Multiple points of singular maps
Part II

by

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Abstract. Here we generalize the theorem from our earlier paper with the same title, Part I, to the maps of nonorientable manifolds. Namely we give some conditions on a singular map which imply that the map cannot have a single multiple point of the highest multiplicity.

§ 0. Introduction. Let us recall some definitions from Part I [7].

Notation. For a generic smooth map $f\colon M^n \to R^{n+1}$ of a closed manifold M^n in R^{n+1} we introduce the following sets:

$$\Delta_k(f) = \{y \in R^{n+1} |\ f^{-1}(y) \text{ contains at least } k \text{ different points}\},$$

$$\Sigma(f) = \{y \in R^{n+1} |\ f^{-1}(y) \text{ contains a singular point}\}.$$

Definition. A map f as above has *singular multiplicity* k if k is the smallest nonnegative integer such that $\Sigma(f)$ and $\Delta_{k+1}(f)$ have disjoint neighbourhoods.

Notation. $\sigma(f)$ will denote the singular multiplicity of f.

Definition. A map as above will be called a *C-map* if it satisfies the following two conditions:

Condition 1. $\sigma(f) < n$.

Condition 2. For any connected component Δ_n' of $\Delta_n(f)$ which is a regular curve in R^{n+1} the ε-neighbourhood of Δ_n' in $\Delta_{n-1}(f)$ consists of *even* number of Möbius bands plus some (may be zero) nontwisted bands $S^1 \times I$, all bands of the width 2ε. (If we throw out the central lines of these bands then the obtained cut bands are embedded subsets of $\Delta_{n-1}(f)$. The central lines lie in Δ_n' but different points of the same central line may lie at the same point of Δ_n'. So these bands are immersed in R^{n+1} and they may have selfintersections only along the central lines.)

Restriction ($R\Sigma$). In what follows "a map" will mean a generic map having only $\Sigma^{1,0}$ type singular points.

THEOREM. *No C-map $f\colon M^n \to R^{n+1}$ (satisfying $R\Sigma$) of an even dimensional closed manifold M^n has a single $(n+1)$-tuple point.*

§ 1. Notation and preliminary remarks. Let G_l denote the group of $l\times l$ matrices with exactly one nonzero entry in each row and column and such that these nonzero entries are $+1$ or -1.

Let G_l^* be the subgroup of G_l consisting of matrices with even number of negative entries.

By an *n-cross* we shall mean the union of the coordinate lines in R^n and an *n-cross bundle* is a bundle with fibre homeomorphic to an n-cross. Notice that the group G_n acts on an n-cross.

Condition 2 above is equivalent to the following condition.

CONDITION 2′. For any component Δ_n' of $\Delta_n(f)$, the intersection of $\Delta_{n-1}(f)$ with an ε-neighbourhood of Δ_n' is an n-cross bundle over Δ_n' with structure group G_n^*.

Since Condition 2′ makes sense even if $\dim\Delta_n(f)\neq 1$, we can extend the notion of C-map

DEFINITION. A map $f\colon V^v \to R^{v+1}$ is a *C-map* if Conditions 1 and 2′ are satisfied.

Notice that

if $v<n-1$ then any map f is a C-map;
if $v=n-1$ then Condition 1 implies Condition 2′.

Notation. Let us denote by $C_l(v)$ the cobordism group of C-maps having no $(l+1)$-tuple points of v-dimensional manifolds in R^{v+1}. (The cobordisms joining the maps must be maps of the same type, i.e. they are C-maps without $(l+1)$-tuple points, their target space is $R^{v+1}\times I$.)

Sketch of the proof. The basic line of the proof is the same as in Part I. Namely, there are three steps.

Step 1. (Analogue of Lemma 1A, B, C, D of Part I.)

LEMMA 1. *If Theorem fails then the natural forgetting map $\varphi\colon C_n(n-1)\to C_{n+1}(n-1)$ is monomorphic.* (Recall that the indices show the maximal allowed multiplicities of selfintersections, the number $(n-1)$ in the brackets shows the dimension of the sources of the maps. The restriction $\sigma(f)<n$ is not expressed by our notation but it always meant.)

