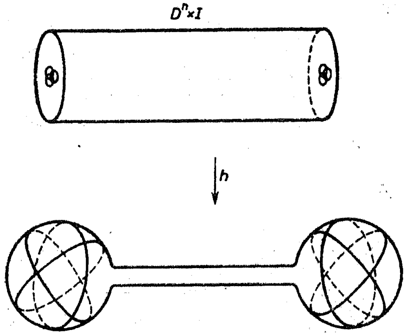

Fig. 3

Now make the following surgery on $f(M)$.

Throw out from $f(M)$ the $2(n+1)$ balls of dimension n which form the intersection $f(M)\cap U$ and add to $(f(M)\setminus U)$ the set

$$h(\{\bigcup S_t|\ t\in[0,1]\}).$$

By thus the number of $(n+1)$-tuple points has been decreased by 2.

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An analytic equivalence relation not arising from a Polish group action

by

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Abstract. We show that the equivalence relation xEy iff $\omega_1^x = \omega_1^y$ can not arise as the orbits of a Polish group action. We also calculate the exact Borel rank of $\{x\colon \omega_1^x = \alpha\}$ for α a countable admissible ordinal.

There are two natural abstractions of Vaught's conjecture to descriptive set theory. The most natural would be the conjecture that any analytic equivalence relation on a Borel set with uncountably many classes, each of which is Borel, admits a perfect set of inequivalent elements. Unfortunately this is easily seen to be false. Let xEy if and only if $\omega_1^x = \omega_1^y$, where ω_1^x is the first ordinal not recursive in x. Then E is Σ_1^1 and all E equivalence classes are Borel, but E has exactly $\aleph_1$ equivalence classes, one for each countable admissible ordinal.

The second abstraction is known as the topological Vaught conjecture. Let G be a Polish group acting continuously on a Polish space X. If G has uncountably many orbits, then there is a perfect subset of X such that any two distinct elements are in different orbits. Kechris asked if the equivalence relation E could arise as the action of a Polish group on the Baire space. In § 1 we show that it cannot. In § 2 we give an exact calculation of the Borel rank of $\{x\in\omega^\omega\colon \omega_1^x = \alpha\}$, for α a countable admissible ordinal.

§ 1. Our main lemma uses several ideas of Vaught's [V].

Definition. Let G be a Polish group acting continuously on a Polish space X. Let $B\subseteq X$. The *Vaught transform* $B^* = \{x\in X\colon \{g\in G\colon gx\in B\}$ is comeager$\}$.

The facts we need about the Vaught transform are summarized in the following lemma.

Lemma 1.1 (Vaught [V]).

(1) *For any $B\subseteq X$, B^* is G-invariant.*

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(2) *If* B *is* G*-invariant,* $B^* = B$.

(3) *If* B *is* $\underset{\sim}{\Pi}^0_\alpha$, *then so is* B^*.

The next lemma is the main step. This lemma was proved independently and earlier by Sami [S]. Sami's proof is far more elementary (i.e. forcing free).

LEMMA 1.2. *Let* G *be a Polish group acting continuously on a Polish space* X. *If* $\mathcal{O}$ *is a nonmeager orbit, then* $\mathcal{O}$ *is* G_δ.

Proof. We first note that every orbit is Borel [R–N], and hence has the Baire property. Thus we can write $\mathcal{O} = X \cup Y$ where X is G_δ and Y is meager and Borel. Consider the Vaught transform X^* of X. Clearly $X^* \subseteq \mathcal{O}^* = \mathcal{O}$. Since X^* is invariant and $\mathcal{O}$ is an orbit, $X^* = \mathcal{O}$ or $X^* = \emptyset$

Suppose for purposes of contradiction that $X^* = \emptyset$. Then for all $x \in \mathcal{O}$ $\{g \in G\colon gx \in X\}$ is not comeager. So for all $x \in \mathcal{O}$ $\{g \in G\colon gx \in Y\}$ is not meager. Let $C = \{(x, g)\colon x \in \mathcal{O}, g \in G \text{ and } gx \in Y\}$. Thus C has the property of Baire. Hence by the Kuratowski–Ulam theorem [O] C is not meager.

