Maximally conjugate sigma-algebras represented as hypergraphs

by

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Abstract. We obtain (Proposition 1.18) a characterization of all maximal conjugates to a sub-σ-algebra generated by a countable partition. It is shown (Proposition 2.2) that every countably generated sub-σ-algebra $\mathscr{C}$ of an analytic space has a maximal conjugate. Some of these maximal conjugates are not countably generated. In Proposition 3.6, we characterize such countably generated maximal conjugates in terms of the structure of an associated combinatorial object (a hypergraph). In Proposition 4.2, we prove that such conjugates exist precisely when there is a measurable selector for the collection of $\mathscr{C}$-atoms. Finally, Section 5 disposes of a natural conjecture by exhibiting a maximal conjugate which is not a complement. Some questions for future investigation conclude the paper.

§ 0. Introduction. We continue to study a measurable space $(X, \mathscr{B})$ through the lattice of all sub-σ-algebras of $\mathscr{B}$. If $\mathscr{C}$ and $\mathscr{D}$ are sub-σ-algebras of $\mathscr{B}$, their infimum in this lattice is $\mathscr{C} \cap \mathscr{D}$, and their supremum is the σ-algebra $\sigma(\mathscr{C}, \mathscr{D})$ generated by $\mathscr{C} \cup \mathscr{D}$. An analysis of this lattice is to be found in [2]. For example, there it is shown that this lattice structure completely determines $\mathscr{B}$. This paper continues the work of [2] and [9].

Let $(X, \mathscr{B})$ be a measurable space and let $\mathscr{C}$ and $\mathscr{D}$ be sub-σ-algebras of $\mathscr{B}$. Say that $\mathscr{D}$ is a *conjugate* for $\mathscr{C}$ if $\mathscr{C} \cap \mathscr{D} = \{\emptyset, X\}$. Say that $\mathscr{D}$ is a *weak complement* for $\mathscr{C}$ if $\sigma(\mathscr{C}, \mathscr{D}) = \mathscr{B}$. Then $\mathscr{D}$ is a *complement* for $\mathscr{C}$ if it is both a conjugate and a weak complement. Say that $\mathscr{D}$ is a *maximal conjugate* for $\mathscr{C}$ if it is a conjugate and is such that whenever $\mathscr{D}' \supseteq \mathscr{D}$ is a conjugate for $\mathscr{C}$, then $\mathscr{D}' = \mathscr{D}$. The notions of minimal weak complement, minimal complement and maximal complement are similarly defined.

A measurable space $(X, \mathscr{B})$ is *separable* if $\mathscr{B}$ is countably generated (c.g.) and contains all singleton subsets of X. An *atom* of a sub-σ-algebra $\mathscr{C}$ of $\mathscr{B}$ is a nonempty set $C \in \mathscr{C}$ such that whenever $C' \subseteq C$ is a $\mathscr{C}$-set, then either $C' = C$ or $C' = \emptyset$. Say that $\mathscr{C}$ is *atomic* if X is a union of $\mathscr{C}$-atoms. Every c.g. $\mathscr{C}$ is atomic.

In order to aid the eye with the reading of fairly long Boolean expressions, the old Polish notation has been adopted: $AB = A \cap B$, $A + B = A \cup B$, and $AB \dotplus C$

$= (A \cap B) \cup C$. It avoids a nasty proliferation of parentheses. If $\mathscr{D}$ is a sub-σ-algebra of $\mathscr{B}(X)$ and $A \subseteq X$, then $\mathscr{D}(A) = \{DA.\ D \in \mathscr{D}\}$. Note also that if $B \subseteq X$, then the σ-algebra $\sigma(\mathscr{D}, B)$ generated by $\mathscr{D} \cup \{B\}$ comprises all sets of the form $D_1 B + D_2 B^c$, where the D_i are $\mathscr{D}$-sets. A 0-1 *measure* μ on $(X, \mathscr{B})$ is a nonzero measure assuming only the values 0 and 1. Given such a measure μ, the collection $\{B \in \mathscr{B}\colon \mu B = 0\}$ is a maximal σ-ideal in $\mathscr{B}$. Every maximal σ-ideal arises in this way.

Let $\mathscr{C}$ be an atomic sub-σ-algebra of $\mathscr{B}(X)$. A set $S \subseteq X$ is a *measurable full selector* for $\mathscr{C}$ if $S \in \mathscr{B}(X)$ and S intersects each $\mathscr{C}$-atom in exactly one point.

Let f be a real-valued function on X and set

$$\mathscr{C} = \{f^{-1}(B)\colon B \subseteq \boldsymbol{R},\ B \text{ Borel}\}.$$

Then $\mathscr{C}$ is the σ-algebra *generated by* f. The method of Marczewski functions [4: p. 7] ensures that every c.g. $\mathscr{C} \subset \mathscr{B}(X)$ is generated by some $f\colon X \to \boldsymbol{R}$. The $\mathscr{C}$-atoms are then sets of the form $f^{-1}(p)$ for $p \in \boldsymbol{R}$.

A separable space $(X, \mathscr{B})$ is *standard* if there is some Polish topology on X whose Borel structure is $\mathscr{B}$. A space $(X, \mathscr{B})$ is *analytic* if it is the measurable image of some standard space. A separable space $(X, \mathscr{B})$ has the *strong Blackwell property* if whenever $\mathscr{C}$ and $\mathscr{D}$ are c.g. sub-σ-algebras of $\mathscr{B}$ with the same atoms, then necessarily $\mathscr{C} = \mathscr{D}$. Every analytic space has the strong Blackwell property [2: p. 21]. A sub-σ-algebra $\mathscr{D}$ of $\mathscr{B}(X)$ is said to *separate* sets $A_1, A_2 \subseteq X$ if there is some $D \in \mathscr{D}$ with $A_1 \subseteq D$ and $A_2 \subseteq D^c$. Lusin's first separation principle [4: p. 32] says that $\mathscr{B}(X)$ separates any two analytic sets $A_1, A_2 \subseteq X$.

Typically, we hold an atomic sub-σ-algebra $\mathscr{C}$ of $\mathscr{B}(X)$ as fixed and search for maximal conjugates for $\mathscr{C}$. Let $\mathscr{D}$ be a sub-σ-algebra of $\mathscr{B}(X)$ and let $B \subseteq X$. Two $\mathscr{C}$-atoms C and C' are *separated by* $\mathscr{D}$ *on* B if there is a $D \in \mathscr{D}$ such that $CB \subseteq D$ and $C'B \subseteq D^c$. The atoms C and C' are *separated by* $\mathscr{D}$ if they are separated by $\mathscr{D}$ on X. Clearly, if $\mathscr{D}$ separates C and C' on B, then it also separate them on any $A \subseteq B$. A $\mathscr{D}$-*cluster* is a collection of $\mathscr{C}$-atoms, no two of which are separated by $\mathscr{D}$. Zorn's lemma implies that every $\mathscr{D}$-cluster is contained in a maximal $\mathscr{D}$-cluster.

A pair $(V, \mathscr{E})$, where V is a nonempty set, and $\mathscr{E}$ is a collection of nonempty subsets of V whose union is V, we term a *hypergraph*. We refer the reader to the text of Berge [1] for further details. In his terminology, our $(V, \mathscr{E})$ is a "simple hypergraph". The elements of V and $\mathscr{E}$ are termed *vertices* and *edges*, respectively. In $(V, \mathscr{E})$, a *chain* is a sequence $v_1 E_1 v_2 E_2 \ldots v_n E_n v_{n+1}$ such that

(1) $v_1 \ldots v_n$ are distinct vertices;

(2) $E_1 \ldots E_n$ are distinct edges;

(3) $v_k, v_{k+1} \in E_k$ for $k = 1, \ldots, n$.

If $n > 1$ and $v_{n+1} = v_1$, then this chain is a *cycle*. Occasionally, the notation of the E_k is suppressed. For vertices v, v' we write $v \sim v'$ if there is a chain starting at v and ending at v'. Then $\sim$ is an equivalence relation whose equivalence classes are the *connected components* of $(V, \mathscr{E})$: see p. 391 in [1]. A *tree* is a connected hypergraph without cycles.

LEMMA 0.1. *Suppose that a hypergraph $(V, \mathscr{E})$ has no cycles. Then no two edges intersect in more than one vertex.*

Proof. If E and F are distinct edges such that $E \cap F$ contains distinct vertices v and v', then $vEv'Fv$ is a cycle, yielding a contradiction. ■

The following fact must certainly be known, but the author has no references for it.

LEMMA 0.2. *Suppose the hypergraph $(V, \mathscr{E})$ is a tree. Given an edge E in $\mathscr{E}$, put $\mathscr{E}_0 = \mathscr{E} \setminus \{E\}$. Then $(V, \mathscr{E}_0)$ has one connected component for each vertex of E.*

Proof. Given a vertex v of E, define $K(v)$ to be the collection of $v' \in V$ for which there is a chain from v to v' not including any vertex of E other than v. Clearly, $v \in K(v)$.

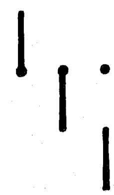
CLAIM 1. $V = \bigcup \{K(v)\colon v \in E\}$. Given $v' \in V$, choose any $v \in E$ and let $v' = v_1 v_2 \ldots v_n v_{n+1} = v$ be a chain. Let k be the least index such that $v_k \in E$. Then $v' \in K(v_{k+1})$ as desired.

CLAIM 2. The sets $K(v)$ are disjoint. Suppose rather $v'' \in K(v) \cap K(v')$ for distinct vertices v', v in E. Let $v'' = u_1 E_1 u_2 E_2 \ldots u_r E_r v$ and $v'' = w_1 F_1 w_2 F_2 \ldots w_s F_s v'$ be chains. Then from the sequence $v'' = u_1 E_1 u_2 E_2 \ldots u_r E_r v E v' F_s w_s F_{s-1} \ldots F_1 w_1 = v''$ it is possible to construct a cycle in $(V, \mathscr{E})$. This contradiction proves the claim.

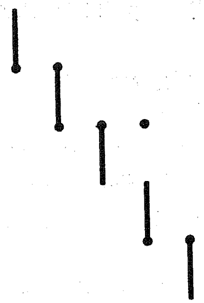
It is easy to check that each $K(v)$ is connected. It remains to show only that there are no chains between disjoint $K(v)$ and $K(v')$. Suppose that $v = v_1 E_1 v_2 \ldots$ $\ldots v_m E_m v_{m+1} = v'$ is a chain in $(V, \mathscr{E}_0)$. Then $v = v_1 E_1 \ldots v_m E_m v_{m+1} E v$ is a cycle in $(V, \mathscr{E})$, a contradiction. ■

Given an atomic sub-σ-algebra $\mathscr{C}$ of $\mathscr{B}(X)$ we associate to every sub-σ-algebra $\mathscr{D}$ of $\mathscr{B}(X)$ a hypergraph $h(\mathscr{D}) = (V, \mathscr{E})$. The vertices of this hypergraph are the $\mathscr{C}$-atoms. Its edges are the maximal $\mathscr{D}$-clusters. It is our purpose to study the maximality of conjugates for $\mathscr{C}$ through their hypergraphs. In previous work (Theorem 5 in [9]), we characterised maximal conjugates for $\mathscr{C}$ which separate no two $\mathscr{C}$-atoms: this is the case where $\mathscr{E} = \{V\}$. As an example of our approach, we offer the following:

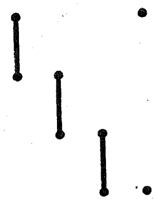
(i)



(ii)

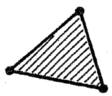


(iii)

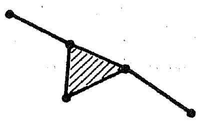


In this figure are represented three subsets of the plane. In each case, the σ-algebras $\mathscr{C}$ and $\mathscr{D}$ are generated by projection onto the lower and left margins, respectively. In Examples 1 and 2, $\mathscr{D}$ is a maximal conjugate for $\mathscr{C}$. In Example 3, $\mathscr{D}$ is a conjugate, but not maximal. The following is a geometric rendering of the hypergraphs $h(\mathscr{D})$ corresponding to each example.