Step 2. (Analogue of Lemma 2A, B, C, D of part I.) Given a natural number l, there exists a space CX_l such that

$$C_l(m)\approx \pi_{m+1}(CX_l)\,.$$

For $l<n$ the space CX_l is the same as X_l in Part I. The space CX_{l+1} can be obtained from CX_l by attaching to it an $(l+1)$-dimensional disc bundle $D\zeta_{l+1}$ by using an attaching map

$$\varrho_l\colon \partial D\zeta_{l+1}\to CX_l\,.$$

Step 3. (Analogue of the Main Lemma A, B, C, D of part I.) The restriction ϱ of the map ϱ_n to a fibre S^n of $\partial D\zeta_{n+1}$ is not null-homotopic if n is even. In order to show this we shall construct a map

$$\theta\colon CX_n\to RP^\infty\vee T\zeta_n \quad\text{where } T\zeta_n \text{ is the Thom space of } \zeta_n,$$

such that the composition $\theta\circ\varrho\colon S^n\to RP^\infty\vee T\zeta_n$ is not null-homotopic.

These three steps imply the Theorem because the map

$\varrho=\varrho_n|_{S^n}$ gives a nonzero element of the kernel of the map $\varphi\colon C_n(n-1)\to C_{n+1}(n-1)$ and so by Lemma 1 the Theorem holds.

Since the proof is quite similar for immersions and for singular maps, we shall concentrate on the proof for immersions and then the general case can be deduced from this special case mainly in the same way as this was done in part I.

Lemma 1 can be proved — mutatis mutandis — in the same way as Lemma 1A, B, C, D in Part I, so we shall deal only with steps 2 and 3.

§ 2. The classifying space for cobordisms of C-immersions and C-maps.

DEFINITION. A codimension 1 immersion $f\colon V^v\to R^{v+1}$ is called a *C-immersion* if it is a C-map.

DEFINITION. The cobordism group of C-immersions (C-maps respectively) of m-manifolds into R^{m+1} having no $(l+1)$-tuple points can be defined replacing in the definition of the cobordism group of embeddings the word "embedding" by the expression "C-immersion (C-maps, respectively) without $(l+1)$-tuple points". This group will be denoted by $C\text{-Imm}_l(m)$ ($C_l(m)$, respectively).

PROPOSITION 1. *For any natural number l there exist spaces $C\Gamma_l$ and CX_l such that*

$$C-\mathrm{Imm}_l(m)\approx\pi_{m+1}(C\Gamma_l)\,,\qquad C_l(m)\approx\pi_{m+1}(CX_l)\,.$$

Proof. Let us denote by Γ_l the space $\Gamma_l(RP^\infty)$. (For the definition of $\Gamma_l(\cdot)$ see § 4 of Part I. Briefly, $\Gamma_l(X)$ is the l-th term in the model of $\Omega^\infty S^\infty X$.) Let $\alpha\colon\Gamma_l\to RP^\infty$ be a map inducing isomorphism of the fundamental groups and let $\beta\colon\tilde\Gamma_l\to\Gamma_l$ be the corresponding double covering map. Now $C\Gamma_l$ can be obtained from $\tilde\Gamma_l$ identifying those points of $\beta^{-1}(\Gamma_{n-1})$ which have the same image at β. (Especially, $C\Gamma_l=\Gamma_l$ if $l\leqslant n-1$.)

An alternative description of $C\Gamma_l$. Let G_l and G_l^* be the groups described in the "Preliminary remarks" and let ζ_l and ζ_l^+ be the universal l-dimensional vector bundles with structure groups G_l and G_l^*. Let $D\zeta_l$, $S\zeta_l$ and $D\zeta_l^*$, $S\zeta_l^*$ be the corresponding disc and sphere bundles: $\partial D\zeta_l = S\zeta_l$, $\partial D\zeta_l^* = S\zeta_l^*$.

The bundle map c

$$\begin{array}{ccc} S\zeta_l^* & \xrightarrow{c} & S\zeta_l \\ \downarrow & & \downarrow \\ BG_l^* & \longrightarrow & BG_l \end{array}$$

corresponding to the inclusion $G_l^* \subset G_l$ is a double covering. Recall from part I that $\Gamma_l = \Gamma_l(RP^\infty)$ can be obtained from Γ_{l-1} by attaching to it a disc bundle $D\zeta_l$ using an attaching map ϱ_l: $S\zeta_l \to \Gamma_{l-1}$.