Let $\boldsymbol{P}_0$ be the conditions for Cohen forcing in X. Let $\boldsymbol{P}_1$ be the conditions for Cohen forcing in G. Using the topological properties of Cohen forcing (see [So] or [Ku]) we can get $\mathring{x}, \mathring{g}$, $\boldsymbol{P}_0 \times \boldsymbol{P}_1$ generic over the ground model s.t. in the generic extension $\mathring{x}, \mathring{g} \in C$ (note: using Borel codes and Shoenfield absoluteness we can extend X, Y, $\mathcal{O}$, C and all their important properties to the generic extension). Let $\mathring{y} = \mathring{g}(\mathring{x})$. Then $\mathring{y} \in Y$.

CLAIM. $\mathring{y}$ *is Cohen generic over the ground model*

Proof. Let $D \subseteq \boldsymbol{P}_0$ be dense. We wish to show $D \cap \mathring{y} \neq \emptyset$. Let

$$D_* = \{(p, q) \in \boldsymbol{P}_0 \times \boldsymbol{P}_1\colon \exists r \in D\ q(p) \leqslant r\}.$$

(Note. Since elements of G are continuous functions on X we determine a neighborhood of the value of $g(x)$.) We claim D_* is dense. Let $(p, q) \in \boldsymbol{P}_0 \times \boldsymbol{P}_1$. Let $r \leqslant r_0 r \in D$. We can extend (p, q) to (p', q') such that $q'(p') \leqslant r$. Since D_* is dense, $\mathring{y} \cap D \neq \emptyset$. Thus $\mathring{y}$ is Cohen generic over the ground model. But this is impossible since $\mathring{y} \in Y$ and Y is a meager Borel set coded in the ground model.

Thus $X^* = \mathcal{O}$. By Lemma 1.1, X^* is G_δ.

Condiser the E class $Y = \{x\colon \omega_1^x = \omega_1^{ck}\}$. By Thomasaon [T] (see also [K]) this set is comeager. Thus we will be done if we can show it is not G_δ.

LEMMA 1.3. Y *is not* G_δ.

Proof. Let $F = \{x \in \omega^\omega\colon \exists n\ \forall m > n\ x(n) = 0\}$. We will show that F is Wadge reducible to Y (i.e. there is $f\colon \omega^\omega \to \omega^\omega$ continuous s.t. $x \in F$ if and only if $f(x) \in Y$). Consider the infinite game where I plays x, II plays y and II wins if and only if $x \in F \leftrightarrow y \in Y$. We will show that II has a winning strategy. This strategy will be the desired reduction.

Let $z \in \omega^\omega$ such that $\omega_1^z \neq \omega_1^{ck}$. If I plays a 0, II plays a 0. If I does not play a 0, let m be the number of times I has failed to play a 0. II plays $z(m-1)+1$. Consider a run of the game where I plays x and II using this strategy plays y.

If $x \in F$, then I eventually plays only zeroes. So II will eventually play only zeroes, in which case y is recursive so $\omega_1^y = \omega_1^{ck}$.

If $x \neq F$ then infinitely often I makes a nonzero play. Thus II will infinitely often play the next element of z plus one. Thus z is recursive in y. So $\omega_1^x \geqslant \omega_1^z > \omega_1^{ck}$.

Thus F is Wadge reducible to Y. But F is F_σ but not G_δ. Thus since the inverse image of a G_δ set under a continuous function is G_δ, Y is not G_δ.

COROLLARY 1.4. *The equivalence relation* $xEy \Leftrightarrow \omega_1^x = \omega_1^y$ *can not arise as the orbits of a Polish group action.*

§ 2. In 1.3 we showed that $\{x\colon \omega_1^x = \omega_1^{ck}\}$ is not G_δ. In this section we will calculate the exact Borel rank of $\{x\colon \omega_1^x = \alpha\}$ for α countable admissible ordinal. Let $Y = \{x\colon \omega_1^x = \alpha\}$.

