$b \in V(-a)$ or $b \notin V(-a)$. It turns out that (G, q, Q) is a quaternionic structure in the sense of [8] and its scheme coincides with S. By [1], CM holds for S, so S is a quaternionic scheme.

References

[1] A. B. Carson and M. A. Marshall, *Decomposition of Witt Rings*, Canad. J. Math. 34 (1982), 1276–1302.

[2] C. Cordes, *The Witt group and the equivalence of fields with respect to quadratic forms*, J. Algebra 25 (1973), 400–421.

[3] — *Quadratic forms over non-formally real fields with a finite number of quaternion algebras*, Pacific J. Math. 63 (1976), 357–365.

[4] M. Kula, *Fields with prescribed quadratic form schemes*, Math. Zeit. 167 (1979), 201–212.

[5] — *Fields with non-trivial Kaplansky's radical and finite square class number*, Acta Arith. 38 (1981), 411–418.

[6] M. Kula, L. Szczepanik and K. Szymiczek, *Quadratic forms over formally real fields with eight square classes*, Manuscripta Math. 29 (1979), 295–303.

[7] Li Delang, *A family of fields with finite square class number*, Sci. Sin. 23 (1980), 1226–1236.

[8] M. Marshall, *Abstract Witt Rings*, Queen's Papers in Pure and Applied Math., No. 57, Queen's Univ., Kingston, Ont. 1980.

[9] M. Marshall and J. Yucas, *Linked quaternionic mappings and their associated Witt rings*, Pacific J. Math. 95 (1981), 411–425.

[10] J. G. Pękała, *On the axioms of quadratic form schemes*, Ann. Math. Silesianae, to appear.

[11] L. Szczepanik, *Fields and quadratic form schemes with the index of the radical not exceeding 16*, Ann. Math. Silesianae 13 (1985) 23–46.

[12] — *Quadratic form schemes with non-trivial radical*, Colloq. Math. 49 (1985), 143–160.

[13] K. Szymiczek, *Grothendieck groups of quadratic forms and G-equivalence of fields*, Proc. Camb. Phil. Soc. 73 (1973), 26–36. Corrig. ibid. 74 (1973), 199.

[14] — *Quadratic forms over fields with finite square class number*, Acta Arith. 28 (1975), 195–221.

[15] E. Witt, *Theorie der quadratischen Formen in beliebigen Körpern*, J. Reine Angew. Math. 176 (1937), 31–44.

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A combinatorial analysis of functions provably recursive in $I\Sigma_n$

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Abstract. We use certain functionals of finite type to define an indicator for $I\Sigma_n$. We show that this indicator is equivalent in $I\Sigma_n$ to an indicator of combinatorial character. The syntactical-combinatorial part is definitely separated from the model — theoretic part. Finally we obtain a simple proof of the estimation of the growth for recursive functions provably total in $I\Sigma_n$.

§ 1. Introduction. This paper is devoted to an application of a family of selected primitive recursive functionals to the investigation of provably recursive functions in $I\Sigma_n$, where $n \geq 1$. We first define the spaces $\bar{F}_k\omega$ on which the above-mentioned functionals are defined Let $\bar{F}_0\omega = \omega$; then we define by induction:

$$\bar{F}_{k+1}\omega = (\bar{F}_k\omega)^{\bar{F}_k\omega}$$

for $k \in \omega$.

We assume that $\bar{I}_1\colon \omega \to \omega$ is the function of the immediate successor and we define the subsequent functionals by

$$\bar{I}_k(f^{k-1}) \dots (f^1)(x) = (f^{k-1})^{x+1}(f^{k-2}) \dots (f_1)(x)$$

for all $x \in \omega$, $f^1 \in \bar{F}_1\omega, \dots, f^{k-1} \in \bar{F}_{k-1}\omega$.

The functionals belonging to the space $\bar{F}_k\omega$ will be said to be of *type k*. In particular, for every $k \in \omega$, $k \geq 1$, the functional $\bar{I}_k$ is of type k.

The idea of using functionals like $\bar{I}_2, \dots, \bar{I}_{n+1}$ is not new. In [4] Paris presents, referring to Aczél, a sketch of proof that for every $\alpha < \omega_{n+1}$ the existence of α-large sets is provable in $I\Sigma_n$. That proof is based on the use of the above-mentioned functionals.

Unfortunately, a considerable difficulty in reading that proof arises from problems connected with the formalization of the above functionals in arithmetic. Moreover, all lemmas are sketched and it is not obvious that they can be formalized in $I\Sigma_n$.

In the present paper we only formalize functionals of type 1, strictly speaking only those of them which are formed of $\bar{I}_1, \bar{I}_2, \dots$ by means of application and superposition. In order to reach this objective we use a kind of miniaturization of all functionals. This topic is discussed in § 2 and § 3.