Now if $l < n$ then put $C\Gamma_l = \Gamma_l$.

If $l \geqslant n$ then we attach to Γ_{n-1} the disc bundle $D\zeta_n^*$ by the attaching map $\varrho_n^* = \varrho_n \circ c$ (where c: $S\zeta_l^* \to S\zeta_l$ is the double covering map mentioned above). The obtained space will be $C\Gamma_n$, i.e.

$$C\Gamma_n = \Gamma_{n-1} \bigcup_{\varrho_n^*} D\zeta_n^* .$$

Then we attach to $C\Gamma_n$ the space $D\zeta_{n+1}^*$ by an attacking map ϱ_{n+1}^*: $S\zeta_{n+1}^* \to C\Gamma_n$ and obtain $C\Gamma_{n+1}$. Then we attach to the space $C\Gamma_{n+1}$ the space $D\zeta_{n+2}^*$ by a map ϱ_{n+2}^* and obtain $C\Gamma_{n+2}$ etc. (The definitions of the maps $\varrho_{n+1}^*, \varrho_{n+2}^*, \ldots$ are analogous to those of the maps ϱ_l from [6].)

In order to show that these two definitions of $C\Gamma_l$ give the same space we have to show only the following

PROPOSITION 2. *The double covering map $\tilde{D}\zeta_l \to D\zeta_l$ induced by the restriction of the map α: $\Gamma_l \to RP^\infty$ to $D\zeta_l$ is the same as the double covering map c: $D\zeta_l^* \to D\zeta_l$ induced by the inclusion $G_l^* \subset G_l$.*

We shall prove this proposition later.

Construction of the space CX_l. Recall that in Part I a space $X_{l,s}(k)$ has been constructed for any natural numbers k, l and s such that $l \geqslant s$ (l denoted the maximal admitted multiplicity of the selfintersections and s denoted the maximal admited singular multiplicity of maps, while k denoted the codimension of the maps. The cobordism groups of the maps of this sort of *oriented* manifolds were isomorphic to the homotopy groups of the space $X_{l,s}(k)$).

Replacing in the definition of this space the group $SO(k)$ by $O(k)$, we obtain a space which we denote by $X'_{l,s}(k)$. For $s = n-1$ and $k = 1$ we shall denote this space by X'_l. Notice that X'_l contains the space Γ_l as a subspace. Now the space CX_l can be defined identifying in the disjoint union $X'_l \cup C\Gamma_l$ the subspaces $\Gamma_{n-1} \subset X'_l$ and $\Gamma_{n-1} \subset C\Gamma_l$.

The proofs of the isomorphisms in Proposition 1 are quite analogous to the proofs of the isomorphisms

$$\mathrm{Imm}_l(n, k) \approx \pi_{n+k}(\Gamma_l(MSO(k))) \quad \text{from [4], [6]}$$

and

$$S(n, k) \approx \pi_{n+k}(X(k)) \quad \text{from [5]}$$

or

$$S_{l,s}(n, k) \approx \pi_{n+k}(X_{l,s}(k)) \quad \text{from part I}.$$

So we shall not repeat them. By thus we consider Step 2 as setdled.

§ 3. Step 3. Let ϱ^* denote the restriction of the attaching map

$$\varrho_{n+1}^*: S\zeta_{n+1}^* \to C\Gamma_n$$

to a fibre S^n and let $T\zeta_n^*$ be the Thom space of ζ_n^*.

PROPOSITION 3. *There exists a map θ: $C\Gamma_n \to RP^\infty \vee T\zeta_n^*$ such that $\theta \circ \varrho^*$ is not null-homotopic if n is even.*

Proof. There exists a map $\tilde{\theta}$: $C\Gamma_n \to RP^\infty$ such that

(a) $\tilde{\theta}_*$: $\pi_1(C\Gamma_n) \to \pi_1(RP^\infty)$ is an isomorphism;

(b) $\tilde{\theta}$ maps the zero section $B\zeta_n^*$ of $D\zeta_n^* \subset C\Gamma_n$ into one point $* \in RP^\infty$.

