We work in $L[a]$. Suppose $Y = \bigcup B_m$ where each $B_m$ is $\mathcal{P}^{2}_{<\omega}$. For $n \in \omega$ let

$$D_n = \{ p \in \mathbb{P}_\alpha : \exists m < n \exists \alpha \in \mathcal{P}_n \dot{\neg} p \in B_m \}.$$ 

Claim 2. $D_n$ is dense.

Let $p \in \mathbb{P}_\alpha$. Fix $\delta > \omega < \kappa^{\omega}_1$. Since $\mathbb{P}_\alpha \subseteq \mathbb{P}_\delta$ and $p \in \mathbb{P}_\alpha$, suppose $\langle T^\delta, H^\delta \rangle \in \mathbb{P}_\delta$ is generic over $L[a]$ and $p \in \langle T^\delta, H^\delta \rangle$. Clearly $\omega_1^\mathbb{P}_\delta > \delta$, thus $T^\delta \notin Y$. Thus $\exists \alpha \in \mathcal{P}_n \dot{\neg} \exists \delta \in \omega \ni \neg \exists \alpha \in \mathcal{P}_n \dot{\neg} p \in B_m$. Thus there is $m < n$ and $r \in \mathbb{P}_\delta$ such that $r \supseteq p$. Let $r \in \mathbb{P}_\alpha$ be the retagging of $r$. By Lemma 2.2, $\mathbb{P}_\alpha \dot{\neg} p \in B_m$, since $\neg B_m$ is $\omega_1^\mathbb{P}_\alpha$. Clearly $r \equiv \not \mathbb{P}_\alpha$ so $D_n$ is dense.

Let $\langle T^\delta, H^\delta \rangle$ be $\mathbb{P}_\alpha$-generic over $L[a]$. Since then $D_n$ are dense $\forall n \in \omega \exists m < n \exists \alpha \in \mathcal{P}_n \dot{\neg} p \in B_m$. Thus $\bigcap B_m = \neg Y$. So $\omega_1^\mathbb{P}_\delta > \delta$, contradicting Lemma 2.2.

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DEPARTMENT OF MATHEMATICS

122 SCIENCE AND ENGINEERING OFFICES

Box 454, Chicago, Illinois 60600

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Representability of $V[\mathcal{A}]$ as intersection of $A$-bounded variation classes

by

Pedro Iñaca (Medellin)

Abstract. It is proved that the generalized bounded variation class $V[\mathcal{A}]$ of Canturija is the intersection of all classes of $A$-bounded variation with $A = \langle \alpha \rangle$ satisfying $28(\langle \alpha \rangle^{1/2} - \langle \alpha \rangle^{1/3}) < \omega$, but it is not the intersection of any countable subcollection of them. As a consequence of this result, a version of Helly's theorem for the classes $V[\mathcal{A}]$ is proved.

1. Two important generalizations of the concept of bounded variation have been given by D. Waterman [4] and Z. A. Canturija [2] by introducing, respectively, the functions of $A$-bounded variation ($ABV$) and the classes $V[\mathcal{A}]$. These spaces have been studied mainly because of their applicability to the theory of Fourier series. An interesting connection between the class of functions of bounded variation ($BV$) and the classes $ABV$ has been pointed out by Perlman [3], who has proved that the space $BV$ is the intersection of all $ABV$ classes but not of any countable collection of them. We shall prove an extension of Perlman's result to study the representability of the classes $V[\mathcal{A}]$ as intersections of $ABV$ classes. This theorem will allow us to prove a version for the classes $V[\mathcal{A}]$ of the well-known Helly's theorem. Let $f$ be a function defined on an interval $[a, b]$. If $\mathcal{I} = \{I_i\}$ is a nondecreasing sequence of positive real numbers such that $\sum \mu(I_i) = \infty$, we say that $f$ is of $A$-bounded variation ($ABV$) on $[a, b]$ if $\sum |f(I_i)|/\mu(I_i) < \infty$ for every $\mathcal{I}$. This is known to imply that the supremum $\sup_{\mathcal{I}}(\sum |f(I_i)|/\mu(I_i))$ of the collection of the above sums is finite [4]. Also, if $f \in ABV$, then $f$ is regulated, i.e., has only simple discontinuities. Let

$$\nu(n, f, [a, b]) = \nu(n, f) = \sup_{\mathcal{I}} \left( \sum_{i=1}^{n} |f(I_i)| \right),$$