The essential part of the paper is § 4, where we show that for $n \geqslant 1$ a formal counterpart Y_n of the function

$$\bar{Y}_n(a, b) = \max_m \bar{I}^m_{n+1}(\bar{I}_n) \dots (\bar{I}_1)(a) \leqslant b$$

is an indicator in models for $I\Sigma_n$ for segments satisfying $I\Sigma_n$ (Theorem 4.16).

To prove that Y_n is an indicator we show that the sentence $\forall a\, \exists b\; Y_n(a, b) \geqslant c$ implies in $I\Sigma_n$ a certain simple combinatorial property (a weakening and modification of FCP in [5], which is known as the Friedman–Pudlák principle), which can easily be used to construct segments which are models for $I\Sigma_n$.

By Theorem 4.16 it follows, in a standard manner, that the family of functions $\{\bar{I}^m_{n+1}(\bar{I}_n) \dots (\bar{I}_1)\colon m \in \omega\}$ is a cofinal set in the class $\mathrm{Rec}(I\Sigma_n)$ — Theorem 4.17, where $\mathrm{Rec}(I\Sigma_n)$ (more generally $\mathrm{Rec}(T)$) denotes the family of provably recursive functions in $I\Sigma_n$ (in T).

In the last section (§ 5) we present a short proof of a result connecting our functionals with Hardy's functions $H_\alpha\colon \alpha < \varepsilon_0$ — Theorem 5.3 and Lemma 5.5. The following equality holds:

$$\bar{I}^m_{n+1}(\bar{I}_n) \dots (\bar{I}_1) = H_{\omega^m_n} \quad \text{for all } m, n \in \omega .$$

It implies, by Theorem 4.17, a subtle version of a well-known theorem of Wainer [7] (§ 5, Corollary 5.16): the hierarchy $H_\alpha\colon \alpha < \omega_{n+1}$ is a cofinal set in the class $\mathrm{Rec}(I\Sigma_n)$.

Moreover we give in § 5 some weaker and easier to prove versions of Lemmas 4.9 and 4.12. Using those versions we can show that the hierarchy $H_\alpha\colon \alpha < \omega_{n+1}$ majorizes the class $\mathrm{Rec}(I\Sigma_n)$ (this is an essential part of Corollary 5.16), without caring about formalizing any notion in $I\Sigma_n$. One of the profits that can be derived from this paper is a simple proof of Wainer's theorem about a hierarchy of functions majorizing $\mathrm{Rec}(I\Sigma_n)$ (the simplest form of this proof is only sketched here). The classical proof-theoretic proof of that theorem is rather long and requires several intermediate steps: cut-elimination, definition of this process in terms of α-recursion (or alternatively, functional interpretation [1] and reduction of primitive recursion for functionals to α-recursion [6]) and bounding the class of functions $< \omega_{n+1}$-recursive by Wainer's or Hardy's hierarchy [7].

Consider now the possibility of some generalizations. Lemmas 4.9 and 4.12 are provable in $I\Sigma_{n+1}$ for $n > 1$. One of the possible generalizations of Theorem 4.16 is the following: Let $M \models \mathrm{PA}$. If $S \in M$ and $\forall m \in \omega\; M \models$ "S is ω^m_n-large" then there exists an $I \subset_e M$ such that $(I, S \cap I) \models I\Sigma_n(R)$, where R is interpreted as $S \cap I$, and the set $S \cap I$ is cofinal in I.

This follows from 4.13, 4.15(2) and the formalization of the proof of Theorem 5.3 in PA.

Finally, let us compare the information included in Theorem 4.16 with that in Corollary 5.6. From Theorem 4.16 we can deduce that every $f \in \mathrm{Rec}(I\Sigma_n)$ is, $I\Sigma_n$-provably, bounded by some function of the form $\bar{I}^m_{n+1}(\bar{I}_n) \dots (\bar{I}_1) = H_{\omega^m_n}$, where $m \in \omega$, whilst 5.6 only says that f is bounded by some $H_{\omega^m_n}$ and not that this fact is provable in $I\Sigma_n$. From 5.6 we cannot directly deduce 4.16.

Indeed, if a family of recursive functions $\{f_m\colon m \in \omega\}$ is such as the family $\{H_{\omega^m_n}\colon m \in \omega\}$ in 5.6, i.e. Σ_1-definable and $f_m \leqslant^* f_n$ ("f_n eventually dominates f_m") for $m \leqslant n$ then the following (easy) equivalence holds: $\{f_m\colon m \in \omega\}$ is cofinal in $\mathrm{Rec}(I\Sigma_n)$ iff the formula defining $\max f_m(a) \leqslant b$ is an indicator for segments satisfying $I\Sigma_n$ in structures elementarily equivalent to N.