Moreover, we can suppose that

(c) there is a neighbourhood U of $B\zeta_n^*$ in $C\Gamma_n$ disjoint form $\Gamma_{n-1} \subset C\Gamma_n$ and diffeomorphic to $D\zeta_n^*$ such that $\theta(U) = * \in RP^\infty$.

Let the point $*$ be the common point in the wedge product $RP^\infty \vee T\zeta_n^*$. (This point is also the singular point of the Thom space $T\zeta_n^*$.)

Let r: $U \to T\zeta_n^* \setminus *$ be a homeomorphism.

Now the map θ can be defined as follows:

$$\theta(x) = \begin{cases} \tilde{\theta}(x) & \text{if } x \notin U, \\ r(x) & \text{if } x \in U. \end{cases}$$

Recall that, quite analogously, a map $\tilde{\theta}$: $\overline{\Gamma}_n \to S^1 \vee S^n$ has been defined in Part I (which was denoted first by θ). The space $\overline{\Gamma}_n$ was the m-th term of the James product of the circle S^1. The m-th homotopy group of this space was isomorphic to the cobordism group of those codimension 1 immersions of oriented $(m-1)$-dimensional manifolds into R^m which had no $(n+1)$-tuple points and which were projections of embeddings into R^{m+1}.

It is easy to see that there is a homotopically commutative diagram

$$\begin{array}{ccc} C\Gamma_n & \xrightarrow{\theta} & RP^\infty \vee T\zeta_n^* \\ \uparrow j & & \uparrow i \\ \overline{\Gamma}_n & \xrightarrow{\bar{\theta}} & S^1 \vee S^n \end{array}$$

where i embeds S^1 and S^n onto a fibre in the Thom spaces $MO(1) = RP^\infty$ and $T\zeta_n^*$, respectively, and j is the "forgetting" map. (It is analogous to the map j of Part I.)

Remark. The degree of a map $f\colon S^n \to RP^\infty \vee T\xi$ for any n dimensional vector bundle ξ can be defined as follows: (This degree will be an element of the group $Z_2 \oplus Z_2$).

Let us identify the space RP^∞ with the Thom space $MO(1)$ and denote by $B\xi$ the base space of ξ. The map f can be supposed to be transversal both to $BO(1) \subset MO(1)$ and $B\xi \subset T\xi$. The preimage $f^{-1}(BO(1))$ is a hypersurface in S^n which divides S^n into components. Divide the set of these components into two groups taking two components into the same group iff a curve joining points from these two components has even number of (transversal) intersection points with $f^{-1}(BO(1))$. Counting modulo 2 the preimages of $B\xi$ in both groups of components, we obtain $\deg f \in Z_2 \oplus Z_2$.

Let us recall how we computed in Part I the degree of the map

$$\bar{\tau} = \bar{\theta} \circ \bar{\varrho}\colon S^n \to S^1 \vee S^n .$$

The source sphere of this map was identified with the boundary of the cube

$$I^{n+1} = \{(x_1, \ldots, x_{n+1}) |\ 0 \leqslant x_i \leqslant 1\}$$

and choose points $P_1 \in S^1$ and $P_2 \in S^n$ so that we had:

$$\bar{\tau}^{-1}(P_1) = \{(x_1, \ldots, x_{n+1}) \in S^n |\ \textstyle\sum x_i \equiv \frac{1}{2} \bmod 1\},$$

$$\bar{\tau}^{-1}(P_2) = \text{the set of the face centers of } I^{n+1} .$$

Now we want to compute the degree of the map $\gamma = i \circ \bar{\tau}\colon S^n \to RP^\infty \vee T\zeta_n^*$. We have

$$\gamma^{-1}(BO(1)) = \bar{\tau}^{-1}(P_1) \quad \text{and} \quad \gamma^{-1}(B\zeta_n^*) = \bar{\tau}^{-1}(P_2) .$$

Both preimage sets $\gamma^{-1}(BO(1))$ and $\gamma^{-1}(B\zeta_n^*)$ are symmetric on the centre of the cube I^{n+1}. The components of $S^n \setminus \gamma^{-1}(BO(1))$ corresponding to each other under this symmetry belong to different groups of components (in the sense of the previous Remark). Hence in both groups the number of the face-centers is $(n+1)$ that is odd.

