter, implies via the Stone representation theory, that **BooSp** is equivalent to $\mathbf{Bool}^{op}$, the opposite category of **Bool**.

From now on we work in Zermelo–Fraenkel set theory **ZF**; thus the axiom of choice is *not* assumed. We shall need the following results of Gleason [3], which are all proved in **ZF**, unless otherwise stated.

1.1. LEMMA. (Lemma 2.3 of [3]). *Any irreducible surjection of a compact Hausdorff space to an extremally disconnected compact Hausdorff space is a homeomorphism.*

1.2. THEOREM. (Theorem 3.2 of [3]). *Assume* **BPI**. *Then any compact Hausdorff space has a Gleason cover.*

§ 2. Equivalents of the Sikorski Extension Theorem. The **Sikorski Extension Theorem** states that, for any Boolean algebra A and any complete Boolean algebra B, any homomorphism of a subalgebra of A into B can be extended to the whole of A. That is,

SET: Any complete Boolean algebra is injective in **Bool**.

Under the Stone equivalence between $\mathbf{Bool}^{op}$ and **BooSp**, complete Boolean algebras correspond to extremally disconnected spaces. Accordingly, *assuming* **BPI**, **SET** is equivalent to the assertion:

A. *Any extremally disconnected compact Hausdorff space is projective in* **BooSp**.

Now consider the additional statements:

B. *Any extremally disconnected compact Hausdorff space is projective in* **CompHaus**.

C. *Any continuous surjection between extremally disconnected compact Hausdorff spaces has an irreducible restriction to a closed subset of its domain.*

D. *Any continuous surjection between Boolean spaces has an irreducible restriction to a closed subset of its domain.*

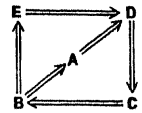
E. *Any continuous surjection between compact Hausdorff spaces has an irreducible restriction to a closed subset of its domain.*

B is (essentially) Theorem 2.5 of [3]

We prove the following

2.1. THEOREM. *Assuming* **BPI**, *assertions* **A–E** *are all equivalent to* **SET**. *Thus* **SET** *is equivalent (over* **ZF**) *to the conjunction of* **BPI** *with each of* **A–E**.

Proof. We establish the complex of implications



First, $\mathbf{E} \Rightarrow \mathbf{D} \Rightarrow \mathbf{C}$ and $\mathbf{B} \Rightarrow \mathbf{A}$ are obvious. We prove $\mathbf{C} \Rightarrow \mathbf{B}$, $\mathbf{B} \Rightarrow \mathbf{E}$ and $\mathbf{A} \Rightarrow \mathbf{D}$.

$\mathbf{C} \Rightarrow \mathbf{B}$. Assume **C**. We first show that

(*) Any continuous surjection $f\colon X \to E$ from a compact Hausdorff space X to an extremally disconnected compact Hausdorff space E has an irreducible restriction to a closed subset of its domain.

To prove (*), by 1.2 let (D, π) be a Gleason cover of X. Then $f \circ \pi$ is a continuous surjection from D to E and so by **C** there is a closed subset Z of D such that $(f \circ \pi)|Z$ is an irreducible surjection from Z to E. Since Z is closed, it is compact (Hausdorff) and so by 1.1 $(f \circ \pi)|Z$ is a homeomorphism.

Now consider $\pi[Z] = Y$. As the continuous image of the compact space Z, Y is compact and hence closed. Also, since $(f \circ \pi)|Z$ is a homeomorphism, $f|Y$ is a homeomorphism of Y with E. Clearly, then, $f|Y$ is irreducible. This proves (*).

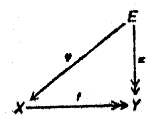
Now to get B, we use (*) to adapt the proof of Theorem 2.5 of [3]. Thus let E be an extremally disconnected compact Hausdorff space, let X and Y be compact Hausdorff spaces; let h be a continuous map of E to Y and let f be a continuous surjection from X to Y. We must show that there is a continuous map g from E to X such that $f \circ g = h$.

In the space $E \times X$ consider $Z = \{\langle a, x\rangle \in E \times X\colon h(a) = f(x)\}$. This set is evidently closed: hence compact (and Hausdorff). Since f is surjective, the projection π of $E \times X$ onto E sends Z onto E. By (*) there is a closed (hence compact) subset W of Z such that $\pi_1|W$ is an irreducible surjection between W and E. By Lemma 1.1, $\varphi = \pi_1|W$ is a homeomorphism between W and E. Define $g = \pi_2 \circ \varphi^{-1}$, where π_2 is the projection of $E \times X$ onto X. Then g is the required map from E to X. For if $a \in E$, then since $\varphi^{-1}(a) \in Z$ we have

$$f(g(a)) = f\big(\pi_2(\varphi^{-1}(a))\big) = h\big(\pi_1(\varphi^{-1}(a))\big) = h(a).$$

Thus $f \circ g = h$ as required. This proves **B**.