§ 2. Introduction to the formalization of functionals. In this section we describe a miniaturization of the spaces $\bar{F}_n\omega$ and the functionals $\bar{I}_n$, consisting in replacing ω by a finite set S and imitating those notions over such set.

In the next section we use this idea to get a formalization in $I\Sigma_n$, where $n \geqslant 1$, of the functionals of type 1 which are formed of $\bar{I}_1, \bar{I}_2, \dots$ by means of application and superposition.

Because the counterpart of $\bar{I}_1$ on $[0, n]$, i.e. $\bar{I}_1 \cap [0, n]^2$, is a partial function for $n \in \omega$, for the miniaturization of $\bar{F}_1\omega$ we take the set $F_1 S$ of all partial functions from S to S. There exist many possibilities of formalizing $F_1 S$ and all functionals of finite type over $F_1 S$ in arithmetic. It will be convenient to base our formalization on an interpretation of a certain fragment of set theory in $I\Delta_0 + \exp$.

Let V_ω denote the family of hereditarily finite sets. Let I be the following standard interpretation of the language L_{ZF} of the model $(V_\omega, \in)$ in the language of arithmetic extended by the exponential function 2^x, $L_A \cup \{2^x\}$:

$$(x \in y)^I = \exists u, v < y\; (y = 2^x(2u+1)+v \wedge v < 2^x) .$$

Since $I\Delta_0 + \exp \vdash$ "ω^I is isomorphic with the universe", this interpretation is unsuitable for our purpose. In order to define an improved interpretation, consider the function $h\colon V_\omega \overset{1-1}{\to} \omega$ defined as follows:

$$h(n) = 2n \text{ for } n \in \omega, \quad h(x) = \sum_{y \in x} 2^{h(y)+1} - 1 \quad \text{for } x \in \mathrm{HF} \setminus \omega .$$

Let $\mathrm{Ev} = \{2n\colon n \in \omega\}$, let $A = \{\sum_{i=0}^{n} 2^{2i+1} - 1\colon n \in \omega\}$ (i.e. A is the set of "bad codes of natural numbers"). It is easy to observe that $m \in h[V_\omega] \cap (\omega - \mathrm{Ev})$ iff $m \notin \mathrm{Ev}$ and $m \notin A$ and for every k_1, if k_1 is an odd exponent in the decomposition of $(m+1)/2$ then $k_1 \notin A$ and for every k_2, if k_2 is an odd exponent in the decomposition of $(k_1+1)/2$ then $k_2 \notin A$ and so on.

Using this observation we construct a $\Delta_0(2^x)$-formula $\omega^J(x)\,[x \in \omega^J]$ defining the set $h[V_\omega]$ in the model $(\omega, <, +, 2^x, 0, 1)$.

The relation $x \in^J y \Leftrightarrow h^{-1}(x) \in h^{-1}(y)$ for $x, y \in \omega^J$ is definable in the model $(\omega, <, +, \cdot, 2^x, 0, 1)$ by the following $\Delta_0(2^x)$-formula:

$$(x, y \in \mathrm{Ev} \wedge x < y) \vee \left((x, y \in \omega^J \wedge y \notin \mathrm{Ev} \wedge \left(x \in \frac{y+1}{2}\right)^I\right)$$

which we denote by $(x\in y)^J$. This formula determines a new interpretation J of the language L_{ZF} in $L_A\cup\{2^x\}$. The image of L_{ZF} under J will be denoted by L^J_{ZF}.

The following axioms of set theory are, in the interpreted form, provable in $I\Delta_0+\exp$: the axioms of equality, pair, sum, power set, separation for Δ_0-formulas and regularity for Δ_0-formulas. Denote by Z_0^- the theory based on the above-mentioned axioms. Of course $(V_\omega,\in)\models Z_0^-$. The second natural model for Z_0^- is $(V_{\omega+\omega},\in)$.

Let $L_{ZF}\cup\{P\}$ be the definitional extension of the language L_{ZF} by the symbol P denoting the power set function. Let $\Delta_0(P)$ denote the class of formulas of the language $L_A^J\cup\{P\}$ with the quantifiers bounded by the terms $P^k(x)$, where $k\in\omega$. Let us mention that $\Delta_0(2^x)$ denotes the class of the formulas of the language $L_A\cup\{2^x\}$ with quantifiers bounded by the terms $2^{\cdot^{\cdot^{2^x}}}$ with k-fold exponentiation, $k\in\omega$. The interpretation J transforms formulas of class $\Delta_0(P)$ into formulas of class $\Delta_0(2^x)$. In the sequel we shall say that J is the natural interpretation of L_{ZF} in $L_A\cup\{2^x\}$.

The following facts are true:

2.1. (a) $I\Delta_0+\exp\vdash\varphi^J\to(V_\omega,\in)\models\varphi$,

(b) The function $g(x)=\frac{1}{2}x$ establishes in $I\Delta_0+\exp$ an isomorphism between $(\omega^J,<^J,+^J,x^J)$ and the universum of the theory $I\Delta_0+\exp$.