Proposition 3 is proved.

Proof of Proposition 2. Let us denote by $\bar{\Gamma}_l = \bar{\Gamma}_l(RP^\infty)$ the l-th term of the James product of RP^∞ (see [3] and [7]). The m-th homotopy group of this space is isomorphic to the cobordism group of those immersions of $(m-1)$ manifolds (not necessarily orientable ones) into R^m which are projections of embeddings into R^{m+1} and have no $(l+1)$-tuple points.

This space can be obtained similarly to $\Gamma_l = \Gamma_l(RP^\infty)$ by induction on l, beginning with $\bar{\Gamma}_1 = RP^\infty$ and obtaining $\bar{\Gamma}_l$ from $\bar{\Gamma}_{l-1}$ by attaching to $\bar{\Gamma}_{l-1}$ an l-dimensional disc bundle $D\bar{\zeta}_l$. But the structure group of the bundle $\bar{\zeta}_l$ is not G_l (as in the case of Γ_l) but the group of diagonal matrices with ± 1 in the diagonal. We shall denote this group by $\bar{G}_l$. Obviously,

$$\bar{G}_l \approx O(1) \oplus \ldots \oplus O(1) \quad (l\text{-summands}) .$$

Similarly to the oriented case treated in part I there is a "forgetting" map $j\colon \bar{\Gamma}_l \to \Gamma_l$ which is an $l!$ — fold covering over

$$BG_l \subset D\zeta_l \subset \Gamma_l \quad \text{and} \quad j^{-1}(BG_l) = B\bar{G}_l .$$

Now consider the following commutative diagram

$$\begin{array}{ccccc}
\bar{B}' & \xrightarrow{\ \bar{p}\ } & B\bar{G}_l \overset{\bar{\imath}}{\subset} \bar{\Gamma}_l & \searrow^{\bar{\alpha}} & \\
\downarrow & & \downarrow \qquad \downarrow{\scriptstyle j} & & RP^\infty \\
B' & \xrightarrow[\ p\ ]{} & BG_l \overset{i}{\subset} \Gamma_l & \nearrow_{\alpha} &
\end{array}$$

Here α and $\bar{\alpha}$ induce isomorphisms of the fundamental groups, $\bar{\imath}$ and i are inclusions, p and $\bar{p}$ are double covering maps induced by $\bar{\alpha} \circ \bar{\imath}$ and $\alpha \circ i$ respectively.

Let us consider also the corresponding diagram of the fundamental groups.

$$\begin{array}{ccccccc}
0 \to & \pi_1(\bar{B}') & \to & \bar{G}_l \approx Z_2 \oplus \ldots \oplus Z_2 & \xrightarrow{\ \bar{\imath}_*\ } & Z_2 & \\
& \downarrow & & \downarrow & & \downarrow \approx & \\
0 \to & \pi_1(B') & \to & G_1 = (Z_2 \oplus \ldots \oplus Z_2) \wr S(l) & \to & Z_2 &
\end{array}$$

Because of the symmetry, $\bar{\imath}_*$ is the "sum map"

$$\bar{\imath}_*(x_1, \ldots, x_l) = \sum x_i$$

and so $\pi_1(\bar{B}') = \{(x_1, \ldots, x_l) \in G_l |\ \sum x_i = 0\}$ or if we think of $\bar{G}_l$ as the diagonal matrices with ± 1 in the diagonal than $\pi_1(\bar{B}')$ is the group of the diagonal matrices with even number of negative entries. Let us denote the latter group by $\bar{G}_l'$.

Then $\bar{B}' = B\bar{G}_l'$.