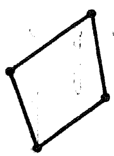
(i)



(ii)



(iii)



The hypergraphs in Examples 1 and 2 are trees. There is a cycle in Example 3. The next section (Proposition 1.17) shows why.

We conclude this introduction by showing that maximal conjugates do not always exist.

EXAMPLE 0.3. Let $\mathscr{B} = \mathscr{B}(0, 1)$ be the usual linear Borel structure. Then the sub-structure $\mathscr{C}$ generated by Borel sets of Lebesgue measure zero has no maximal conjugate in $\mathscr{B}$.

To see this, suppose that $\mathscr{D}$ is such a maximal conjugate. Then we

CLAIM. $\mathscr{D}$ is an atomless structure. For suppose that D is a $\mathscr{D}$-atom. Then has Lebesgue measure $0 < mD < 1$ and may therefore be written as the disjoint union $D = B_1 + B_2$ of two Borel sets of positive measure. Consider the σ-algebra $\mathscr{D}_0 = \sigma(\mathscr{D}, B_1)$, a strict enlargement of $\mathscr{D}$. It must be that for some sets D_1, D_2 in $\mathscr{D}$, we have $D_1 B_1 + D_2 B_1^c$ of Lebesgue measure zero but not empty. If $D_1 B_1$ is not empty, then $D_1 B_1 = B_1$ is of positive measure; if $D_2 B_1^c$ is nonempty, then either $B_2 \subseteq D_2 B_1^c$ is of positive measure or $D_2 B_1^c \in \mathscr{D}$ is of positive measure. These conclusions are contradictory, and the claim is established.

Since $\mathscr{D}$ is atomless, we may write X as the disjoint union of uncountably many $\mathscr{D}$-sets (see p. 29 in [2]), each of which is of positive measure. This impossibility shows that no such $\mathscr{D}$ exists.

§ 1. **Conjugation for discrete structures.** Throughout this section, $(X, \mathscr{B})$ will denote a measurable space and V a partition of X into countably many $\mathscr{B}$-sets. Let $\mathscr{C} = \sigma(V)$ and suppose that $\mathscr{D}$ is some sub-σ-algebra of $\mathscr{B}$. As described in the introduction, each substructure $\mathscr{D}$ gives rise to a hypergraph $h(\mathscr{D}) = (V, \mathscr{E})$, where $\mathscr{E}$ is the collection of all maximal $\mathscr{D}$-clusters of $\mathscr{C}$-atoms.

LEMMA 1.1. *The sub-σ-algebra $\mathscr{D}$ is a conjugate for $\mathscr{C}$ if and only if $h(\mathscr{D}) = (V, \mathscr{E})$ is connected.*

Proof. Suppose that $\mathscr{D}$ is not a conjugate for $\mathscr{C}$ and select some nontrivial D from $\mathscr{C} \cap \mathscr{D}$. Then there are $\mathscr{C}$-atoms $C \subseteq D$ and $C' \subseteq D^c$. If $(V, \mathscr{E})$ is connected, then there is a chain $C = C_0 E_0 C_1 E_1 \dots E_{n-1} C_n = C'$ connecting C to C'. Now each $\mathscr{C}$-atoms is contained in D or D^c. Let i be the least index such that $C_i \subseteq D^c$. The fact that $C_{i-1} \subseteq D$ yields a contradiction. It must be that $(V, \mathscr{E})$ is not connected.

Now suppose that $(V, \mathscr{E})$ is not connected. Let $V_0 \subseteq V$ be one of the connected components. For each $C \in V_0$ and $C' \in V \setminus V_0$ we choose a $\mathscr{D}$-set $D[C, C']$ such that $C \subseteq D[C, C']$ and $C' \subseteq D[C, C']^c$. Then put

$$D = \bigcup_{C} \bigcap_{C'} D[C, C'] .$$

The set D is a nontrivial set in $\mathscr{C} \cap \mathscr{D}$. ■

Note that only the second half of this proof used the countability of V. However, its use there is essential, as we shall see in the next section (Example 2.1). The next two technical lemmas (especially 1.2) constitute the *pons asinorum* of our approach.

LEMMA 1.2. *Suppose that $\mathscr{D}$ is a sub-σ-algebra of $\mathscr{B}(X)$ and that $\mathscr{C} = \sigma(V)$. Suppose also that $D \in \mathscr{C}$ and that $C_0 \in \mathscr{C}$. Define $\mathscr{D}_0 = \sigma(\mathscr{D}, D \cap C_0)$.*

(1) *If C and C' are $\mathscr{C}$-atoms separated by $\mathscr{D}_0$, then C and C' are separated by $\mathscr{D}$ on D^c.*

(2) *If C and C' are $\mathscr{C}$-atoms separated by $\mathscr{D}_0$ but not by $\mathscr{D}$, then precisely one of C, C' is a subset of C_0.*

Proof. (1) For some $\mathscr{D}$-sets D_1 and D_2 we have $C \subseteq D_1(DC_0) + D_2(DC_0)^c = D_1 DC_0 + D_2 C_0^c + D_2 C^c$ and $C' \subseteq D_1^c DC_0 + D_2^c C_0^c + D_2^c D^c$. Then $CD^c \subseteq D_2$ and $C'D^c \subseteq D_2^c$.

(2) Suppose that for D_1, D_2 in $\mathscr{D}$ we have $C \subseteq D_1 DC_0 + D_2 C_0^c + D_2 D^c$ and $C' \subseteq D_1^c DC_0 + D_2^c C_0^c + D_2^c D^c$. If C, $C' \subseteq C_0^c$, then $C \subseteq D_2$ and $C' \subseteq D_2^c$, a contradiction. If C, $C' \subseteq C_0$, then $C \subseteq D_1 D + D_2 D^c$ and $C' \subseteq D_1^c D + D_2^c D^c = (D_1 D + D_2 D^c)^c$, another contradiction. The only alternative is Statement 2. ■

LEMMA 1.3. *Suppose that $\mathscr{D}$ is a maximal conjugate for $\mathscr{C} = \sigma(V)$, that C is a $\mathscr{C}$-atom, and that $B \subseteq C$ is a set in $\mathscr{B} \setminus \mathscr{D}$. Put $\mathscr{D}_0 = \sigma(\mathscr{D}, B)$.*

(1) *If C_1 and C_2 are $\mathscr{C}$-atoms not separated by $\mathscr{D}$, but separated by $\mathscr{D}_0$, then one of C_1, C_2 equals C.*

(2) *There is a $\mathscr{C}$-atom C' such that $\mathscr{D}$ does not separate C and C', but $\mathscr{D}_0$ does. Also, C and C' are separated by $\mathscr{D}$ on B^c.*

Proof. (1) Let $C_1 \subseteq D_1 B + D_2 B^c$ and $C_2 \subseteq D_1^c B + D_2^c B^c$ provide a separation by $\mathscr{D}_0$. If neither C_i equals C, then because $B \subseteq C$, we have $C_1 \subseteq D_2 B^c \subseteq D_2$ and $C_2 \subseteq D_2^c B^c \subseteq D_2^c$. This is a contradiction.

(2) By Lemma 1.1, $\mathscr{D}_0$ must separate two $\mathscr{C}$-atoms not separated by $\mathscr{D}$. From the first part of this lemma, we see that one of these $\mathscr{C}$-atoms is C and that they are separated by $\mathscr{D}$ on B^c. ■

Let $u_1 u_2 u_3 \ldots$ be a decreasing sequence of positive reals such that, for each n, $u_n > u_{n+1} + u_{n+2} + \ldots$ (e.g. $u_n = 3^{-n}$). Define u to be a one-one map assigning to each pair of $\mathscr{C}$-atoms $\{C, C'\}$ not separated by $\mathscr{D}$ a number $u\{C, C'\}$ in the sequence. If there be infinitely many such pairs, then take u to be a one-one correspondence; if only n such pairs, then use $u_1 u_2 \ldots u_n$. For $D \in \mathscr{D}$, we define the set function

$$\psi(D) = \sum u\{C, C'\},$$

where the sum is taken over all pairs $\{C, C'\}$ such that C and C' are not separated by $\mathscr{D}$ on D. Clearly, ψ is monotone (isotone). We shall prove that when $\mathscr{D}$ is a maximal conjugate, ψ is a measure on $\mathscr{D}$. This will enable us to produce a useful decomposition of X into rather "simple" $\mathscr{D}$-sets.

LEMMA 1.4. *If $D_n \uparrow D$ in $\mathscr{D}$, then $\psi(D_n) \uparrow \psi(D)$.*

Proof. If C and C' are $\mathscr{C}$-atoms not separated on D, then we claim that there is some n with C, C' not separated on D_n. Otherwise, we choose $\mathscr{D}$-sets $D(n)$ such that $C \cap D_n \subseteq D(n)$ and $C' \cap D_n \subseteq D(n)^c$. Put $D_0 = \limsup D(n)$. Then $C \cap D \subseteq D_0$ and $C' \cap D \subseteq D_0^c = \liminf D(n)^c$. The lemma follows from the claim. ■

A $\mathscr{D}$-set D is *completely reduced* if every pair of $\mathscr{C}$-atoms is separated by $\mathscr{D}$ on D; equivalently, $\psi(D) = 0$. A $\mathscr{D}$-set $D' \subseteq D$ *reduces* D if neither D' nor $D \setminus D'$ is completely reduced. A $\mathscr{D}$-set which cannot be reduced but is not completely reduced is *irreducible*.

LEMMA 1.5. *Suppose that C and C' are distinct $\mathscr{C}$-atoms and that $D \in \mathscr{D}$, where $\mathscr{D}$ is a maximal conjugate for $\mathscr{C}$. Then C and C' are separated by $\mathscr{D}$ on either D or D^c.*

Proof. Suppose rather that C and C' are not separated on either D or D^c. Let V_0 be the collection of those $\mathscr{C}$-atoms C'' for which there is a sequence $C' = C_0 C_1 C_2 \ldots C_n = C''$ of $\mathscr{C}$-atoms such that

(1) C_i and C_{i+1} are not separated on D for $i = 0, 1, \ldots, n-1$;

(2) C does not occur in the sequence.

We also include C' in V_0. Consider the σ-algebra $\mathscr{D}_0 = \sigma(\mathscr{D}, D \cap C_0)$, where C_0 is the union of the atoms in V_0. Since $\mathscr{D}_0$ separates C and C' on D, it strictly enlarges $\mathscr{D}$. We claim that $h(\mathscr{D}_0)$ is connected. This will contradict Lemma 1.1 and complete this proof.