(c) It is a theorem of $I\Delta_0+\exp$ that g maps in a one-one manner the family of subsets $\subseteq\omega$ in the sense of L^J_{ZF} into the family of bounded codable sets of natural numbers.

Since the theory Z_0^- is interpretable in $I\Delta_0+\exp$, every definition in Z_0^- can be regarded up to the interpretation J as a definition in $I\Delta_0+\exp$.

2.2. DEFINITION (Z_0^-). Let S be a set $\subseteq\omega$. Let $F_0S=S$, $F_1S=\bigcup_{A\subseteq S}S^A$. For $n\geqslant 2$ the sets F_nS are defined by induction: $F_nS=(F_{n-1}S)^{F_{n-1}S}$, just as the sets $\bar{F}_n\omega$. Next, let $I_1^S(a)$ denote the immediate successor of a in S with respect to the order $<$ on ω. The functional I_1^S is defined for $a\in S\setminus\{\max S\}$, and for $n\geqslant 2$ the functionals $I_n^S\colon F_{n-1}S\to F_{n-1}S$ are defined by

$$I_n^S(f^{n-1})\dots(f_1)\simeq(f^{n-1})^{x+1}(f^{n-2})\dots(f^1)(x)$$

for $x\in S$, $f^1\in F_1S,\dots,f^{n-1}\in F_{n-1}S$, where

$$f(x)\simeq g(x)\Leftrightarrow\big(f(x)\downarrow\wedge g(x)\downarrow\wedge f(x)=g(x)\vee f(x)\uparrow\vee g(x)\uparrow\big),$$

$f(x)\downarrow$ – $f(x)$ defined, $f(x)\uparrow$ – $f(x)$ undefined.

Remark. This definition is, of course, correct in set theory, i.e. it is correct in the model $(V_{\omega+\omega},\in)$. It is also correct in the model $(V_\omega,\in)$. The proofs of its correctness in these models can be based on the axioms of Z_0^-. Hence the definition is correct in $I\Delta_0+\exp$.

Moreover, observe that for arbitrary $1\leqslant m,n\in\omega$ we have $I_n^m(I_{n-1})$ (I_1) $=I_n^\omega(I_{n-1}^\omega)\dots(I_1^\omega)$.

To analyse the properties of the functions $(I_{n+1}^S)^m(I_{n-1}^S)\dots(I_1^S)$ we shall need terms for all functionals that can be obtained from $I_1^S,\dots,I_n^S$ through application and superposition.

2.3. DEFINITION. The symbols $T_1^n,\dots,T_n^n$ denote Δ_1-definable classes for which the following conditions of forming terms are provable in $Z_0^-+\Sigma_1$-Ind:

(1) $I_i\in T_i^n$ for $i=1,\dots,n$,

(2) $\forall s,t\ (s,t\in T_i^n\to(s\circ t)\in T_i^n)$ for $i=1,\dots,n$,

(3) $\forall s,t\ (s\in T_{i+1}^n\wedge t\in T_i^n\to s(t)\in T_i^n)$ for $i=1,\dots,n-1$,

(4) $t\in T_i^n\to t$ is formed in a finite number of steps as a result of applying $(1)_i$, $(2)_i$ where $i=1,\dots,n$ and $(3)_i$ where $i=1,\dots,n-1$.

Σ_1-Ind denotes mathematical induction for Σ_1-formulas in the language L_{ZF}.

2.4. Remark. (a) Since the interpretation J maps theorems of the theory $Z_0^-+\Sigma_1$-Ind to theorems of the theory $I\Sigma_1$, the above definition is also correct in $I\Sigma_1$.

(b) It can be proved in $Z_0^-+\Sigma_1$-Ind that the classes $T_1^n,\dots,T_n^n$ are the smallest among the Δ_1-definable classes satisfying (1), (2) and (3).

Below, we shall denote by T the class $T_1^n\cup\dots\cup T_n^n$ and by FS the sum $F_1S\cup\dots\cup F_nS$.

2.5. DEFINITION ($Z_0^-+\Sigma_1$-Ind). The *value* of the term $t\in T$ in the model FS, t^S is defined by the following inductive conditions:

(1) For $i=1,\dots,n$, I_i^S is a functional defined in 2.2,

(2) $(t_1\circ t_2)^S=t_1^S\circ t_2^S$ for $t_1,t_2\in T_i^n$, $1\leqslant i\leqslant n$, $S\subseteq\omega$,

(3) $t_1(t_2)^S=t_1^S(t_2^S)$ for $t_1\in T_{n+1}^n$, $t_2\in T_i^n$, $1\leqslant n-1$, $S\subseteq\omega$.