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the supremum being taken over all finite collections \( \{ I_{n} \}_{n=1}^{\infty} \). For a nondecreasing, concave function \( h \) on the positive integers, satisfying \( h(0) = 0 \), let

\[
V[h, [a, b]] = V[h] = \{ f \in V : f \text{ is } h(0) < \infty \).
\]

For \( h \) bounded, \( V[h] \) is BV, and thus \( h(n) \to \infty \) for all other classes \( V[h] \). It may be shown that \( V[h] \) consists only of regulated functions if and only if \( h(n) = o(n) \), and therefore we will make this assumption on \( h \), since our interest is to represent \( V[h] \) as intersection of ABV classes, which contain only regulated functions. We will also assume that \( h(n) \to \infty \) as \( n \to \infty \).

2. The following theorems establish some properties of the classes \( V[h] \).

**Theorem 1.** If \( f \in V[h, [a, b]] \) and \( f \in V[h, [b, c]] \), then \( f \in V[h, [a, c]] \).

The proof of Theorem 1 is trivial.

**Theorem 2.** If \( f \notin V[h, [a, b]] \), then there exists \( x \in [a, b] \) such that \( f \notin V[h, [a, b]] \) for any closed interval \( J = [a, b] \) containing a neighborhood of \( x \).

**Proof.** We split \( [a, b] \) into two closed intervals of equal length \( I_1 \) and \( I_2 \), and observe that, by Theorem 1, for one of \( I_1 \) or \( I_2 \), say \( I_1 \), \( f \notin V[h, I_1] \). Dividing \( I_1 \), we did \([a, b], \) by an inductive procedure, we obtain a nested sequence \( J_{1} \supseteq J_{2} \supseteq \ldots \) of closed intervals of length approaching zero, and such that \( f \notin V[h, J_i] \) for \( i = 1, 2, \ldots \). The intersection of the \( J_i \)'s is a single point \( x \) which satisfies the requirements of the conclusion of the theorem.

Theorem 2 implies the existence of a point \( x \in [a, b] \) such that either \( f \notin V[h, [x, x + \delta]] \) for all \( \delta > 0 \), or \( f \notin V[h, [x - \delta, x]] \) for all \( \delta > 0 \).

It is observed from the definitions that \( f \in V[h] \) if and only if there is a constant \( C \) such that

\[
\left( \sum_{i=1}^{n} |f(U_i)| \right) h(n) \leq C
\]

for all collections \( \{ I_{n} \}_{n=1}^{\infty} \) and all \( n \). The next theorem shows, however, that only requiring the above expression (as a sequence of \( n \)) to be bounded for each particular collection \( \{ I_{n} \} \) is sufficient to assure its uniform boundedness.

**Theorem 3.** (i) \( f \in V[h, [a, b]] \) if and only if

\[
\sum_{i=1}^{n} |f(U_i)| = O(h(n))
\]

for each collection \( \{ I_{n} \}_{n=1}^{\infty} \).

(ii) If \( f \) is regulated, then \( f \in V[h, [a, b]] \) if and only if (i) is true for each collection \( \{ I_{n} \}_{n=1}^{\infty} \) satisfying \( |f(U_i)| \leq 1 \) as \( n \to \infty \).

**Proof.** We prove (ii) ((i) being similar). The "only if" part follows immediately from the definitions. For the "if" part of the theorem, suppose that \( f \notin V[h, [a, b]] \).