We show that every $\mathscr{C}$-atom is connected to C' in $h(\mathscr{D}_0)$. Note that this is certainly true for the $\mathscr{C}$-atoms in V_0: this follows from Lemma 1.2 — no new separations between these atoms have been introduced in $\mathscr{D}_0$. Now suppose that $C' = C_1 C_2 \ldots C_n$ is a sequence of $\mathscr{C}$-atoms such that C_i and C_{i+1} are not separated by $\mathscr{D}$ for $i = 1, \ldots, n-1$. Let i be the least index such that C' and C_{i+1} lie in different connected components of $h(\mathscr{D}_0)$. Thus C_i and C_{i+1} are separated by $\mathscr{D}_0$, but not by $\mathscr{D}$. By Lemma 1.2, exactly one of these is contained in C_0.

It cannot be that $C_i \subseteq C_0^c$ and $C_{i+1} \subseteq C_0$: this in view of the earlier remark that the $\mathscr{C}$-atoms in C_0 are connected to C' in $h(\mathscr{D}_0)$. So $C_i \subseteq C_0$ and $C_{i+1} \subseteq C_0^c$. Again using Lemma 1.2, C_i and C_{i+1} are separated by $\mathscr{D}$ on D^c, but not on D. Since $C_{i+1} \subseteq C_0^c$, it must be that $C_{i+1} = C$. Now C and C' are assumed not to be separated by $\mathscr{D}$ on D^c. Lemma 1.2 implies that C and C' are not separated by $\mathscr{D}_0$. This violates the choice of i and shows that $h(\mathscr{D}_0)$ is connected. This contradiction (Lemma 1.1) concludes the proof. ■

LEMMA 1.6. *If $\mathscr{D}$ is a maximal conjugate for $\mathscr{C} = \sigma(V)$, then ψ is a finite measure on $\mathscr{D}$.*

Proof. Keeping Lemma 1.4 in mind, we need only prove that ψ is finitely additive. Suppose that D_1 and D_2 are disjoint $\mathscr{D}$-sets. Then any $u\{C, C'\}$ can only occur in one of the sums defining $\psi(D_1)$ and $\psi(D_2)$: this is the import of Lemma 1.5. It follows that $\psi(D_1 + D_2) = \psi(D_1) + \psi(D_2)$. ■

LEMMA 1.7. *Suppose that* $\mathscr{D}$ *is a maximal conjugate for* $\mathscr{C} = \sigma(V)$ *and that* $D' \subseteq D$ *are* $\mathscr{D}$*-sets such that* D' *reduces* D. *Then* $0 < \psi(D') < \psi(D)$ *and* $0 < \psi(D \setminus D') < \psi(D)$.

Proof. Easily deduced from Lemma 1.6 and the definitions. ■

Lemma 1.6 enables us to decompose X into irreducible $\mathscr{D}$-sets. These sets are ψ-atoms and aid in the combinatorial analysis of $h(\mathscr{D})$.

LEMMA 1.8. *Let* $\mathscr{D}$ *be a maximal conjugate for* $\mathscr{C} = \sigma(V)$. *Then there is a decomposition of* $X = D_1 + D_2 + \ldots$ *into pairwise disjoint irreducible* $\mathscr{D}$*-sets. If* V *is finite, then so will be any such decomposition.*

Also, this decomposition can be chosen so that for each $\mathscr{C}$*-atom* C, *if* $CD_n \neq \emptyset$, *then there is some other* $\mathscr{C}$*-atom* C' *such that* C *and* C' *are not separated on* D_n.

Proof. Our choice of the numbers u_n means that the range of the set function ψ is totally disconnected. A standard decomposition theorem (see for instance 5.2.13 in [3]) allows one to write $X = D_0 + D_1 + D_2 + \ldots$, where the D_n are pairwise disjoint $\mathscr{D}$-sets, each D_n is a ψ-atom for $n \geqslant 1$, and the restriction of ψ to D_0 is nonatomic. However, the Liapounov Convexity Theorem (e.g. 11.4.5 in [3]) implies that $\psi(D_0) = 0$. Note also that if V is finite, then there are only finitely many ψ-atoms. The lemma follows from these remarks. ■

This next result on "transitivity" shows that the structure of $\mathscr{D}$ on each ψ-atom is rather simple.

LEMMA 1.9. *Let* D *be an irreducible* $\mathscr{D}$*-set. If* C, C', C'' *are* $\mathscr{C}$*-atoms with the pairs* $\{C, C'\}$ *and* $\{C', C''\}$ *not separated on* D, *then* C *and* C'' *are not separated on* D.

Proof. Suppose contrariwise and choose a $\mathscr{D}$-set D_0 such that $CD \subseteq D_0$ and $C''D \subset D_0^c$. Then DD_0 reduces D: C' and C'' are separated on DD_0, but not on $D \setminus D_0$; also, C and C' are separated on $D \setminus D_0$, but not on DD_0. This contradiction proves the lemma ■

Before proceeding with the analysis of $h(\mathscr{D})$, we complete the description of the measure ψ.

LEMMA 1.10. *Let* $X = D_1 + D_2 + \ldots$ *be a decomposition into irreducible* $\mathscr{D}$*-sets as in Lemma* 1.8. *For each* n,

$$\mathscr{I}_n = \{D \in \mathscr{D} \colon DD_n \text{ is completely reduced}\}$$

is a maximal σ*-ideal in* $\mathscr{D}$.

Proof. Lemma 1.4 shows that $\mathscr{I}_n$ is closed under countable unions. Now suppose that for some $D \in \mathscr{D}$, neither D nor D^c is an element of $\mathscr{I}_n$. Then neither DD_n nor D^cD_n is completely reduced. This contradicts the irreducibility of D_n. ■

It is not hard to show that if μ_n is the 0-1 measure on $\mathscr{D}$ corresponding to $\mathscr{I}_n$, then $\psi = \alpha_1\mu_1 + \alpha_2\mu_2 + \ldots$, where the α_n are positive.

LEMMA 1.11. *Let* $\mathscr{D}$ *be a maximal conjugate for* $\mathscr{C} = \sigma(V)$. *Let* $X = D_1 + D_2 + \ldots$ *be a decomposition of* X *into irreducible* $\mathscr{D}$*-sets as in Lemma* 1.8. *If* E *is an edge in* $h(\mathscr{D})$, *then there is a unique* D_n *such that* $C, C' \in E$ *if and only if* C, C' *are not separated on* D_n. *Also, every irreducible* D_n *is obtained in this way.*

Note. This lemma establishes a one-one correspondence between maximal $\mathscr{D}$-clusters and the irreducible $\mathscr{D}$-sets in the decomposition.

Proof. We first show that for each $\mathscr{C}$-atom C and irreducible D, the collection E comprising all $\mathscr{C}$-atoms not separated from C on D is a maximal $\mathscr{D}$-cluster. First, note that no two such $\mathscr{C}$-atoms are separated from each other on D: This by Lemma 1.9. So E is a $\mathscr{D}$-cluster contained in some maximal $\mathscr{D}$-cluster E_0. Let C' be an element of E distinct from C and let C'' be a $\mathscr{C}$-atom in $E_0 \setminus E$. Let D' be the unique irreducible on which C' and C'' are not separated: we know that $D \neq D'$. Also, C and C'' are not separated on some irreducible D''. Since $C'' \notin E$, we know that $D'' \neq D$. Additionally, $D'' \neq D'$ because of Lemmas 1.9 and 1.5. Consider $\mathscr{D}_0 = \sigma(\mathscr{D}, C''D'')$. Since $\mathscr{D}_0$ enlarges D, $h(\mathscr{D}_0)$ is not connected. We show that it is connected and derive a contradiction. Let $C_1C_2 \ldots C_n = C''$ be a chain in $h(\mathscr{D})$ which is no longer a chain in $h(\mathscr{D}_0)$. By Lemma 1.2, $C_1C_2 \ldots C_{n-1}$ is still a chain in $h(\mathscr{D}_0)$, and C_{n-1} and C'' are separated by $\mathscr{D}_0$, but not by $\mathscr{D}$. By Lemma 1.2, it must be that $C_{n-1}C''$ are not separated by $\mathscr{D}$ on D''. Since C'' and C are not separated by $\mathscr{D}$ on D'', transitivity (1.9) implies that C_{n-1} and C are not separated on D'' by $\mathscr{D}$ or $\mathscr{D}_0$. Then $C_1C_2 \ldots C_{n-1}CC'C''$ provides a connection between C_1 and C'' in $h(\mathscr{D}_0)$. It follows that $h(\mathscr{D}_0)$ is connected, as desired.

We can now prove that if C is a $\mathscr{C}$-atom and E is the maximal $\mathscr{D}$-cluster corresponding to C and an irreducible $\mathscr{D}$-set D as in the previous paragraph and if F is another maximal $\mathscr{D}$-cluster containing C, then $E \cap F$ has no elements other than C. Suppose instead that C' is another element of $E \cap F$. Let $C'' \in E \setminus F$ and $C''' \in F \setminus E$. Consider $\mathscr{D}_0 = \sigma(\mathscr{D}, CD)$, a strict enlargement of $\mathscr{D}$. By showing that $h(\mathscr{D}_0)$ is connected, we will obtain a contradiction that will establish our claim. We show that all $\mathscr{C}$-atoms are connected to C' in $h(\mathscr{D}_0)$. Let $C_1C_2 \ldots C_n = C'$ be a chain of $\mathscr{C}$-atoms in $h(\mathscr{D})$ but not in $h(\mathscr{D}_0)$. This means that at least one of these equals C. Let i be the least index such that $C_i = C$. If C_{i-1} and C_i are separated by $\mathscr{D}_0$, then C_{i-1} and C_i are not separated on D. But then C_{i-1} and C' are not separated by $\mathscr{D}$ on D and hence not from each other: so $C_1C_2 \ldots C_{i-1}C'$ is a chain in $h(\mathscr{D}_0)$. Now suppose that C_{i-1} and $C_i = C$ are not separated by $\mathscr{D}_0$. Then we claim that $C_1C_2 \ldots C_{i-1}CC'''C'$ provides a connection between C_1 and C' in $h(\mathscr{D}_0)$: C and C''' are not separated by $\mathscr{D}$ on D^c ($C''' \notin E$). Our claim is proved.

Now given E, choose $C, C' \in E$ and let $D[E]$ be the unique decomposition element such that C and C' are not separated by $\mathscr{D}$ on D. The mapping $E \to D[E]$ is well defined. To prove this, note that the set of $\mathscr{C}$-atoms not separated by $\mathscr{D}$

on $D[E]$ is a maximal $\mathscr{D}$-cluster F with $C, C' \in E \cap F$. We know that $E = F$. So if C'' and C''' are elements of E which are separated on an irreducible D', we know that $D' = D[E]$.

The mapping $E \to D[E]$ is also one-one: this follows from the irreducibility of $D[E]$. To prove that it is surjective, let D be any irreducible decomposition element and let C, C' be $\mathscr{C}$-atoms not separated on D. Then $C, C' \in E$ for some maximal $\mathscr{D}$-cluster E. Clearly, $D[E] = D$. ■

LEMMA 1.12. *If $\mathscr{D}$ is a maximal conjugate for $\mathscr{C} = \sigma(V)$, then no two maximal $\mathscr{D}$-cluster intersect in more than one $\mathscr{C}$-atom.*

Proof. Let E and F be maximal $\mathscr{D}$-clusters. If $C, C' \in E \cap F$, let D be the unique element of the irreducible decomposition of X on which C and C' are not separated. Using Lemma 1.11, we see that $D[E] = D = D[F]$, which implies that $E = F$. ■

LEMMA 1.13. *Let $\mathscr{D}$ be a maximal conjugate for $\mathscr{C} = \sigma(V)$. Then $h(\mathscr{D})$ is an acyclic hypergraph.*

Proof. Suppose that $C_1 E_1 C_2 E_2 \ldots C_n E_n C_{n+1} = C_1$ is a cycle of length $n \geq 2$. Using Lemma 1.8, we find an irreducible $\mathscr{D}$-set D such that C_n and $C_{n+1} = C_1$ are not separated on D. Put $\mathscr{D}_0 = (\mathscr{D}, C_n D)$. We prove that $h(\mathscr{D}_0)$ is connected. Since $\mathscr{D}_0$ strictly enlarges $\mathscr{D}$, this is a contradiction which will establish the lemma.