It can be proved that there is a mapping $t\in T$, $S\subseteq\omega\mapsto t^S$ of class Δ_1 in the theory $Z_0^-+\Sigma_1$-Ind for which conditions (1), (2), (3) are provable in $Z_0^-+\Sigma_1$-Ind. To this end we first construct a mapping $t\in T_n^n$, $S\subseteq\omega\mapsto t^S$ of class Δ_1 satisfying (1) and (2) for $i=n$ and then a mapping $t\in T_{n-1}^n$, $S\subseteq\omega\mapsto t^S$ of class Δ_1 for which conditions (1), (2), (3) are provable for $i=n-1$, etc.

§ 3. Formalization of functionals $\bar{t}$ of type 1. The aim of this section is to obtain a formalization of the functionals $\bar{t}$, where $t\in T_1^{n+1}$, in the theory $I\Sigma_n$ (Lemma 3.9) using the miniaturization t^S of those functionals described in §2.

In the first part of this section we show that the miniaturization in question is adequate (Theorem 3.1) and has good properties (Theorem 3.2).

Using Lemma 3.3 on which Theorem 3.2 is founded we also prove that functions definable by "more complicated terms" are growing faster (Corollary 3.6), whence we infer a result which is simple but important for further combinatorial considerations (Lemma 3.8).

3.1. THEOREM (adequacy of the miniaturization). *For every n*, $1 \leqslant n \in \omega$, *and for every* $t \in T_1^n$

$$t^{\omega}(a) = b \Leftrightarrow t^{[a,b]}(a) = b \quad \text{for all } a, b \in \omega .$$

For $S_1 \subseteq S_2$, $f_1 \in F_1 S_1$, $f_2 \in F_1 S_2$ let $f_1 \subseteq_p f_2$ denote the relation $f_1 = f_2 \cap (S_1 \times S_2)$. Theorem 3.1 is a particular case of the next theorem.

3.2. THEOREM (good properties of the miniaturization). *Let* $1 \leqslant n \in \omega$.

(1) $\forall S \subseteq \omega\ \forall t \in T_1^n\ \forall a, b \in S\ t^{S \cap [a,b]} \subseteq_p t^S$.

(2) *The sentence in* (1) *is provable in* $I\Sigma_1$.

(3) *The sentences*: $\forall S \in \mathrm{Fin}\ \forall a, b \in S\ I_n^m(I_{n-1}) \dots (I_1)^{S \cap [a,b]} \subseteq_p I_n^m(I_{n-1}) \dots (I_1)$ *are provable in* $I\Delta_0 + \exp$ *for all* $m \in \omega$, *where* Fin *denotes the class of all finite subsets of* ω.

Since the structure $(V_{\omega+\omega}, \in)$ is a model for $Z_0^- + \Sigma_1$-Ind, in order to prove (1) it is sufficient to prove it within the system $Z_0^- + \Sigma_1$-Ind. Hence follows also (2).

In fact, (1) is provable in Z_0^-, but we shall not need this.

Proof. Observe that the first part of the proof of (1) in $Z_0^- + \Sigma_1$-Ind given below runs within the system Z_0^-; this will permit us to deduce (3).

Part 1 (Z_0^-). Given a set $S \subseteq \omega$ and $a, b \in S$, the set $S \cap [a, b]$ is an interval in S. Further, denote S by S_2 and the resulting interval in S_2 by S_1. We define:

$$f_1 \subseteq_1 f_2 \Leftrightarrow f_1 \in F_1 S_1 \wedge f_2 \in F_2 S_2 \wedge f_1 \subseteq_p f_2 \wedge \forall x (f_2(x){\downarrow} \to x \leqslant f_2(x))$$

$$f_1 \subseteq_{i+1} f_2 \Leftrightarrow f_1 \in F_{i+1} S_1 \wedge f_2 \in F_{i+1} S_2 \wedge$$
$$\wedge \forall g_1, g_2 (g_1 \subseteq_i g_2 \to f_1(g_1) \subseteq_i f_2(g_2)), \quad \text{for } i = 1, \dots, n-1 .$$

Observe that if $f_1 \subseteq_1 f_2, g_1 \subseteq_1 g_2$, then $(f_1 \circ g_1) \subseteq_1 (f_2 \circ g_2)$. Moreover, $I_1^{S_1} \subseteq_1 I_1^{S_2}$. The following facts are also true:

(i) $f_1 \subseteq_j f_2 \wedge g_1 \subseteq_j g_2 \to (f_1 \circ g_1) \subseteq_j (f_2 \circ g_2)$,

(ii) $I_j^{S_1} \subseteq_j I_j^{S_2}$ for $j = 1, \dots, n$.