$\mathbf{B} \Rightarrow \mathbf{E}$. Assume **B** and let $X \overset{f}{\twoheadrightarrow} Y$ be a continuous surjection between compact Hausdorff spaces X, Y. By 1.2, let (E, π) be a Gleason cover of Y. By **B**, E is projective in **CompHaus**. So there is a continuous map $\varphi\colon E \to X$ such that the diagram



commutes. Now consider $Z = \varphi[E]$. Clearly Z is compact and hence closed in X. Moreover, $f[Z] = f[\varphi[E]] = Y$. We claim that $f|Z$ is irreducible. For let W be any closed subset of Z such that $f[W] = Y$. Consider the closed subset $\varphi^{-1}[W]$ of E. We have (since $W \subseteq \varphi[E]$)

$$\pi[\varphi^{-1}[W]] = f[\varphi[\varphi^{-1}[W]]] = f[W] = Y.$$

Since $\pi|E$ is irreducible, it follows that $\varphi^{-1}[W] = E$, whence $W = \varphi[E] = Z$. Therefore $f|Z$ is irreducible as claimed, and **E** follows.

Finally, **A** $\Rightarrow$ **D** is proved as in **B** $\Rightarrow$ **E**, replacing "compact Hausdorff" by "Boolean" everywhere.

Since, assuming **BPI**, **A** is equivalent to **SET** we have our theorem. ■

It is proved in [1] that **SET** is not a consequence of **BPI** (in **ZF**). It follows from this and 2.1 that

2.2. Corollary. *None of the statements* **A–E** *is a consequence of* **BPI**. ■

We now reformulate statement **D** in terms of Boolean algebras. Let $m: A \to B$ be a monomorphism between Boolean algebras A and B. An epimorphism $p: B \twoheadrightarrow C$ to a Boolean algebra C is said to be *m-minimal* if the following two conditions hold: (1) the composition $p \circ m$ is monic; (2) if $C \overset{q}{\twoheadrightarrow} D$ is any epimorphism to a Boolean algebra D such that $q \circ p \circ m$ is monic, then q is an isomorphism.

Now consider the following assertions.

F. If $m: A \rightarrowtail B$ is a monomorphism between Boolean algebras, then there is an m-minimal epimorphism $p: B \twoheadrightarrow C$ to some Boolean algebra C.

G. If A is a subalgebra of a Boolean algebra B, then there is a $\subseteq$-maximal proper filter F in B such that $F \cap A = \{1\}$.

It is easy to show that **F** and **G** are equivalent, and that both of them imply **BPI**. Moreover, under the Stone equivalence between **BooSp**^{op} and **Bool**, statement **D** clearly corresponds to statement **F**. Theorem 2.1 therefore yields the following result.

2.3. Corollary. *Statements* **F** *and* **G** *are each equivalent to* **SET**, *and are therefore not consequences of* **BPI**. ■

This leads to a negative solution of a problem of Rubin and Rubin [4]. On p. 101 of that volume, they consider the statement

S. If B is a Boolean algebra and $S \subseteq B$ such that S is closed w.r.t. $\wedge$ then there is a $\subseteq$-maximal proper ideal I such that $I \cap S = \{0\}$.

They state that it is not known whether **BPI** implies **S**. Now **S** is equivalent to the dual assertion **S′** obtained by replacing "$\wedge$", "ideal", "$\{0\}$" by "$\vee$", "filter", "$\{1\}$", respectively, and **S′** evidently implies **G**. Therefore from Corollary 2.3, we infer the

2.4. Corollary. **S** *is not a consequence of* **BPI**.

§ 3. **A statement equivalent to the axiom of choice.** As we have pointed out, it is unknown whether **SET** or any of the statements **A–E** imply the axiom of choice. However, by dropping "Hausdorff" for the domain space of statement **E** we get an assertion equivalent to the axiom of choice.

3.1. Theorem. *The assertion*:

E′. *Any continuous surjection between compact spaces where the range space is Hausdorff has an irreducible restriction to a closed subset of its domain*

implies the axiom of choice.

Proof. We adapt an argument of Franklin and Thomas [2]; see Thm 8.9 of Rubin and Rubin [4].

Let $\{X_i: i \in I\}$ be a family of non-empty pairwise disjoint sets, and for each $i \in I$ assign X_i the cofinite topology (a subset is open iff it is empty or its complement is finite). Each X_i is then compact. Let $X = \bigcup_{i \in I} X_i$ be the free union of the topological spaces X_i: that is, $U \subseteq X$ is open iff $U \cap X_i$ is open in X_i for all $i \in I$. Let $X' = X \cup \{*\}$ be the one-point compactification of X. Assign I the discrete topology and let $I' = I \cup \{*\}$ be the one-point compactification of I. Then I' is compact Hausdorff.

Define $f: X' \to I'$ by

$$f(x) = \text{unique } i \in I \text{ such that } x \in X_i$$

if $x \in X$; $f(*) = *$. It is easy to check that f is a continuous surjection. Therefore, by **E′**, there is a closed subset C of X' such that $f|C$ is an irreducible surjection. Clearly the intersection of C with each X_i must be a singleton. The axiom of choice follows. ■

References

[1] J. L. Bell, *On the strength of the Sikorski extension theorem for Boolean algebras*, J. Symbolic Logic 48 (1983), 841–846.

[2] S. P. Franklin and B. V. S. Thomas, *Another topological equivalent of the axiom of choice*, Amer. Math. Monthly 78 (1976), 1109–1110.

[3] A. M. Gleason, *Projective topological spaces*, Ill. J. Math. 2 (1958), 482–489.

[4] H. Rubin and J. E. Rubin, *Equivalents of the Axiom of Choice*, II, Studies in Logic and the Foundations of Mathematics Vol. 116, North Holland, Amsterdam 1985.

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