In particular, we show that every $\mathscr{C}$-atom is connected to C_1 in $h(\mathscr{D}_0)$. Given a $\mathscr{C}$-atom K, let $K = K_1 K_2 \ldots K_m = C_1$ be a chain in $h(\mathscr{D})$ connecting K to C_1. If C_n does not appear in the chain, then this is also a chain $h(\mathscr{D}_0)$ (Lemma 1.2). Otherwise, let $i \geq 1$ be the smallest index such that $K_i = C_n$. If K_{i-1} and $K_i = C_n$ are separated by $\mathscr{D}_0$, they are not separated by $\mathscr{D}$ on D. Now C_n and C_1 are also not separated by $\mathscr{D}$ on D. By Lemma 1.9, K_{i-1} and C_1 are not separated by $\mathscr{D}$ on D. So $K = K_1 K_2 \ldots K_{i-1} C_1$ is a chain in $h(\mathscr{D}_0)$. Now suppose that K_{i-1} and $K_i = C_n$ are not separated by $\mathscr{D}_0$. Then we claim that $K = K_1 K_2 \ldots K_{i-1} C_n C_{n-1} C_{n-2} \ldots C_1$ is a chain in $h(\mathscr{D}_0)$. The only thing to check is whether C_n and C_{n-1} are separated by $\mathscr{D}_0$. If so, then C_n and C_{n-1} are not separated by $\mathscr{D}$ on D. So (as in the proof of Lemma 1.11), $D = D[E_{n-1}]$. But this forces $E_n = E_{n-1}$, a contradiction. ■

In the next few lemmas, we show that maximal conjugates for $\sigma(V)$ are actually (maximal) complements as well.

LEMMA 1.14. *Suppose that D is a completely reduced set in the maximal conjugate $\mathscr{D}$. Then $\mathscr{D}(D) = \mathscr{B}(D)$.*

Proof. Suppose that $B \subseteq D$ is a set in $\mathscr{B}(D) \setminus \mathscr{D}$. Then there is some $\mathscr{C}$-atom C such that $BC \notin \mathscr{D}$. Since $\mathscr{D}_0 = \sigma(\mathscr{D}, BC)$ strictly enlarges $\mathscr{D}$, there is a $\mathscr{C}$-atom C' such that $\mathscr{D}_0$ separates C and C', but $\mathscr{D}$ does not (Lemma 1.3). Also, C and C' are separated on B^c and therefore also on D^c. But D is completely reduced, so that C, C' are separated on D. A contradiction ensues. ■

Recall from Lemma 1.10 that each irreducible $\mathscr{D}$-set D gives rise to a 0-1 measure μ on $\mathscr{D}$ such that $\mu D_0 = 0$ if and only if $D_0 D$ is completely reduced.

LEMMA 1.15. *Let $\mathscr{D}$ be a maximal conjugate for $\mathscr{C} = \sigma(V)$. Suppose that D is an irreducible $\mathscr{D}$-set with corresponding 0-1 measure μ. For each $\mathscr{C}$-atom C and $\mathscr{B}$-set $B \subseteq DC$, we have $\mu^*(CD \setminus B) = 1$ implies $B \in \mathscr{D}$.*

Proof. Suppose rather that both $\mu^*(CD \setminus B) = 1$ and $B \notin \mathscr{D}$. We consider the enlargement $\mathscr{D}_0 = \sigma(\mathscr{D}, B)$ and find a $\mathscr{C}$-atom C' such that C and C' are separated by $\mathscr{D}_0$, but not by $\mathscr{D}$. Also, C and C' are separated by $\mathscr{D}$ on B^c. So find a $\mathscr{D}$-set D_0 with $CB^c \subseteq D_0$ and $C'B^c \subseteq D_0^c$ (so that $C' \subseteq D_0^c$). We have $CD \setminus B \subseteq D_0 D$, so that $\mu(D_0 D) = 1$. Now C and C' are separated on $D^c \subseteq B^c$, but not on D. Since $\mu(D_0 D) = 1$ also C and C' are not separated on D_0. But $C' = C'B^c \subseteq D_0^c$, a contradiction. ■

LEMMA 1.16. *Let D be an irreducible set in the maximal conjugate $\mathscr{D}$. For each $\mathscr{C}$-atom C, we have that $\mathscr{C}(CD) = \mathscr{B}(CD)$. In fact, for each $B \subseteq CD$ a $\mathscr{B}$-set, we have either B or $CD \setminus B$ actually a $\mathscr{D}$-set.*

Proof. From Lemma 1.15, for such a B with neither B nor $CD \setminus B$ a $\mathscr{D}$-set, we must have $\mu^* B = \mu^*(CD \setminus B) = 0$, where μ is the 0-1 measure corresponding to D. So there is a $\mathscr{D}$-set D_0 with $CD \subseteq D_0$ and DD_0 completely reduced. By Lemma 1.14, $\mathscr{D}(DD_0) = \mathscr{B}(DD_0)$. Since $B \subseteq CD \subseteq DD_0$, we have $B \in \mathscr{D}$, a contradiction. ■

LEMMA 1.17. *If $\mathscr{D}$ is a maximal conjugate for $\mathscr{C} = \sigma(V)$, then $\mathscr{D}$ is a complement for $\mathscr{C}$.*

Proof. Let $X = D_1 + D_2 + \ldots$ be the irreducible decomposition of Lemma 1.8. Then for each $\mathscr{C}$-atom C, Lemma 1.16 says that $\mathscr{D}(CD_i) = \mathscr{B}(CD_i)$. It follows that $\sigma(\mathscr{C}, \mathscr{D}) = \mathscr{B}$. ■

The following is the major result of this section.

PROPOSITION 1.18. *Let $\mathscr{C}$ be the substructure of $\mathscr{B}(X)$ generated by the countable partition V. Then the following conditions are equivalent:*

(1) *$\mathscr{D}$ is a maximal conjugate for $\mathscr{C}$;*

(2) *$\mathscr{D}$ is a maximal complement for $\mathscr{C}$;*

(3) *the hypergraph $h(\mathscr{D}) = (V, \mathscr{E})$ is a tree, and there are for each E in $\mathscr{E}$ and C in E certain 0-1 measures $\mu(C, E)$ on $\mathscr{B}$ such that $\mu(C, E)(C) = 1$, and*

$$D = \{B \in \mathscr{B}:\ \mu(C, E)(B) = \mu(C', E)(B) \text{ for all } C,\ C' \in E \in \mathscr{E}\}.$$

Demonstration $1 \Rightarrow 2$: This follows from Lemma 1.17.

$2 \Rightarrow 1$: Immediate.

$1 \Rightarrow 3$: Lemmas 1.1 and 1.13 show that $h(\mathscr{D})$ is a tree. Let $X = D_1 + D_{2n} + \ldots$ be the irreducible decomposition of Lemma 1.8. We claim that for each n, $\mathscr{D}(D_n)$ is a maximal conjugate for $\mathscr{C}(D_n)$ in $\mathscr{B}(D_n)$ separating no two $\mathscr{C}(D_n)$-atoms.

To see this, note that if CD_n and $C'D_n$ are $\mathscr{C}(D_n)$-atoms, and $D \subseteq D_n$ is a $\mathscr{D}$-set with $CD_n \subseteq D$ and $C'D_n \subseteq D_n \setminus D$, then we may find $\mathscr{C}$-atoms C'' and C''' such that

CD_n and $C''D_n$ are not separated on D and such that $C'D_n$ and $C'''D_n$ are not separated on $D_n \setminus D$. It follows that D reduces D_n, a contradiction. This implies that $\mathscr{C}(D_n)$ and $\mathscr{D}(D_n)$ are conjugate.

Suppose that $\mathscr{D}(D_n)$ is not a maximal conjugate for $\mathscr{C}(D_n)$. Then there is some B in $\mathscr{B}(D_n)\setminus\mathscr{D}(D_n)$ such that $\mathscr{F} = \sigma(\mathscr{D}(D_n), B)$ is also conjugate to $\mathscr{C}(D_n)$. Now $B \in \mathscr{B}\setminus\mathscr{D}$ and $\mathscr{D}$ is maximal, so there are $\mathscr{D}$-sets D and D' such that $DB + D'B^c = C_0$ is a nontrivial $\mathscr{C}$-set. Since $\mathscr{F}$ is a conjugate, $DD_nB + D'D_nB^c = C_0 D_n$ is trivial in $\mathscr{C}(D_n)$; say $C_0 D_n = D_n$, so that $D_n \subseteq C_0$. The equation $D_n + D' = C_0$ yields a contradiction.

So $\mathscr{D}(D_n)$ is a maximal conjugate not separating any two atoms of $\mathscr{C}(D_n)$. Such σ-algebras have been characterized (Theorem 5 in [9]). Using this, we find for each $E \in \mathscr{E}$ and $C \in E$ a 0-1 measure $\mu(C, E)$ on $\mathscr{B}(D[E])$ such that $\mu(C, E)(CD[E]) = 1$ and such that

$$\mathscr{D}(D[E]) = \{B \in \mathscr{B}(D[E]) \colon \mu(C, E)(B) = \mu(C', E)(B) \text{ for all } C,\ C' \text{ in } E\}.$$

Now each $\mu(C, E)$ has a unique extension to a 0-1 measure on $\mathscr{B}(X)$. We preserve the notation for these extensions. Then

$$\mathscr{D} = \{B \in \mathscr{B}(X) \colon \mu(C, E)(B) = \mu(C', E)(B) \text{ all } E \in \mathscr{E} \text{ and } C,\ C' \in E\}.$$

$3 \Rightarrow 1$: Suppose that the $\mu(C, E)$ and $h(\mathscr{D})$ have the properties described. Since $h(\mathscr{D}) = (V, \mathscr{E})$ is connected, Lemma 1.1 says that $\mathscr{D}$ is a conjugate for $\mathscr{C}$.

Suppose that $\mathscr{D}$ is not maximal. Then there is some B in $\mathscr{B}\setminus\mathscr{D}$ such that $\mathscr{D}_0 = \sigma(\mathscr{D}, B)$ is also conjugate to $\mathscr{C}$. Then for some $E \in \mathscr{E}$ and C-atoms C, C' in E, we have $\mu(C, E)(B) \neq \mu(C', E)(B)$. Write $E = \{C_1, C_2, \ldots\}$ and put

$$P = \{i \colon \mu(C_i, E)(B) = 1\}$$

and $N = \{i \colon \mu(C_i, E)(B) = 0\}$. For each C in E, choose a set $A[C] \in \mathscr{B}$ such that

$$A[C_i] \subseteq C_i B \quad \text{whenever } i \in P$$

and

$$A[C_i] \subseteq C_i B^c \quad \text{whenever } i \in N$$

and such that

$$\mu(C, E)(A[C]) = 1$$

and

$$\mu(C, F)(A[C]) = 0$$

for all $F \neq E$. Such a choice is possible because $\mu(C, E) \neq \mu(C, F)$ whenever $F \neq E$ (otherwise $\mathscr{D}$ could not separate any $\mathscr{C}$-atom in F from any $\mathscr{C}$-atom in E).