Instead of proving these facts directly it will be more advantageous to formulate and prove in Z_0^- a general lemma which implies them. First note that (i) and (ii) imply $(I_{n+1}^m)^{S_1} \subseteq_{n+1} (I_{n+1}^m)^{S_2}$, whence in view of (ii) and the definition of $\subseteq_{n+1}, \dots, \subseteq_1$

$$I_{n+1}^m(I_n) \dots (I_1)^{S_1} \subseteq_1 I_{n+1}^m(I_n) \dots (I_1)^{S_2} .$$

Hence (3) will be proved when we prove the following:

3.3. LEMMA (Z_0^-). *For arbitrary sets* S_1, S_2 *such that* $S_1 \subseteq S_2$, *if* $r_1, \dots, r_n$ *is a sequence of relations such that for all* f_1, f_2, g_1, g_2

(1) $r_1 \subseteq F_1 S_1 \times F_1 S_2 \wedge f_1 r_1 f_2 \wedge g_1 r_1 g_2 \to (f_1 \circ g_1) r_1 (f_2 \circ g_2)$,

(2) $f_1 r_{i+1} f_2 \Leftrightarrow f_1 \in F_{i+1} S_1 \wedge f_2 \in F_{i+1} S_2 \wedge$
$\forall g_1 g_2 (g_1 r_i g_2 \to f_1(g_1) r_i f_2(g_2))$ *for* $i = 1, \dots, n-1$,

(3) $f_1 \in F_1 S_1 \wedge f_2 \in F_1 S_2 \wedge$
$\wedge \forall a \in S_1 \exists g_1 g_2 (g_1 r_1 g_2 \wedge f_1(a) \simeq g_1(a) \wedge f_2(a) \simeq g_2(a)) \to f_1 r_1 f_2$,

then

(4) $f_1 r_j f_2 \wedge g_1 r_j g_2 \to (f_1 \circ g_1) r_j (f_2 \circ g_2)$ *and*

(5) $I_j^{S_1} r_j I_j^{S_2}$ *for* $j = 2, 3, \dots, n$.

(1) and (2) are of course satisfied for the relations $\subseteq_1, \dots, \subseteq_n$. Let $f_i \in F_i S_i$ for $i = 1, 2$. If for $a \in S_1$ there are g_1, g_2 such that $g_1 \subseteq_1 g_2, f_1(a) \simeq g_1(a), f_2(a) \simeq g_2(a)$ then $f_1(a) \simeq g_1(a)$, under the condition that $f_1(a){\downarrow}$ or $f_2(a) \in S_1$. Therefore, if the assumption of (3) is true, then $f_1 \subseteq_1 f_2$. Thus (3) is satisfied so Lemma 3.3 indeed implies the validity of (i) and (ii).

Proof of Lemma 3.3. We apply induction with respect to j, $1 \leqslant j \leqslant n$. For $j = 1$, (4) coincides with (1). Assume therefore that (4) is true for $1 \leqslant j < n$. Let $f_1 r_{j+1} f_2$ and $g_1 r_{j+1} g_2$; of course $f_1 \circ g_1 \in F_{j+1} S_1$, $f_2 \circ g_2 \in F_{j+1} S_2$.

Let h_1, h_2 be such that $h_1 r_j h_2$. Thus, by (2), we have $g_1(h_1) r_j g_2(h_2)$, whence, also by (2), $f_1(g_1(h_1)) r_j f_2(g_2(h_2))$. Therefore $(f_1 \circ g_1)(h_1) r_j (f_2 \circ g_2)(h_2)$, and we conclude by (2) that $(f_1 \circ g_1) r_{j+1} (f_2 \circ g_2)$.

(5) Suppose that $2 \leqslant j \leqslant n$. Take an arbitrary sequence of pairs

$$(f_1, g_1), \dots, (f_{j-1}, g_{j-1})$$

such that $f_k r_k g_k$ for $k = 1, \dots, j-1$.

Let $a \in S_1$. Thus, by (4), $f_{j-1}^{a+1} r_{j-1} g_{j-1}^{a+1}$. Hence, by (2), we have

$$f_{j-1}^{a+1}(f_{j-2}) \dots (f_1) r_1 g_{j-1}^{a+1}(g_{j-2}) \dots (g_1) .$$

Denote the function on the left of r_1 by f_j, and the one on the right by g_j. By the definition of the functionals $I_j^{S_1}$, $I_j^{S_2}$ we have $I_j^{S_1} \in F_j S_1$, $I_j^{S_2} \in F_j S_2$ and

$$I_j^{S_1}(f_{j-1}) \dots (f_1)(a) = f_j(a), I_j^{S_2}(g_{j-1}) \dots (g_1)(a) = g_j(a) .$$