Applying Theorem 2, we may assume that \( f \notin V[h, [x, x+\delta]] \) for some \( x \) and all \( \delta > 0 \). Let \( M = \sup |f(U_i)| \). Let \( \delta_1 > 0 \). We choose \( n_1 \) such that \( h(n_1+1) > 2M \) and \( \delta_1 + 1 > 2h(n_1+1) \). There are subintervals of \( [x, x+\delta_1] \), \( I_1, \ldots, I_{n_1} \), \( I_{n_1+1} \) (ordered from right to left, i.e., \( I_{n_1+1} \) lies to the left of \( I_1 \)) such that

\[
\sum_{i=1}^{n_1+1} |f(U_{i})| > 2h(n_1+1).
\]

We can assume that \( I_{n_1+1} \) has nonempty interior. Thus \( x \notin I_{n_1} \). Now

\[
\sum_{i=1}^{n_1+1} |f(U_{i})| = \sum_{i=1}^{n_1+1} |f(U_{i})| - |f(U_{n_1+1})| > \sum_{i=1}^{n_1+1} -2M \\
> 2h(n_1+1) - h(n_1+1) = h(n_1+1) > h(n_1).
\]

Having chosen \( n_1, \ldots, n_{n_1-1} \), and \( I_1, \ldots, I_{n_1-1} \subseteq [x, b] \), ordered from right to left and \( x \notin I_{n_1-1} \), let \( I_{n_1} = [a_{n_1-1}, b_{n_1-1}] \) and \( \delta_2 = \min((a_{n_1-1} - x), 1/k) \), and choose \( n_k \) such that

\[
h(n_k+1) > 2M(n_k-1+1),
\]

and

\[
v(n_k+1, [x, x+\delta_2]) > (1+k)h(n_k+1).
\]

Therefore, there are intervals \( I_1, \ldots, I_{n_1} \subseteq [x, x+\delta_2] \), having nonempty interior and ordered from right to left, such that

\[
\sum_{i=1}^{n_1} |f(U_{i})| > (1+k)h(n_k+1).
\]

Let \( J_1 = I_1 \) for \( n_{k-1} < i < n_k \), then \( x \notin I_1 \) and

\[
\sum_{i=1}^{n_1} |f(U_{i})| > \sum_{i=1}^{n_{k-1}+1} |f(U_{i})| - \sum_{i=n_{k-1}+1}^{n_k} |f(U_{i})| \\
> (1+k)h(n_k+1) - 2M - 2Mn_{k-1} > (1+k)h(n_k+1) - h(n_k+1) = kh(n_k).
\]

Since \( \delta_k \to 0 \) as \( k \to \infty \), we have that \( f(U_i) \to 0 \) as \( i \to \infty \) and therefore \( \{ I_i \} \) can be rearranged into \( \{ I_i^* \} \) such that \( |f(U_i^*)| \leq 1 \) as \( n_k \to \infty \) and

\[
\sum_{i=1}^{n_k} |f(U_{i}^*)| \geq \sum_{i=1}^{n_k} |f(U_{i})| = O(h(n_k)).
\]

Throughout the rest of the paper, sequences \( \lambda = (\lambda_i) \) are assumed to satisfy the condition \( \lambda_i \to \infty \). For a sequence \( \{ a_i \} \) we will write \( a_{n_i} = a_i - a_{i-1} \). The following relation between classes \( V[h] \) and \( ABV \) is due to Avdispahić [1].

**Theorem 4.** If \( \sum h(\lambda) \delta(1/\lambda_i) < \infty \), then \( V[h] \subseteq ABV \).
Proof. We first observe that
\[ h(n)/\lambda_n = h(n) \sum_{k=1}^{n} A(1/\lambda_k) \leq \sum_{k=1}^{n} k h(k) A(1/\lambda_k), \]
and therefore $h(n)/\lambda_n \to 0$ as $n \to \infty$. Now, for $f \in \mathcal{V}[h]$ and a collection $\{l_i\}$ we have
\[ \sum_{i=1}^{n} f(l_i)/\lambda_{l_i} \leq \sum_{i=1}^{n} \sum_{k=1}^{l_i} [f(k) A(1/\lambda_k) + \sum_{k=1}^{l_i} f(k)/\lambda_k] \]
\[ \leq CV(f)(\sum_{k=1}^{n} k h(k) A(1/\lambda_k) + h(n)/\lambda_n) \leq CV(f) \]
for some constant $C$ and all $n$. ■

3. We now state the generalization of Perlman's result.

**Theorem 5.** $\mathcal{V}[h]$ is the intersection of all ABV classes satisfying
\[ \sum_{k=1}^{n} h(k) A(1/\lambda_k) < \infty. \]

To prove this theorem we need the following results:

**Lemma 1.** Suppose $h, g \geq 0$ are nondecreasing concave functions of the positive integers. Then the function $p$ defined by $p(n) = (h(n)g(n))^{1/2}$ is also nondecreasing and concave.