Now we know from Lemma 0.2 that removal of E will break V into connected components, one for each $C \in E$. Let K_i be the connected component containing C_i. Then define

$$D^+ = \bigcup \{C \in K_i \colon i \in P \text{ and } C \neq C_i\} \cup \bigcup \{C_i \setminus A[C_i] \colon i \in P\}$$

and

$$D^- = \bigcup \{C \in K_i \colon i \in N \text{ and } C \neq C_i\} \cup \bigcup \{C_i \setminus A[C_i] \colon i \in N\}.$$

It is not hard to verify that these are $\mathscr{D}$-sets and that $D^+ + B\setminus D^-$ is the union of all the $\mathscr{C}$-atoms in the components K_i for $i \in P$. We have produced a nontrivial element of $\sigma(\mathscr{D}, B) \cap \mathscr{C}$. ■

If $\mathscr{B}(X)$ is a separable structure, so that 0-1 measures are concentrated at points, then the structure of any maximal conjugate for $\sigma(V)$ may be localised to a countable subset of X.

COROLLARY 1.19. *If $\mathscr{B}(X)$ is separable, and $\mathscr{D}$ is a maximal conjugate for $\mathscr{C} = \sigma(V)$, then there is a countable $X_0 \subseteq X$ such that $\mathscr{D}(X_0)$ is a maximal conjugate for $\mathscr{C}(X_0)$ in $\mathscr{B}(X_0)$ and such that $\mathscr{D}(X_0^c) = \mathscr{B}(X_0^c)$.*

COROLLARY 1.20. *If $\mathscr{B}(X)$ is countably generated, then any maximal conjugate for $\mathscr{C} = \sigma(V)$ is also c.g.*

§ 2. **Conjugation for continuous structures.** In this section, we consider maximal conjugation for sub-σ-algebras $\mathscr{C}$ which are not generated by a countable partition. If $\mathscr{C}$ is atomic, however, we may still couch our discussion in terms of a hypergraph $h(\mathscr{D})$ whose vertices are $\mathscr{C}$-atoms. Unfortunately, there are serious problems caused by the failure of Lemma 1.1 in this context.

EXAMPLE 2.1. Let X be the square $[0, 1]\times[0, 1]$ under the usual Borel structure $\mathscr{B}$. Let $\mathscr{C}$ be the sub-σ-algebra generated by projection onto the first co-ordinate. Let $[0, 1] = L_0 + L_1$ be a partition of the interval into non-Borel sets. Put $G_0 = \{(x, 0) \colon x \in L_0\}$ and $G_1 = \{(x, 1) \colon x \in L_1\}$. Define the σ-algebra $\mathscr{D} = \{B \in \mathscr{B} \colon \text{for each } i, \text{ either } G_i \subseteq B \text{ or } G_i \subseteq B^c\}$. Then $\mathscr{D}$ is a conjugate for $\mathscr{C}$, but this hypergraph $h(\mathscr{D})$ has two connected components corresponding to L_0 and L_1.

As we have seen (Example 0.3), maximal conjugates do not always exist, even for atomic structures. We do, however, have one positive result for the continuous case.

PROPOSITION 2.2. *Let $(X, \mathscr{B})$ be analytic. Every countably generated sub-σ-algebra $\mathscr{C}$ of $\mathscr{B}$ has a maximal conjugate in $\mathscr{B}$.*

Demonstration. Case 1. There is an uncountable $\mathscr{C}$-atom C_0. Let φ be a choice function for the $\mathscr{C}$-atoms: for each $\mathscr{C}$-atom C, we have $\varphi(C) \in C$. Let ψ be a one-one correspondence between a set $P \subset C_0$ and the collection of all $\mathscr{C}$-atoms other than C_0. (If C_0 is the only $\mathscr{C}$-atom, the entire problem is trivial: $C_0 = X$ and $\mathscr{D} = \mathscr{B}(X)$ is a maximal conjugate for $\mathscr{C}$.) Define $f \colon P \to X$ by $f(p) = \varphi(\psi(p))$. Put

$$\mathscr{D} = \{B \in \mathscr{B} \colon \{p, f(p)\} \subseteq B \text{ or } \{p, f(p)\} \subseteq B^c \text{ all } p \in P\}.$$

Any $\mathscr{D}$-set containing C_0 must intersect all $\mathscr{C}$-atoms. Thus $\mathscr{D}$ is a conjugate for $\mathscr{C}$.

Now suppose that $B \in \mathscr{B} \setminus \mathscr{D}$. Taking a complement if necessary, we find some $p \in P$ such that $f(p) \in B$ and $p \notin B$. Now $D_1 = \{p, f(p)\}$ and $D_2 = \psi(p) \setminus \{f(p)\}$ are $\mathscr{D}$-sets. We see that $BD_1 + D_2 = \psi(p)$ is a $\mathscr{C}$-set. This shows that $\mathscr{D}$ is maximal.

Case 2. There is a $\mathscr{B}$-measurable full selector for the $\mathscr{C}$-atoms: there is some $G \in \mathscr{B}$ which intersects each $\mathscr{C}$-atom at precisely one point. Then put

$$\mathscr{D} = \{B \in \mathscr{B} \colon G \subseteq B \text{ or } G \subset B^c\}.$$

Any $\mathscr{D}$-set containing a single point of G intersects every $\mathscr{C}$-atom. So $\mathscr{D}$ is a conjugate for $\mathscr{C}$.

Suppose that $B \in \mathscr{B} \setminus \mathscr{D}$. Let $p \colon X \to R$ be a measurable function generating $\mathscr{C}$, i.e., $\mathscr{C} = p^{-1}(\mathscr{B}(R))$. Then $p(GB)$ and $p(GB^c)$ are disjoint, nonempty analytic subsets of R. By a separation theorem of Lusin [5; p. 218] there is some linear Borel set A with $p(GB) \subseteq A$ and $p(GB^c) \subseteq R \setminus A$. Then $D_1 = p^{-1}(A) \setminus G$ and G are both $\mathscr{D}$-sets. We see that $D_1 + GB = p^{-1}(A)$ is a nontrivial $\mathscr{C}$-set. This proves that $\mathscr{D}$ is maximal.

Case 3. Every $\mathscr{C}$-atom is countable, and there is no full selector as in Case 2. Let $p \colon X \to R$ generate $\mathscr{C}$ as above. A selection theorem (found e.g. on p. 11 of [2]) says that in this case X may be partitioned into $\mathscr{B}$-sets $X = A_1 + A_2 + \ldots$ in such a way that p is one-one on each A_n, but not one-one on $A_n + A_m$ whenever $n \neq m$. Since there is no full selector for the $\mathscr{C}$-atoms, it must be that for some $n \neq m$ we have that $p(A_n) \cap p(A_m)$ is uncountable. Let $B_0 \subseteq p(A_n) \cap p(A_m)$ be an uncountable linear Borel set. Noting that p restricted to either A_n or A_m is a Borel-isomorphism, we see that $B_1 = p^{-1}(B_0) \cap A_n$ and $B_2 = p^{-1}(B_0) \cap A_m$ are standard (absolute Borel) sets.

Using once more the fact that there is no full selector for the $\mathscr{C}$-atoms, we see that $B_0 = p(B_1)$ is a proper subset of $p(X)$. Now $p(X) \setminus B_0$ is analytic, and so there is a Borel isomorphism $\psi \colon K \to p(X) \setminus B_0$ taking an analytic set $K \subseteq B_2$ onto $p(X) \setminus B_0$. Let φ be a choice function for the $\mathscr{C}$-atoms: for each $\mathscr{C}$-atom C, we have $\varphi(C) \in C$. Define $f \colon K \to X$ by $f(a) = \varphi(p^{-1}\varphi(a))$. Put

$$\mathscr{D} = \{B \in \mathscr{B} \colon B_1 \subseteq B \text{ or } B_1 \subseteq B^c, \text{ and } \{a, f(a)\} \subseteq B \text{ or } \{a, f(a)\} \subseteq B^c \text{ for each } a \in K\}.$$

Any $\mathscr{D}$-set which intersects B_1 must contain it. Every $\mathscr{C}$-set containing B_1 must contain B_2. Any $\mathscr{D}$-set containing B_2 must meet every $\mathscr{C}$-atom disjoint from B_2. These statements together imply that $\mathscr{C} \cap \mathscr{D}$ is trivial. So $\mathscr{D}$ is a conjugate for $\mathscr{C}$.

Now suppose that $B \in \mathscr{B} \setminus \mathscr{D}$. Suppose first that for some $a \in K$, we have $f(a) \in B$ and $a \notin B$. Now $D_1 = \{a, f(a)\}$ and $D_2 = p^{-1}(\psi(a)) \setminus \{f(a)\}$ are $\mathscr{D}$-sets. We have $D_1 B + D_2 = p^{-1}(\psi(a))$ a nontrivial $\mathscr{C}$-set. Next, suppose that both $B_1 B$ and $B_1 B^c$ are nonvoid. Now $p(B_1 B)$ is a standard linear subset, so that $p^{-1}(p(B_1 B))$ is a $\mathscr{C}$-set. Put $K_0 = p^{-1}(p(B_1 B)) \cap K$ and note that $\psi(K_0)$ is relatively Borel in $p(X) \setminus B_0$ and therefore also in $p(X)$. (It is possible that $K_0 = \emptyset$.) So $p^{-1}(\psi(K_0))$ is a $\mathscr{C}$-set. Also, $D = [p^{-1}(p(B_1 B)) \setminus B_1] + p^{-1}(\psi(K_0))$ and B_1 are $\mathscr{D}$-sets. We see that

$C = D + B_1 B = p^{-1}(p(B_1 B)) + p^{-1}(\psi(K_0))$ is a $\mathscr{C}$-set. Since $B_1 B \subseteq C$ and $B_1 B^c \subseteq C^c$, C is nontrivial. Therefore $\mathscr{D}$ is maximal. ■

QUESTION 2.3. Is the preceding proposition also true for co-analytic spaces?

EXAMPLE 2.4. Assuming the continuum hypothesis, there is a separable space X and a c.g. sub-σ-algebra $\mathscr{C}$ of $\mathscr{B}(X)$ such that $\mathscr{C}$ has no maximal conjugate in $\mathscr{B}(X)$.

Construction. A standard transfinite induction (with CH) establishes the existence of a subset X of the square $S = [0, 1] \times [0, 1]$ with the properties

(1) X is uncountable;

(2) every horizontal and vertical section of X is either singleton or void;

(3) $X \cap N$ is countable whenever N is a Borel subset of S with two-dimensional Lebesgue measure $\lambda^2(N) = 0$.

Let $\mathscr{B}(X)$ be the Borel structure that X inherits as a subspace of S and let $p \colon X \to [0, 1]$ be projection to the first factor. Define $\mathscr{C} \subseteq \mathscr{B}(X)$ to be the spectral σ-field for p:

$$\mathscr{C} = \{p^{-2}(B) \colon B \subseteq [0, 1],\ B \text{ Borel}\}.$$

Note that $\mathscr{C}$ contains all countable subsets of X. Note also that Condition 3 implies that any family of uncountable pair-wise disjoint sets in $\mathscr{B}(X)$ is countable.