This implies by (3) that $I_j^{S_1}(f_{j-1}) \dots (f_1) r_1\ I_j^{S_2}(g_{j-1}) \dots (g_1)$. Hence, by (2), it is easy to show by induction with respect to $k = 1, \dots, j-1$ that

$$I_j^{S_1}(f_{j-1}) \dots (f_k) r_k I_j^{S_2}(g_{j-1}) \dots (g_k)$$

for any $f_k, g_k, \dots, f_{j-1}, g_{j-1}$ satisfying the assumption. Thus $I_j^{S_1}(f_{j-1}) r_{j-1} I_j^{S_2}(g_{j-1})$ for every pair f_{j-1}, g_{j-1} such that $f_{i-1} r_{j-1} g_{j-1}$. Hence $I_j^{S_1} r_j I_j^{S_2}$. ■

Remark. The proof of the lemma will not change if we weaken the assumption by supposing that instead of the sets $F_1 S_1, F_1 S_2, F_2 S_1, F_2 S_2, \dots$, etc. we are given arbitrary sets $G_1 S_1, G_1 S_2, G_2 S_1, G_2 S_2, \dots$, etc. contained, with the order preserved, in the preceding sets, closed under superposition and such that $I_i^S \in G_i S_j$ for $i = 1, 2, \dots, n$, $j = 1, 2$, and $f \in G_{i+1} S_j \wedge g \in G_i S_j$ implies $f(g) \in G_i S_j$.

Part 2 of the proof of Theorem 3.2; carried over in $Z_0^- + \Sigma_1$-Ind; proof of (1) and (2).

For $j = 1, \ldots, n$ we have the equivalences

$$t^{S_1} \subseteq_j t^{S_2} \Leftrightarrow \exists f_1, f_2 (f_1 = t^{S_1} \wedge f_2 = t^{S_2} \wedge f_1 \subseteq_j f_2) \Leftrightarrow$$
$$\Leftrightarrow \forall f_1, f_2 (f_1 = t^{S_1} \wedge f_2 = t^{S_2} \rightarrow f_1 \subseteq_j f_2).$$

Since the relations $\{(f, t, S): f \in FS \wedge t \in T^n \wedge S \subseteq \omega\}$, $\subseteq_1, \ldots, \subseteq_n$ are Δ_1-definable (cf. Def. 2.5), the classes $A_j = \{t \in T_j^n\colon t^{S_1} \subseteq_j t^{S_2}\}$ are Δ_1-definable.

By (i), (ii) and the definition of the relations $\subseteq_1, \ldots, \subseteq_n$ we obtain:

(a) A_j is closed under superposition,

(b) $I_j \in A_j$ for $j = 1, \ldots, n$,

(c) $s \in A_{j+1} \wedge t \in A_j \rightarrow s(t) \in A_j$ for $j = 1, \ldots, n-1$.

Hence, by Remark 2.4(b), $A_n = T_n^n$, $A_{n-1} = T_{n-1}^n, \ldots, A_1 = T_1^n$. Thus $\forall S_1, S_2 \subseteq \omega\ \forall t \in T_1^n$ (S_1 is an interval in $S_2 \rightarrow t^{S_1} \subseteq_1 t^{S_2}$), i.e. $\forall S \subseteq \omega\ \forall t \in \omega\ \forall a, b \in S\ t^{S \cap [a,b]} \subseteq_p t^S$. ■

In the sequel we shall need, loosely speaking, the following fact: if a term t_2 is "more complicated" than t_1, then $\forall a \in S\ t_1^S(a) < t_2^S(a)$. To obtain this we define a relation of majorization $\leqslant_1$ in $F_1 S$ and extend it to relations $\leqslant_i$ on some functionals belonging to $F_i S$, where $i = 2, 3, \ldots$ Then we show our result by induction for terms of type $n, n-1, \ldots, 1$.

3.4. Definition (Z_0^-). If f, g are partial, then $f(x) \leqslant g(y)$ means that $g(y)\downarrow$ implies $f(x)\downarrow \wedge f(x) \leqslant g(y)$. For every $S \subseteq \omega$ we define:

$$G_1 S = \{f \in F_1 S\colon f \text{ increasing} \wedge \forall x\, f(x)\downarrow \rightarrow x \in f(x)\},$$

$f \leqslant_1 g \Leftrightarrow f, g \in G_1 S \wedge \forall x \in S\ f(x) \leqslant g(x)$ and further by induction:

$$f \in G_{i+1} S \Leftrightarrow f \in F_{i+1} S \wedge \forall g \in G_i S\, (\forall x \leqslant \min S)\, g^{x+1} \leqslant_i f(g) \wedge$$
$$\wedge \forall g_1, g_2 (g_1 \leqslant_i g_2 \rightarrow f(g_1) \leqslant_i f(g_2)),$$

$$f \leqslant_{i+1} g \Leftrightarrow f, g \in G_{i+1} S \wedge \forall f_1, g_1 (f_1 \leqslant_i g_1 \rightarrow f(f_1) \leqslant_i g(g_1)) \quad \text{for } 1, 2, \ldots$$