**Proof.** $p$ is obviously increasing. If $0 \leq t \leq 1$ and $r = mn+(1-t)m$ is an integer, then by Hölder's inequality we have that
\[ p(r) = (h(r)g(r))^{1/2} \geq (h(n)+(1-t)h(m))^{1/2}((g(n)+(1-t)g(m))^{1/2} \]
\[ \geq t(h(n)g(n))^{1/2} + (1-t)(h(m)g(m))^{1/2} = tp(n) + (1-t)p(m). \]

A consequence of this lemma is

**Theorem 6.** No $\mathcal{V}[h]$ contains the class of regulated functions.

**Proof.** Let $p(n) = (nh(n))^{1/2}$, $p$ is nondecreasing and concave by Lemma 1, and also $p(n) = o(n)$. Then the sequence defined by $b_n = p(n) - p(n-1)$ is decreasing and converges to zero. Consider the function $f$ defined on $[a, b]$ by $f(x) = 0$,
\[ f(x) = \sum_{i=1}^{k} (-1)^{i+1} b_i \] for $a+(b-a)(k+1) \leq x < a+(b-a)k$, $k = 1, 2, ...$

and $f(a) = \sum_{i=1}^{k} (-1)^{i+1} b_i$. Then $f$ has only simple discontinuities and $\mathcal{V}(n, f, [a, b]) = p(n)$. But $p(n) = o(n)$ and therefore $f \notin \mathcal{V}[h]$.

**Theorem 7.** Suppose $f$ has only simple discontinuities. If $f \notin \mathcal{V}[h]$, then $f \notin \mathcal{ABV}$ for some $A = \{\lambda_i\}$ satisfying (2).
and \( h(n)/\lambda_n < \varepsilon \) for \( n > M \). Then, for \( n > M \) we have
\[
\text{(3) } \sum_{i=1}^{n} 1/\lambda_i = \frac{h(n)}{n} \sum_{i=1}^{n} 1/\lambda_i = \frac{h(n)}{n} \sum_{n} 1/\lambda_i = 1 + I_2.
\]

\( I < \varepsilon \), and applying summation by parts

\[
II = \frac{h(n)}{n} \sum_{i=n}^{n-1} (i+1-N) \lambda_i + \frac{h(n)(n+1-N)}{n} \lambda_n\]

\[
\leq \sum_{i=n}^{n-1} \frac{h(i)}{n} \lambda_i + \frac{h(n)}{n} \lambda_n
\]

\[
\leq \sum_{i=1}^{n} h(i) \Delta(1/\lambda_i) + \varepsilon,
\]

since \( h(n)/n \leq h(i)/i \) for \( i \leq n \). Therefore the left side of (3) is less than \( 3\varepsilon \) if \( n > M \), and the conclusion follows.

**Theorem 8.** Let \( A = \sum_{i=1}^{n} \lambda_i \), \( l = 1, 2, \ldots \) be a collection of sequences such that

\[
\sum_{i=1}^{n} h(i) \lambda_i(1/\lambda_i - 1/\lambda_{i+1}) < \infty \lambda_i \lambda_{i+1} \infty \text{ as } n \to \infty \text{ and } \sum_{i=1}^{n} 1/\lambda_i = \infty \text{ for all } l.
\]

Then there exists a function \( f \) in \( L^1(1/\lambda_i - 1/\lambda_{i+1}) \) for all \( l \) and all \( i \). Also, \( \gamma_l/\lambda_i \infty \) as \( n \to \infty \), and

\[
\sum_{i=1}^{n} h(i) \lambda_i(1/\lambda_i - 1/\lambda_{i+1}) < \infty \cdot
\]