The $S(l)$ extension $0 \to \bar{G}_l \to G_l \to S(l) \to 0$ induces an $S(l)$ extension of the group $\bar{G}_l'$:

$$0 \to \bar{G}_l' \to \pi_1(B') \to S(l) \to 0 .$$

Hence $\pi_1(B') =$ the group of matrices with 1 nonzero entry in each row and column, these nonzero entries are ± 1 and the number of negative entries is even. In other words, $\pi_1(B') \approx G_l^*$. This proves that $B' = BG_l^*$. ■

Thus, the Theorem is proved for immersions. The extension of the result to singular maps goes in the same way as in Part I. Namely, the map

$$\theta\colon C\Gamma_n \to RP^\infty \vee T\zeta_n^*$$

can be extended to a map

$$\theta^{x}\colon CX_n \to RP^{\infty} \vee T\zeta_n^{*}$$

by the same argument as in part I.

So we have the commutative diagram

$$\begin{array}{ccc} S^n \xrightarrow{\ \varrho\ } & C\Gamma_n & \xrightarrow{\ \theta\ } RP^{\infty}\vee T\zeta_n^{*} \\ {\scriptstyle \varrho^{x}}\searrow & \downarrow & \nearrow{\scriptstyle \theta^{x}} \\ & CX_n & \end{array}$$

Here ϱ^{x} is the restriction to a fibre of the attaching map by using of which we obtain CX_{n+1} from CX_n (i.e. ϱ^{x} here is the analogue of the map ϱ^{x} from Part I). From this diagram we deduce that ϱ^{x} is not null-homotopic. Now by Lemma 1 the Theorem follows. ■

§ 4. Final remarks.

(1) The restriction ($R\Sigma$) can be released as soon as the appropriate classifying spaces for the cobordisms of maps with higher singularities are constructed (see the corresponding remark in § 8 of Part I).

(2) For immersion T. Banchoff in [1] described a procedure of pairwise elimination of the multiple points of the highest multiplicity. His procedure applied to a C-immersion gives again a C-immersion. Hence we have the following

COROLLARY. *No C-immersion of an even dimensional manifold M^n into R^{n+1} has odd number of $(n+1)$-tuple points.*

(3) As a matter of fact the multiple points of the highest multiplicity can be eliminated pairwisely for an arbitrary C-map as well (although not by Banchoff's procedure since that procedure may increase the singular multiplicity). So in the previous corollary "C-immersion" can be replaced by "C-map".

The elimination procedure of $(n+1)$-tuple points of a map $f\colon M^n \to R^{n+1}$ is shown on Figs. 1, 2, 3.

(1) Take small balls B_1 and B_2 in R^{n+1} centered around $(n+1)$-tuple points. These balls can be joined by a thin tube T which

(a) avoids the image of the singular points and the double points,

(b) its central line has even number of transversal intersection points with $f(M)$ (see Fig. 1).

(2) Making a surgery of $f(M)$, we can achieve that the tube does not intersect the image set of the map at all (see Fig. 2).

(3) Now in a big disc D of R^n take $(n+1)$ small ε-balls $B_1^n, B_2^n, \ldots, B_{n+1}^n$ such that

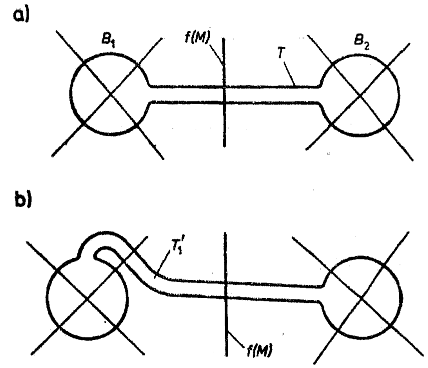

Fig. 1

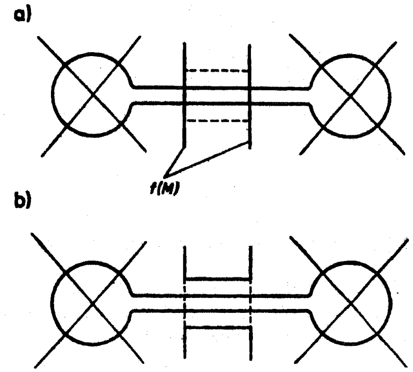

Fig. 2

(a) their centers are in general position and the distance between any two of them is ε,

(b) the boundary of any of these balls contains the centers of all the other balls.