LEMMA 2.1. Y *is* $\underset{\sim}{\Pi}^0_{\alpha+2}$.

Proof. $y \in Y \Leftrightarrow \forall e(\{e\}^y \notin \{T \in WF\colon |T| = \alpha\})$, where $\{e\}^y$ is the e-th partial recursive function with oracle y, WF is the set of well founded trees and for T a tree $|T|$ is the height of T. Stern [Ste] has showed that $\{T \in WF\colon |T| = \alpha\}$ is $\underset{\sim}{\Pi}^0_{\alpha+1}$. Thus Y is $\underset{\sim}{\Pi}^0_{\alpha+2}$. (This also follows from a result of Sami [S].)

The proof of the lower bound uses Steel forcing [St]. Let $\delta > \alpha$. Let $\boldsymbol{P}_\alpha$ and $\boldsymbol{P}_\delta$ be conditions for Steel forcing, i.e. $\boldsymbol{P}_\gamma = \{\langle t', h\rangle\colon t$ a finite tree on ω^ω and $h\colon t \to \omega \cdot \gamma \cup \{\infty\}$ such that $h(\emptyset) = \infty$ and if $\sigma, \tau \in t$, $\sigma \subset \tau$ and $h(\sigma) < \infty$, then $h(\tau) < h(\sigma)\}$ ordered by extension. If $p \in \boldsymbol{P}_\delta$ and $p = \langle t, h\rangle$, we can define $\bar{p} \in \boldsymbol{P}_\alpha$ by $\bar{p} = \langle t, h'\rangle$ where

$$h'(\sigma) = \begin{cases} h(\sigma) & h(\sigma) < \alpha, \\ \infty & h(\sigma) > \alpha. \end{cases}$$

We call $\bar{p}$ the retagging of p.

We will use two basic facts about Steel forcing.

LEMMA 2.2. (1) (Steel [St]) If $\langle T, H\rangle \subseteq \boldsymbol{P}_\alpha$ is generic over $L[a]$ and α is a-admissible, then $\omega_1^T = \alpha$

(2) (Stern [Ste]) *If* X *is* $\underset{\sim}{\Pi}^0_\alpha$, $p \in P_\delta$ *and* $\mathring{T}$ *is a canonical name for the generic tree, then*

$$p \Vdash_{\boldsymbol{P}_\delta} \mathring{T} \in X \quad \text{iff} \quad \bar{p} \Vdash_{\boldsymbol{P}_\alpha} \mathring{T} \in X.$$

THEOREM 2.3. Y *is* $\underset{\sim}{\Pi}^0_{\alpha+2}$, *but not* $\underset{\sim}{\Sigma}^0_{\alpha+2}$.

Proof. Fix $a \in \omega^\omega$ such that $\omega_1^a = \alpha$. Suppose y is $\underset{\sim}{\Sigma}^0_{\alpha+2}$.

CLAIM 1. $L[a] \models Y$ *is* $\underset{\sim}{\Sigma}^0_{\alpha+2}$.

Proof. Let $b \in L[a]$ s.t. $\omega_1^b > \alpha$ and $a \leqslant_T b$. Then Y is $\underset{\sim}{\Sigma}^0_{\alpha+2} \Leftrightarrow \exists x$ (x is a $\underset{\sim}{\Sigma}^0_{\alpha+2}$-code $\wedge$ $\forall y$ ($y \in Y$ is in the Borel set coded by x)). This is easily seen to be $\Sigma^1_2(b)$. Thus by Shoenfield absoluteness $L[a] \models Y$ is $\underset{\sim}{\Sigma}^0_{\alpha+2}$.