CLAIM 1. Let D be an uncountable set in $\mathscr{B}(X)$. Then $\mathscr{C}(D)$ is a proper sub-σ-algebra of $\mathscr{B}(D)$. To see this, suppose instead that $\mathscr{C}(D) = \mathscr{B}(D)$. Then the restriction p_0 of p to D is a Borel-isomorphism. Let $F \colon p(D) \to D$ be the inverse of p_0 and let $q \colon S \to [0, 1]$ be projection to the second factor. Then $f = q \circ F$ is a measurable mapping of $p(D)$ onto $q(D)$ which extends to Borel function $\bar{f} \colon [0, 1] \to [0, 1]$. Now if $G = \operatorname{graph}(\bar{f})$, we have $\lambda^2(G) = 0$. But $G \cap X$ contains the uncountable set D, contradicting Condition 3 and establishing the claim.

CLAIM 2. The σ-algebra $\mathscr{C}$ has no maximal conjugate in $\mathscr{B}(X)$. We suppose contrariwise with $\mathscr{D}$. First, note that $\mathscr{D}$ cannot be atomless: if so, we may partition X into uncountably many $\mathscr{D}$-sets [2; p. 29], contradicting our earlier observations. So let D be a $\mathscr{D}$-atom; necessarily D is uncountable. Claim 1 implies that $\mathscr{C}(D)$ is a proper sub-σ-algebra of $\mathscr{B}(D)$. Select $B \in \mathscr{B}(D) \setminus \mathscr{C}(D)$ and form $\mathscr{D}_0 = \sigma(\mathscr{D}, B)$, a strict enlargement of $\mathscr{D}$. The supposed maximality of $\mathscr{D}$ implies that there are $\mathscr{D}$-sets D_1 and D_2 such that $D_1 B + D_2 B^c = C$ is a nontrivial $\mathscr{C}$-set. Taking a complement if necessary, one may assume that $D \subseteq D_1$. Then $C = B + D_2 B^c$.

Case 1. $D \subseteq D_2$, then $C = B + D_2 B^c = D_2$, a contradiction.

Case 2. If $D \subseteq D_2^c$, then $CD = B$ and $B \in \mathscr{C}(D)$, another contradiction.

Thus, Claim 2 has been established.

§ 3. Maximal conjugates for analytic structures. These results may be summarized as follows. Let $(X, \mathscr{B})$ be an analytic space and let $\mathscr{C}$ and $\mathscr{D}$ be c.g. sub-σ-algebras of $\mathscr{B}$. Then $\mathscr{D}$ is a conjugate for $\mathscr{C}$ if and only if $h(\mathscr{D})$ is connected, and $\mathscr{D}$ is a maximal conjugate if and only if it is a complement for $\mathscr{C}$, and $h(\mathscr{D})$ is a tree. Some technical results are needed:

LEMMA 3.1. *Let $\mathscr{C}$ and $\mathscr{D}$ be c.g. sub-σ-algebras of an analytic structure $\mathscr{B}(X)$. Let f and g be real-valued functions on X generating $\mathscr{C}$ and $\mathscr{D}$, respectively.*

(1) *Two $\mathscr{C}$-atoms C and C' are not separated by $\mathscr{D}$ if and only if there are points $x \in C$ and $x' \in C'$ such that $g(x) = g(x')$.*

(2) *There is a chain in the hypergraph $h(\mathscr{D})$ connecting $\mathscr{C}$-atoms C and C' if and only if there are points $x_0 x_1 \dots x_{2n-1}$ $(n \geq 1)$ of X such that*

(i) $x_0 \in C$ *and* $x_{2n-1} \in C'$;

(ii) $g(x_{2k}) = g(x_{2k+1})$ *and* $f(x_{2k+1}) = f(x_{2k+2})$ *for* $k = 0, 1, \dots, n-1$.

(3) *If the hypergraph $h(\mathscr{D})$ has a cycle, then there are points $x_0 \dots x_{2n}$ $(n \geq 1)$ of X such that*

(i) $x_0 = x_{2n}$ *and* $x_0 x_1 \dots x_{2n-1}$ *are distinct;*

(ii) $g(x_{2k}) = g(x_{2k+1})$ *and* $f(x_{2k+1}) = f(x_{2k+2})$ *for* $k = 0, 1, \dots, n-1$.

The converse holds so long as $\sigma(\mathscr{C}, \mathscr{D})$ separates points of X.

Proof. (1) Suppose that such points x and x' exist. Since $g(x) = g(x')$, x and x' belong to the same $\mathscr{D}$-atom. So x and x', and therefore C and C' cannot be separated by $\mathscr{D}$. Conversely, suppose that no such points x, x' exist. In other words, $g(C)$ and $g(C')$ are disjoint analytic sets. Use Lusin's separation principle to find a linear Borel set B with $g(C) \subseteq B$ and $g(C') \subseteq B^c$. Then $D = g^{-1}(B)$ is a $\mathscr{D}$-set separating C and C'.

(2) Suppose that $C = C_0 E_0 C_1 E_1 \dots C_{n-1} E_{n-1} C_n = C'$ is a chain in $h(\mathscr{D})$. Since C_k and C_{k+1} are not separated by $\mathscr{D}$, there are (part 1) points $x_{2k} \in C_k$ and $x_{2k+1} \in C_{k+1}$ with $g(x_{2k}) = g(x_{2k+1})$, where $k = 0, 1, \dots, n-1$. Note that x_{2k+1} and x_{2k+2} belong to the same $\mathscr{C}$-atom C_{k+1}, so that $f(x_{2k+1}) = f(x_{2k+2})$, as desired. The converse is proved similarly.

(3) Suppose that $C = C_0 E_0 C_1 E_1 \dots C_{n-1} E_{n-1} C_n = C$ is a cycle in $h(\mathscr{D})$. Let $x_0 \dots x_{2n-1}$ be as in part 2. Since $C_0 C_1 \dots C_{n-1}$ are distinct, the sets $\{x_{2k}, x_{2k+1}\}$ are pair-wise disjoint for $k = 0, \dots, n-1$. If $x_{2k} = x_{2k+1}$ for some k, then C_{k-1} and C_{k+1} are not separated by $\mathscr{D}$, and there is some edge E in $h(\mathscr{D})$ containing C_{k-1} and C_{k+1}. So $C = C_0 E_0 \dots C_{k-1} \dots C_n = C$ is again a cycle, of shorter length. By such successive shortenings of the cycle, one may ensure that the points $x_0 \dots x_{2n-1}$ are distinct.

Conversely, suppose that $\sigma(\mathscr{C}, \mathscr{D})$ separates X and that the points $x_0 x_1 \dots x_{2n}$ exist. Let $C_0 C_1 \dots C_{n-1} C_n = C$ be the $\mathscr{C}$-atoms containing $x_0 x_2 x_4 \dots x_{2n}$, respectively. The condition $g(x_{2k}) = g(x_{2k+1})$ implies that C_k and C_{k+1} are not separated by $\mathscr{D}$. So there are maximal $\mathscr{D}$-clusters $E_0 E_1 \dots E_{n-1}$ such that C_k and C_{k+1} belong to E_k. This implies the existence of a cycle in $h(\mathscr{D})$. ■

Under the conditions of the lemma, the map $F\colon X \to \mathbf{R} \times \mathbf{R}$ defined by $F(x) = (f(x), g(x))$ is a Borel isomorphism onto its range and provides a "planar representation" of the pair $(\mathscr{C}, \mathscr{D})$. It is useful to view statements about $\mathscr{C}$ and $\mathscr{D}$ in terms of the geometry of $F(X)$. The above lemma enables one to do this accurately.

LEMMA 3.2. *Let $\mathscr{C}$ and $\mathscr{D}$ be c.g. sub-σ-algebras of an analytic structure $\mathscr{B}(X)$. Suppose that the hypergraph $h(\mathscr{D})$ is a tree and that $\sigma(\mathscr{C}, \mathscr{D})$ separates points of X. Then there is a one-one correspondence between $\mathscr{D}$-atoms of cardinality ≥ 2 and maximal $\mathscr{D}$-clusters. This correspondence is obtained by pairing each nonsingleton $\mathscr{D}$-atom D with the collection of all $\mathscr{C}$-atoms C with $C \cap D \neq \emptyset$.*

Proof. Let f and g be real-valued functions on X generating $\mathscr{C}$ and $\mathscr{D}$, respectively. Suppose that $D = g^{-1}(p)$ is a nonsingleton $\mathscr{D}$-atom and define E to be the collection of all $\mathscr{C}$-atoms C with $C \cap D \neq \emptyset$. Clearly, E is a $\mathscr{D}$-cluster. If E is not maximal, there is some $\mathscr{C}$-atom $C_0 \notin E$ not $\mathscr{D}$-separated from any $\mathscr{C}$-atom in E. Let C_1 and C_2 be distinct $\mathscr{C}$-atoms in E. Using Lemma 3.1.1, find points $x_0, x_1, x_2, x_3, x_4, x_5$ such that $x_0 \in C_0$, $x_1 \in C_1$, $g(x_0) = g(x_1)$, $x_2 \in C_1$, $x_3 \in C_2$, $g(x_2) = g(x_3)$, $x_4 \in C_2$, $x_5 \in C_0$, $g(x_4) = g(x_5)$. Since $C_0 \in E$, we have $x_1 \neq x_2$, $x_3 \neq x_4$, and $x_0 \neq x_5$. Again using Lemma 3.1.3, we see that $h(\mathscr{D})$ has a cycle, which is a contradiction. So, in fact, E is maximal (an edge in $h(\mathscr{D})$).

Now suppose that E is a maximal $\mathscr{D}$-cluster. Since $h(\mathscr{D})$ is connected, E contains at least two $\mathscr{C}$-atoms $C \neq C'$. By Lemma 3.1.1, there are points $x \in C$ and $x' \in C'$ with $g(x) = g(x') = p$. Let F be the set of $\mathscr{C}$-atoms intersecting the $\mathscr{D}$-atom $D = g^{-1}(p)$. The first part of the proof shows that F is a maximal $\mathscr{D}$-cluster. But $h(\mathscr{D})$ is a tree, and $E \cap F$ has at least two elements C and C': Lemma 0.1 shows that $E = F$, as desired. ■

LEMMA 3.3. *Let $\mathscr{C}$ and $\mathscr{D}$ be c.g. sub-σ-algebras of an analytic structure $\mathscr{B}(X)$. Let C be a $\mathscr{C}$-atom and define K to be the union of all $\mathscr{C}$-atoms in the connected component of $h(\mathscr{D})$ that contains C. Then K is analytic.*

Proof. Let f and g be real-valued functions on X generating the σ-algebras $\mathscr{C}$ and $\mathscr{D}$, respectively. From Lemma 3.1.2, we see that K is the union of $K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots$, where $K_0 = C$, and

$$K_{n+1} = \begin{cases} g^{-1}(g(K_n)) & n \text{ even}, \\ f^{-1}(f(K_n)) & n \text{ odd}. \end{cases}$$

The sets K_n are analytic. ■

We are now ready for the first of our advertised results.