By Lemma 2 and the fact that \( h(n)/|a(n)| \) there exists integers \( n = n_0, n_1, \ldots \) such that \( a_k = kh(n)/n_k = n_k - n_{k-1} \) is a decreasing sequence of \( k \) which converges to zero, \( n_k > n_{k-1} \), and

\[
\frac{h(n_k)}{n_k} \sum_{i=1}^{n_k} 1/\lambda_i \leq \frac{1}{k2^{k+1}},
\]

for \( k = 1, 2, \ldots \). Let \( b_i = a_i \) for \( n_{k-1} < i \leq n_k \). Clearly, \( b_i \downarrow 0 \) as \( i \to \infty \). Also

\[
\sum_{i=1}^{n_k} b_i \geq \sum_{i=n_{k-1}+1}^{n_k} a_i = k h(n_k).
\]

Hence \( \sum_{i=1}^{n_k} b_i = O(h(n)) \). But, since \( a_i < 2kh(n)/n_k \), and \( \gamma_l \geq \gamma_l/ \) for \( k \geq l \), it follows that

\[
\sum_{i=n_{k-1}+1}^{n_k} a_i \leq \sum_{i=n_{k-1}+1}^{n_k} 2kh(n)/n_k \sum_{i=n_{k-1}+1}^{n_k} 1/\lambda_i < \infty \cdot
\]

Thus \( \sum_{i=n_{k-1}+1}^{n_k} b_i \leq \infty \). Finally, by using the sequence \( \{b_i\} \) we define \( f \) as we did in the proof of Theorem 6, and the procedure above shows that \( f \) is contained in the intersection of all \( A BV \), \( l = 1, 2, \ldots \), but \( f \notin V[A] \).

**Proof.** For \( f \in ABV \) let

\[
\|f\|_A = [\|f(a)\| + V_A(f)]
\]

If \( f \in V[A] \), let

\[
\|f\|_A = [\|f(a)\| + V_A(f)]
\]

\( V_A \) and \( V_x \) as defined in §1. It is easy to see that \( ABV \) and \( V[A] \) are Banach spaces under the norms \( \|\|_A \) and \( \|\|_A \), respectively.

As an application of Theorem 5 we will prove an analogue of the well-known Helly's Theorem for the classes \( V[A] \).

**Theorem 9.** Let \( \{f_k\} \) be a sequence in \( V[A] \) such that \( \|f_k\|_A < M \) for some \( M \), \( k = 1, 2, \ldots \). Then there exists a subsequence \( \{f_{k_0}\} \) converging pointwise to some \( f \) in

\( V[A] \) with \( \|f\|_A < M \).

**Proof.** Theorem 5 guarantees the existence of a class \( ABV \) satisfying (2). For each collection \( \{f_i\} \) and each \( k \), by an argument similar to that given in the proof of Theorem 4, we have that

\[
\sum_{i=1}^{n_k} |f_i(a_k)|/|a_k| \leq CV_A(f_k)
\]

for some \( C > 0 \) independent of \( k \). Thus

\[
\|f\|_A = [\|f(a)\| + V_A(f)] \leq [\|f(a)\| + CV_A(f)] \leq (C+1)\|f\|_A \leq (C+1)M.
\]

By the analogue of Helly's Theorem for the classes \( ABV \) ([4], Theorem 5), there is a subsequence \( \{f_{k_0}\} \) converging pointwise to some \( f \). For a finite collection \( \{f_i\} \), consisting of \( n \) elements, we have

\[
\|f_{k_0}(a) + (\sum_{i=1}^{n_k} |f_i(a_k)|)/h(n) \|_A \leq \|f_{k_0}\|_A < M.
\]

By letting \( j \to \infty \), we observe that

\[
\left( \sum_{i=1}^{n_k} |f_i(a_k)|/h(n) \right) \leq M - \|f\|_A,
\]

and thus \( V_A(f) \leq M - \|f\|_A \), which is the same as \( \|f\|_A = M \).
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DEPARTMENT OF MATHEMATICS
UNIVERSIDAD NACIONAL DE COLOMBIA
AA 566
Medellín, Colombia

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Alphabetic index of Volumes 121—130
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1 — Fundamenta Mathematicae 130, 3