Let S denote the union of the boundaries of these balls

$$S = \bigcup_{i=1}^{n+1} \partial B_i^n \subset D .$$

Let us denote by S_t the subset $S\times\{t\}$ in $D\times\{t\}$ for $t\in[0, 1]$. There exists a diffeomorphism h of $D\times[0, 1]$ onto the set $U = B_1\cup T\cup B_2$ such that $h(S_0\cup S_1) = (\partial U)\cap f(M)$ (see Fig. 3)

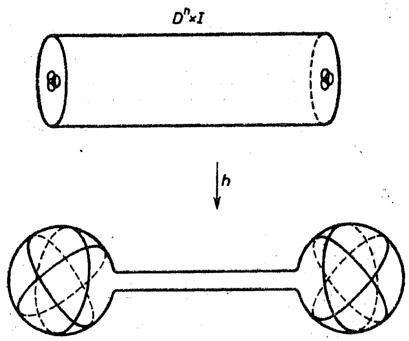

Fig. 3

Now make the following surgery on $f(M)$.

Throw out from $f(M)$ the $2(n+1)$ balls of dimension n which form the intersection $f(M) \cap U$ and add to $(f(M) \setminus U)$ the set

$$h(\{\bigcup S_t |\ t \in [0, 1]\}).$$

By thus the number of $(n+1)$-tuple points has been decreased by 2.

References

[1] T. Banchoff, *Triple points and surgery of immersed surfaces*, Proc. Amer. Math. Soc. 46 No 3 (1974).

[2] M. G. Barratt and P. J. Eccles, *Γ^+-structures* I, Topology 13 (1974), 25–45.

[3] I. M. James, *Reduced product spaces*, Ann. of Math. 62 (1) (1955), 170–197.

[4] A. Szűcs, *Cobordism groups of l-immersion* I (in Russian), Acta Math. Acad. Sci. Hungar. 27 (1976), 343–358.

[5] — *Cobordism of maps with simplest singularities*, Lecture Notes in Math. 788. Springer 1980, 223–244.

[6] — *Cobordism groups of immersions with restricted selfintersections*, Osaka J. Math. 21 (1984), 71–80.

[7] — *Multiple points of singular maps*, Math. Proceedings of Cambridge Philos. Soc. 100 (1986), 331–346.

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An analytic equivalence relation not arising from a Polish group action

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Abstract. We show that the equivalence relation xEy iff $\omega_1^x = \omega_1^y$ can not arise as the orbits of a Polish group action. We also calculate the exact Borel rank of $\{x: \omega_1^x = \alpha\}$ for α a countable admissible ordinal.

There are two natural abstractions of Vaught's conjecture to descriptive set theory. The most natural would be the conjecture that any analytic equivalence relation on a Borel set with uncountably many classes, each of which is Borel, admits a perfect set of inequivalent elements. Unfortunately this is easily seen to be false. Let xEy if and only if $\omega_1^x = \omega_1^y$, where ω_1^x is the first ordinal not recursive in x. Then E is Σ_1^1 and all E equivalence classes are Borel, but E has exactly $\aleph_1$ equivalence classes, one for each countable admissible ordinal.

The second abstraction is known as the topological Vaught conjecture. Let G be a Polish group acting continuously on a Polish space X. If G has uncountably many orbits, then there is a perfect subset of X such that any two distinct elements are in different orbits. Kechris asked if the equivalence relation E could arise as the action of a Polish group on the Baire space. In § 1 we show that it cannot. In § 2 we give an exact calculation of the Borel rank of $\{x \in \omega^\omega: \omega_1^x = \alpha\}$, for α a countable admissible ordinal.

§ 1. Our main lemma uses several ideas of Vaught's [V].

Definition. Let G be a Polish group acting continuously on a Polish space X. Let $B \subseteq X$. The *Vaught transform* $B^* = \{x \in X: \{g \in G: gx \in B\}$ is comeager$\}$.

The facts we need about the Vaught transform are summarized in the following lemma.

Lemma 1.1 (Vaught [V]).

(1) *For any $B \subseteq X$, B^* is G-invariant.*

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