PROPOSITION 3.4. *Let $\mathscr{C}$ and $\mathscr{D}$ be c.g. sub-σ-algebras of an analytic structure $\mathscr{B}(X)$. Then $\mathscr{D}$ is a conjugate for $\mathscr{C}$ if and only if the corresponding hypergraph $h(\mathscr{D})$ is connected.*

Demonstration. Suppose that $\mathscr{D}$ is not a conjugate for $\mathscr{C}$ and let $B \in \mathscr{C} \cap \mathscr{D}$ be nontrivial. Let $C \subseteq B$ and $C' \subseteq B^c$ be $\mathscr{C}$-atoms. There is no chain in $h(\mathscr{D})$ connecting C and C': if there is such a chain $C = C_0 C_1 \dots C_n = C'$, let i be the largest index such that $C_i \subseteq B$. Then $0 \leqslant i \leqslant n-1$, and $C_{i+1} \subseteq B^c$. But then C_i and C_{i+1} are separated by $\mathscr{D}$, a contradiction.

Now suppose that $h(\mathscr{D})$ is not connected. Let C and C' be $\mathscr{C}$-atoms in different connected components of $h(\mathscr{D})$. Let K and K' be, respectively, the union of all the $\mathscr{C}$-atoms in these components. By Lemma 3.3, K and K' are analytic.

Let f and g be real-valued functions on X generating the σ-algebras $\mathscr{C}$ and $\mathscr{D}$, respectively. It is not hard to see that $f(K)$ and $f(K')$ are disjoint analytic sets, as are $g(K)$ and $g(K')$. So one may choose linear Borel sets A_0 and B_0 such that $f(K) \subseteq A_0$, $f(K') \subseteq A_0^c$, $g(K) \subseteq B_0$, $g(K') \subseteq B_0^c$. We now define two decreasing sequences $A_0 \subseteq A_1 \subseteq \ldots$ and $B_0 \subseteq B_1 \subseteq \ldots$ of linear Borel sets by a recursive rule: Suppose that $A_0 A_1 \ldots A_n$ and $B_0 B_1 \ldots B_n$ have been defined so that $f(K) \subseteq A_n$ and $g(K) \subseteq B_n$. Then choose $f(K) \subseteq A_{n+1} \subseteq A_n$ such that $f\left(f^{-1}(A_n) \cap g^{-1}(B_n^c)\right) \subseteq A_{n+1}^c$. Also choose $g(K) \subseteq B_{n+1} \subseteq B_n$ such that $g\left(f^{-1}(A_{n+1}^c) \cap g^{-1}(B_n)\right) \subseteq B_{n+1}^c$. These choices are possible, since the pairs

$$f\left(f^{-1}(A_n) \cap g^{-1}(B_n^c)\right) \quad \text{and} \quad f(K),$$

$$g\left(f^{-1}(A_n^c) \cap g^{-1}(B_n)\right) \quad \text{and} \quad g(K)$$

are disjoint analytic sets. Now put $A = \bigcap A_n$ and $B = \bigcap B_n$. We claim that $f^{-1}(A) = g^{-1}(B)$ is a nontrivial set in $\mathscr{C} \cap \mathscr{D}$. Suppose rather that $x \in f^{-1}(A) \setminus g^{-1}(B)$. Then for some n, $x \in g^{-1}(B_n^c)$. But $x \in f^{-1}(A_n)$, so that

$$f(x) \in f\left(f^{-1}(A_n) \cap g^{-1}(B_n^c)\right) \subseteq A_{n+1}^c;$$

at the same time $x \in f^{-1}(A_{n+1})$, a contradiction. A similar argument shows that $g^{-1}(B) \setminus f^{-1}(A)$ is empty. The set $f^{-1}(A) = g^{-1}(B)$ is nontrivial, since it contains K and misses K'. Thus, $\mathscr{D}$ is not a conjugate for $\mathscr{C}$. ∎

The requirement that X be analytic was important in the second half of this proof. Indeed, without such a condition, the result becomes false:

EXAMPLE 3.5. Let K_1 and K_2 be disjoint linear sets which cannot be separated by Borel sets. (It is know that K_1 and K_2 can be chosen co-analytic [10].) Put $X = K_1 \cup K_2$ and let $\mathscr{C}(X)$ be the relative linear Borel structure on X. Also set $\mathscr{B}(X) = \sigma(\mathscr{C}(X), K_1)$ and $\mathscr{D}(X) = \sigma(K_1)$. Then $\mathscr{D}(X)$ is a conjugate for $\mathscr{C}(X)$ in $\mathscr{B}(X)$, but $h(\mathscr{D}(X))$ has two connected components, corresponding to K_1 and K_2.

And now the second major result of this section, which solves the c.g. maximal conjugate problem for analytic spaces.

PROPOSITION 3.6. *Let $\mathscr{C}$ and $\mathscr{D}$ be c.g. sub-σ-algebras of an analytic structure $\mathscr{B}(X)$. Then the following are equivalent:*

(1) *$\mathscr{D}$ is a maximal conjugate for $\mathscr{C}$;*

(2) *$\mathscr{D}$ is a maximal complement for $\mathscr{C}$;*

(3) *the hypergraph $h(\mathscr{D})$ is a tree, and $\sigma(\mathscr{C}, \mathscr{D})$ separates points of X.*

Demonstration. 1 → 2: We show first that $\sigma(\mathscr{C}, \mathscr{D})$ separates points of X: suppose that p and q are points of X not separated by $\sigma(\mathscr{C}, \mathscr{D})$. Then $\mathscr{D}_0 = \sigma(\mathscr{D}, \{p\})$ is an enlargement of $\mathscr{D}$. Therefore, there are $\mathscr{D}$-sets D_1 and D_2 such that $(D_1 \cap \{p\}) \cup (D_2 \setminus \{p\}) = C$ is a nontrivial $\mathscr{C}$-set. Without loss of generality, assume that $p \in C$. Then $\{p\} \cup (D_2 \setminus \{p\}) = C$ and $q \in C$. Also, $q \in D_2$ so that $p \in D_2$. We conclude $D_2 = C$, a contradiction.

Since $\sigma(\mathscr{C}, \mathscr{D})$ is c.g. and separates points of X, the strong Blackwell property for analytic spaces implies that $\sigma(\mathscr{C}, \mathscr{D}) = \mathscr{B}$, as desired.

2 → 3: Suppose that $\mathscr{D}$ is a maximal complement for $\mathscr{C}$. Of course $\sigma(\mathscr{C}, \mathscr{D}) = \mathscr{B}$ separates X. By Proposition 3.4, $h(\mathscr{D})$ is connected. Now suppose that there is a cycle in $h(\mathscr{D})$. By Lemma 3.1.3, there are points $x_0 x_1 \ldots x_{2n}$ in X such that $g(x_{2k}) = g(x_{2k+1})$ and $f(x_{2k+1}) = f(x_{2k+2})$ for $k = 0, \ldots, n-1$, where f and g are real-valued functions on X generating $\mathscr{C}$ and $\mathscr{D}$, respectively; also, $x_0 = x_{2n}$, while $x_0 \ldots x_{2n-1}$ are distinct.

Put $\mathscr{D}_0 = \sigma(\mathscr{D}, \{x_{2n}\})$, a strict enlargment of $\mathscr{D}$, since $\mathscr{D}$ does not separate $x_{2n} = x_0$ and x_1. Then there are $\mathscr{D}$-sets D_1 and D_2 such that

$$(D_1 \cap \{x_{2n}\}) \cup (D_2 \setminus \{x_{2n}\}) = C$$

is a nontrivial $\mathscr{C}$-set. Without loss of generality, we assume $x_{2n} \in C$, so that $C = \{x_{2n}\} \cup (D_2 \setminus \{x_{2n}\})$. Since C contains x_{2n}, it must also contain x_{2n-1}; so D_2 contains x_{2n-1} and therefore x_{2n-2}. Since C contains x_{2n-2}, it must also contain x_{2n-3}. Continuing in this fashion, we find that C and therefore D_2 contains x_1. So D_2 and C both contain $x_0 = x_{2n}$. This means that $C = D_2$, a contradiction.

So $h(\mathscr{D})$ is connected and contains no cycles: $h(\mathscr{D})$ is a tree.

3 → 1: Suppose that $\sigma(\mathscr{C}, \mathscr{D})$ separates X and that $h(\mathscr{D})$ is a tree. Since $h(\mathscr{D})$ is connected, $\mathscr{D}$ is a conjugate. If now $\mathscr{D}$ is not maximal, then there is some B in $\mathscr{B}(X) \setminus \mathscr{D}$ such that $\sigma(\mathscr{D}, B)$ is also a conjugate for $\mathscr{C}$. Since X is analytic, and $\mathscr{D}$ and $\sigma(\mathscr{D}, B)$ are c.g., it must be that $\mathscr{D}$ and $\sigma(\mathscr{D}, B)$ do not have the same atoms. So there is some $\mathscr{D}$-atom D that may be partitioned $D = D_1 \cup D_2$ into disjoint, non-empty sets D_1 and D_2 in $\sigma(\mathscr{D}, B)$. Put $\mathscr{D}_0 = \sigma(\mathscr{D}, D_1)$, a conjugate of $\mathscr{C}$ strictly larger than $\mathscr{D}$.

Let E_i be the collection of all $\mathscr{C}$-atoms intersecting D_i, $i = 1, 2$. Set $E = E_1 \cup E_2$. Considering the edge sets of the hypergraphs associated with $\mathscr{D}$ and D_0, we have

$$h(\mathscr{D}) \setminus \{E\} \subseteq h(\mathscr{D}_0) \subseteq \left(h(\mathscr{D}) \setminus \{E\}\right) \cup \{E_1, E_2\}.$$

Tacitly, this uses Lemma 3.1. We now claim that $h(\mathscr{D}_0)$ is not connected, which will be a contradiction. To see this, use Lemma 0.2 to see that $h(\mathscr{D}) \setminus \{E\}$ has one component for each $\mathscr{C}$-atom intersecting D. Let C_1 and C_2 be $\mathscr{C}$-atoms with $C_1 \cap D \neq \emptyset$ and $C_2 \cap D \neq \emptyset$ ($C_1 \neq C_2$ since $\sigma(\mathscr{C}, \mathscr{D})$ separates X). Then there is no chain in $h(\mathscr{D}_0)$ connecting C_1 to C_2. ∎

§ 4. **Measurable selections.** In Proposition 2.2, it was shown that every c.g. sub-σ-algebra $\mathscr{C}$ of an analytic structure has a maximal conjugate. However, some of these maximal conjugates are not countably generated. In this section we prove that a c.g. maximal conjugate exists just in case there is a measurable full selector for $\mathscr{C}$.

First, a technical lemma:

LEMMA 4.1. *Let p be a measurable function defined on an analytic set A. Suppose that $G \subseteq A$ is analytic, that p is one-one on G, and that $p(G) = p(A)$. Then $A \setminus G$ is analytic.*

Proof. Let p_0 be the restriction of p to G. Then p_0 is a Borel-isomorphism of G onto $p(G) = p(A)$, and p_0 extends to a Borel-isomorphism defined on $G_0 \supseteq G$ with $G_0 \in \mathscr{B}(A)$. We claim that $G_0 = G$: if $x \in G_0$, choose $y \in G$ with $p(x) = p(y)$. Then $x = y$. So G and $A \setminus G$ belong to $\mathscr{B}(A)$, so that $A \setminus G$ is analytic. ■

PROPOSITION 4.2. *Let $\mathscr{C}$ be a c.g. sub-σ-algebra of an analytic structure $\mathscr{B}(X)$. Then $\mathscr{C}$ has a c.g. maximal conjugate if and only if there is a measurable full selector for $\mathscr{C}$.*

Demonstration. Suppose that G is such a selector. Let z be a point in G and define $g\colon X \to X$ by the rule

$$g(x) = \begin{cases} z & \text{if } x \in G \\ x & \text{if } x \notin G \end{cases}.$$

Then g generates the c.g. σ-algebra $\mathscr{D} = \{B \in \mathscr{B}(X)\colon G \subseteq B \text{ or } G \subseteq B^c\}$, which is, as in [9; 222], a maximal conjugate for $\mathscr{C}$.