We work in $L[a]$. Suppose $Y = \bigcup_n \bigcap_m B_{nm}$ where each B_{nm} is $\underset{\sim}{\Sigma}^0_{\alpha+2}$. For $n \in \omega$ let

$$D_n = \{p \in \boldsymbol{P}_\alpha \colon \exists m p \Vdash_{\boldsymbol{P}_\alpha} \mathring{T} \notin B_{nm}\}.$$

Claim 2. *D_n is dense.*

Let $p \in \boldsymbol{P}_\alpha$. Fix $\delta > \alpha \delta < \aleph_1^{L[\alpha]}$. Since $\boldsymbol{P}_\alpha \subseteq \boldsymbol{P}_\delta$, $p \in \boldsymbol{P}_\delta$. Suppose $\langle T^*, H\rangle \subseteq \boldsymbol{P}_\delta$ is generic over $L[a]$ and $p \in \langle T^*, H\rangle$. Clearly $\omega_1^{T^*} \geqslant \delta$, thus $T^* \notin Y$. Thus $p \not\Vdash_{\boldsymbol{P}_\delta} \bigvee_n \bigwedge_m \mathring{T} \in B_{nm}$. Thus $p \not\Vdash_{\boldsymbol{P}_\delta} \bigvee_m \mathring{T} \in B_{nm}$. Thus there is $m \in \omega$ and $r \leqslant p$ such that $r \Vdash_{\boldsymbol{P}_\delta} \mathring{T} \notin B_{nm}$. Let $\bar{r} \in \boldsymbol{P}_\alpha$ be the retagging of r. By Lemma 2.2 $\bar{r} \Vdash_{\boldsymbol{P}_\alpha} \mathring{T} \notin B_{nm}$, since $\neg B_{nm}$ is $\underset{\sim}{\Pi}^0_\alpha$. Clearly $r \leqslant \bar{p}$ so D_n is dense.

Let $\langle T, H\rangle$ be $\boldsymbol{P}_\alpha$-generic over $L[a]$. Since then D_n are dense $\forall n \in \exists m \in \omega \exists p \in \in \langle T, H\rangle p \Vdash_{\boldsymbol{P}_\alpha} \mathring{T} \notin B_{nm}$. Thus $T \in \bigcap_n \bigcup_m \neg B_{nm} = \neg Y$. So $\omega_1^T \neq \alpha$, contradicting Lemma 2.2.

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Representability of $V[h]$ as intersection of Λ-bounded variation classes

by

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Abstract. It is proved that the generalized bounded variation class $V[h]$ of Čanturija is the intersection of all classes of Λ-bounded variation with $\Lambda = \{\lambda_i\}$ satisfying $\Sigma h(i)(\lambda_i^{-1} - \lambda_{i+1}^{-1}) < \infty$, but it is not the intersection of any countable subcollection of them. As a consequence of this result, a version of Helly's theorem for the classes $V[h]$ is proved.

1. Two important generalizations of the concept of bounded variation have been given by D. Waterman [4] and Z. A. Čanturija [2] by introducing, respectively, the functions of Λ-bounded variation (ΛBV) and the classes $V[h]$. These spaces have been studied mainly because of their applicability to the theory of Fourier series. An interesting connection between the class of functions of bounded variation (BV) and the classes ΛBV has been pointed out by Perlman [3], who has proved that the space BV is the intersection of all ΛBV classes but not of any countable collection of them. We shall prove an extension of Perlman's result to study the representability of the classes $V[h]$ as intersections of ΛBV classes. This theorem will allow us to prove a version for the classes $V[h]$ of the well-known Helly's theorem.

Let f be a function defined on an interval $[a, b]$. If $I = [x, y]$, we write $f(I) = f(y) - f(x)$. Let $\{I_i\}$ be a collection of nonoverlapping intervals $I_i \subseteq [a, b]$.

If $\Lambda = \{\lambda_i\}$ is a nondecreasing sequence of positive real numbers such that $\sum 1/\lambda_i = \infty$, we say that f is of Λ-bounded variation (ΛBV) on $[a, b]$ if $\sum |f(I_i)|/\lambda_i < \infty$ for every $\{I_i\}$. This is known to imply that the supremum $V_\Lambda(f)$ of the collection of the above sums is finite [4]. Also, if $f \in \Lambda BV$, then f is regulated, i.e., has only simple discontinuities.

Let

$$v(n, f, [a, b]) = v(n, f) = \sup \sum_{i=1}^{n} |f(I_i)|,$$

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