Conversely, suppose that $\mathscr{D}$ is a c.g. maximal conjugate for $\mathscr{C}$. Let f and g be real-valued functions on X generating $\mathscr{C}$ and $\mathscr{D}$, respectively. We define G_0 to be some $\mathscr{C}$-atom and put

$$G_{n+1} = \begin{cases} g^{-1}(g(G_n)) \setminus G_n & n \text{ even}, \\ f^{-1}(f(G_n)) \setminus G_n & n \text{ odd}. \end{cases}$$

For each $n \geqslant 0$, define real-valued functions h_n and k_n on X by

$$h_n = \begin{cases} f & n \text{ odd}, \\ g & n \text{ even}, \end{cases} \qquad k_n = \begin{cases} g & n \text{ odd}, \\ f & n \text{ even}. \end{cases}$$

CLAIM 1. For each $n \geqslant 0$, h_n is one-one on G_n. We proceed by induction on n: For $n = 0$, the claim follows from the fact that $\mathscr{C}$ and $\mathscr{D}$ are complements (Proposition 3.6). Now assume the claim true for integers $< n$. Suppose contrariwise that there are points $p_0 \neq q_0$ in G_n with $h_n(p_0) = h_n(q_0)$. Define sequences $p_0 p_1 \ldots$ and $q_0 q_1 \ldots$ recursively so that $p_0 \ldots p_{k-1} q_0 \ldots q_{k-1}$ are distinct and satisfy

$$\left.\begin{array}{l} p_i, q_i \in G_{n-i} \\ h_{n-i}(p_i) = h_{n-i}(q_{i+1}) \\ h_{n-i}(q_i) = h_{n-i}(q_{i+1}) \end{array}\right\} i = 0, 1, \ldots, k-1.$$

At the $i = k-1$ stage, we stop to ask whether $k_{n-i}(p_{i+1}) = k_{n-i}(q_{i+1})$. If they are equal, then the selection of points comes to an end with $p_0 \ldots p_k q_0 \ldots q_k$. If they are not equal, selection continues (it must terminate at $i = n$, because $k_0 = f$ is constant on G_0). We have tacitly used the induction hypothesis.

Lemma 3.1.3 allows one to construct from the sequences $p_0 \ldots p_m q_m q_{m-1} \ldots q_0 p_0$ a cycle in $h(\mathscr{D})$. This contradicts Proposition 3.6 and establishes the claim.

Claim 1 allows an application of Lemma 4.1, which shows (again *via* an induction argument) that each G_n is analytic. Connectivity of $h(\mathscr{D})$ shows that

$$X = G_0 \cup G_1 \cup G_2 \cup \ldots$$

Let p be a point in G_0.

CLAIM 2. $S = \{p\} \cup G_1 \cup G_3 \cup G_5 \cup \ldots$ is a measurable full selector for $\mathscr{C}$. Now S is analytic, so that once S is shown to be a full selector, the usual theorem on analytic graphs [4; p. 34] will establish that S is measurable.

We prove, using induction, that for $k \geqslant 0$, f is one-one on $\{p\} \cup G_1 \cup \ldots \cup G_{2k+1}$. Of course, Claim 1 has already shown that f is one-one on each G_{2k+1}. Likewise, Claim 1 shows that f is one-one on $\{p\} \cup G_1$: $G_0 \cap G_1 = \emptyset$, and G_0 is a $\mathscr{C}$-atom (a level curve of f). Suppose that f is one-one on $\{p\} \cup G_1 \cup \ldots \cup G_{2k-1}$ but that there are points $r_0 \in G_{2k+1}$ and $q_0 \in G_{2m+1}$ with $0 \leqslant m < k$ and $f(r_0) = f(q_0)$. (The case $f(q_0) = f(p)$ is easily eliminated.) We find a point $q_1 \in G_{2m}$ with $g(q_0) = g(q_1)$ and a point $r_1 \in G_{2k}$ with $g(r_0) = g(r_1)$. Continue recursively defining $q_0 q_1 \ldots$ and $r_0 r_1 \ldots$ so that

$$q_i \in G_{2m+1-i}, \qquad r_i \in G_{2k+1-i},$$
$$h_i(q_i) = h_i(q_{i+1}), \qquad h_i(r_i) = h_i(r_{i+1}).$$

The process stops at stage s when $h_s(q_s) = h_s(r_s)$.

Lemma 3.1.3 allows one to construct from the sequence $p_0 p_1 \ldots p_s q_s q_{s-1} \ldots q_0 p_0$ a cycle in $h(\mathscr{D})$. This contradicts Proposition 3.6, and establishes that f is one-one on S.

We must now show that S is a *full* selector. Given any $\mathscr{C}$-atom C, choose $x \in C$. We know that $x \in G_n$ for some n (connectedness of $h(\mathscr{D})$ is used here). So if n is odd, there is nothing to prove. If $n = 0$, we have $f(x) = f(p)$. If $x \in G_{2k}$, find $x' \in G_{2k-1}$ with $f(x) = f(x')$. We see that S hits every $\mathscr{C}$-atom. ■

Note. The same proof may be applied to show that $G_0 \cup G_2 \cup G_4 \cup \ldots$ is a measurable full selector for $\mathscr{D}$. So we see that every c.g. maximal conjugate for $\mathscr{C}$ has a measurable full selector.

§ 5. Examples and conjectures. Let $(X, \mathscr{B})$ be a measurable space and let $\mathscr{C}$ be a sub-σ-algebra of $\mathscr{B}$. It is known [7; Theorem 2] that every minimal weak complement of $\mathscr{C}$ is actually a complement. Dually, it is natural to guess that every maximal conjugate of $\mathscr{C}$ is also a complement. However, this guess is incorrect:

EXAMPLE 5.1. Let S be an uncountable standard space. There are sub-σ-algebras $\mathscr{C}$ and $\mathscr{D}$ of $\mathscr{B}(S)$ such that

(1) $\mathscr{C}$ is countably generated;
(2) $\mathscr{D}$ is a maximal conjugate for $\mathscr{C}$;
(3) $\mathscr{D}$ is not a complement for $\mathscr{C}$.

Construction. We realize S as the union of two line segments in the plane: $S = L_1 \cup L_2$

$$L_1 = \{(x, 0)\colon 0 < x < 1\}, \quad L_2 = \{(0, y)\colon 0 < y < 1\}.$$

Let $\mathscr{C}$ be the structure generated by $p\colon S \to \mathbf{R}$, projection onto the x-axis. Partition $L_1 = R_1 \cup R_2$ into two non-Borel sets, each of power c. Similarly, partition $L_2 = P_1 \cup P_2$ into two uncountable Borel sets. Let $f\colon L_2 \to L_1$ be a one-one correspondence such that $f(P_1) = R_1$ and $f(P_2) = R_2$. We define

$$\mathscr{D} = \{B \in \mathscr{B}(S)\colon \{a, f(a)\} \subseteq B \text{ or } \{a, f(a)\} \subseteq B^c, \text{ for each } a \in L_2\}.$$

Then the only $\mathscr{D}$-set containing the $\mathscr{C}$-atom L_2 is S: $\mathscr{D}$ is a conjugate for $\mathscr{C}$. Also, suppose that for some B in $\mathscr{B}(S)$ and some $a \in L_2$ we have $a \in B$ and $f(a) \in B$. Then put $D = \{a, f(a)\} \in \mathscr{D}$ and note that $D \cup B = \{f(a)\} \in \mathscr{C}$. Thus, $\mathscr{D}$ is a maximal conjugate for $\mathscr{C}$.

We now show that $\mathscr{D}$ is not a complement by proving that $P_1 \notin \sigma(\mathscr{C}, \mathscr{D})$. Suppose contrariwise that $P_1 \in \sigma(\mathscr{C}, \mathscr{D})$. Then $P_1 \in \sigma(\mathscr{C}, \mathscr{D})(L_2) = \mathscr{D}(L_2)$, since L_2 is a $\mathscr{C}$-atom. So for some $D \in \mathscr{D}$ we have $P_1 = D \cap L_2$. Since D contains P_1, D also contains $f(P_1) = R_1$. Since R_1 is not Borel, D must intersect R_2 and therefore also P_2. This is a contradiction.

$\mathscr{D}$ is not a complement for $\mathscr{C}$, nor is it contained in any such complement.

We have mentioned that every c.g. sub-σ-algebra of an analytic structure has a maximal (not necessarily c.g.) conjugate. The question arises of characterizing separable spaces with this property. In Example 2.4 an example was constructed (using the continuum hypothesis) of a space where this property fails. In particular, we have almost no idea of what happens for co-analytic spaces. For example, does Proposition 4.2 hold for these spaces?

EXAMPLE 5.2. Assume Martin's Axiom (MA) and let X be a linear subset of cardinality strictly between $\aleph_0$ and c (not-CH). The relative Borel structure $\mathscr{B}(X)$ is just the power set algebra $\mathscr{P}(X)$: in fact, every subset of X is a G_δ subset of X [8; p. 497].

Every c.g. sub-σ-algebra $\mathscr{C}$ of $\mathscr{B}(X)$ has a c.g. maximal conjugate: simply let S contain exactly one point from each $\mathscr{C}$-atom. Automatically, $S \in \mathscr{B}(X)$. Define

$$\mathscr{D}(X) = \{B \subseteq X\colon S \subseteq B \text{ or } S \subseteq B^c\},$$

a c.g. maximal conjugate for $\mathscr{C}$.

The question arises whether such a set X can be co-analytic. It is in fact consistent with ZFC set theory to assume that every set of cardinality $\aleph_1$ is co-analytic and that the only separable Borel structure on a set X of cardinality $\aleph_1$ is the power set algebra $\mathscr{P}(X)$. For details, see [6; pp. 162–165].

QUESTION 5.3. Let $(X, \mathscr{B})$ be a separable space and let $\mathscr{C}$ be a proper sub-σ-algebra of $\mathscr{B}$. Suppose that $\mathscr{C}(X)$ is separable. Can $\mathscr{C}$ have a maximal conjugate in $\mathscr{B}$?

QUESTION 5.4. Let $\mathscr{B}(\mathbf{R})$ be the usual linear Borel structure. Does the countable/co-countable σ-algebra have a maximal conjugate in $\mathscr{B}(\mathbf{R})$? in the power set algebra $\mathscr{P}(\mathbf{R})$? (Certainly, such a maximal conjugate cannot be c.g.)

We have shown that if $(X, \mathscr{B})$ is separable, and $\mathscr{C}$ is generated by a partition, then every maximal conjugate for $\mathscr{C}$ is c.g. Are there other natural conditions on $\mathscr{C}$ which guarantee this property? For example:

QUESTION 5.5. Let X be the union of the lines $y = 0$ and $y = 1$ in the plane. Let $\mathscr{C}$ be the sub-σ-algebra of $\mathscr{B}(X)$ generated by projection onto the x-axis. Is every maximal conjugate for $\mathscr{C}$ c.g.?

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