Immediately from the definition it follows that

$$Z_0^- \vdash f \in G_{i+1} S \rightarrow f \leqslant_{i+1} f.$$

It is also obvious that the relation $\leqslant_1$ is transitive in Z_0^-. Assume that $\leqslant_i$ is transitive in Z_0^-. We shall prove the transitivity of $\leqslant_{i+1}$. Let f, g, h be such that $f \leqslant_{i+1} g$, $g \leqslant_{i+1} h$. Take f_1, g_1 such that $f_1 \leqslant_i g_1$. Hence $f_1 \in G_i S$ and $f_1 \leqslant_i f_1$. By the definition of $\leqslant_{i+1}$ we have $f(f_1) \leqslant_i g(f_1)$ and $g(f_1) \leqslant_i h(g_1)$, whence by the inductive assumption $f(f_1) \leqslant_i h(g_1)$. This shows that $f \leqslant_{i+1} h$. Hence we have shown that for each i, $Z_0^- \vdash$ "$\leqslant_i$ is transitive".

3.5. Lemma (Z_0^-) (properties of the majorization). *For every S and the relations $\leqslant_i$ corresponding to it*:

(1) $f_1 \leqslant_{i+1} g_1 \wedge f_2 \leqslant_i g_2 \rightarrow f_1(f_2) \leqslant_i g_1(g_2)$,

(2) $f \in G_{i+1} S \wedge g \in G_i S \rightarrow g^{x+1} \leqslant_i f(g)$ *for each* $x \leqslant \min S$,

(3) $g \in G_i S \rightarrow \forall a \in \omega\, (\forall x)_{\leqslant a}\, g^{x+1} \leqslant_i g^{a+1}$,

(4) $f_1 \leqslant_i g_1 \wedge f_2 \leqslant_i g_2 \rightarrow f_1 \circ f_2 \leqslant_i g_1 \circ g_2$,

(5) $I_i^S \leqslant_i I_i^S$,

(6) $Z_0^- + \Sigma_1$-Ind $\vdash \forall t \in T_1^n\ \forall S \subseteq \omega$ "*t^S is increasing*".

Proof. (1) and (2) are explicitly contained in the definition of $\leqslant_{i+1}$. Since for every function $g \in G_1 S$ and $x \leqslant a$ we have $g^x \leqslant_1 g^{a+1}$, in the proof of (3) we can assume that $i \geqslant 2$.

Let $g \in G_i S$, $x \leqslant a$. Take f_1, f_2 such that $f_1 \leqslant_{i-1} f_2$. Thus by (1) and the reflexivity of $\leqslant_i$ we have $g^{x+1}(f_1) \leqslant_{i-1} g^{x+1}(f_2)$. By (2), $\forall f \in G_{i-1} S\ f \leqslant_{i-1} g(f)$. Since $g(f_2) \in G_{i-1} S$ it follows by Δ_0-induction that

$$g(f_2) \leqslant_{i-1} g^2(f_2) \leqslant_{i-1} \ldots \leqslant_{i-1} g^{a+1}(f_2).$$

By the transitivity of $\leqslant_{i-1}$, $g^{x+1}(f_2) \leqslant_{i-1} g^{a+1}(f_2)$. Hence $g^{x+1}(f_1) \leqslant_{i-1} g^{a+1}(f_2)$, and we conclude that $g^{x+1} \leqslant_i g^{a+1}$.

To prove (4) and (5) we first prove in Z_0^- that $G_{n+1} S$ is closed under superposition and that $I_{n+1}^S \in G_{n+1} S$ where n is an arbitrary natural number.

Assume that $f_1, f_2 \in G_{n+1} S$. Let $g \in G_n S$, $x \leqslant \min S$. To prove that $f_1 \circ f_2 \in G_{n+1} S$ it is enough to verify that $g^{x+1} \leqslant_n (f_1 \circ f_2)(g)$. We have $g^{x+1} \leqslant_n f_2(g)$ and $f_2(g) \in G_n S$. Similarly, $f_2(g) \leqslant_n f_1(f_2(g))$. Hence $g^{x+1} \leqslant_n (f_1 \circ f_2)(g)$.

To prove that $I_{n+1}^S \in G_{n+1} S$ it suffices to show that $g^{\min S+1} \leqslant_n I_{n+1}^S(g)$ for $g \in G_n S$ and that $g_1 \leqslant_n g_2$ implies $I_{n+1}^S(g_1) \leqslant_n I_{n+1}^S(g_2)$.

To see this let $g \in G_n S$ and $g_1' \leqslant_n g_2'$ and take a sequence of pairs of functions such that $f_1 \leqslant_1 g_1, \ldots, f_{n-1} \leqslant_{n-1} g_{n-1}$. By (3), $g^{\min S+1} \leqslant_n g^{a+1}$ for $a \in S$. Thus we have $\forall a \in S\ g^{\min S+1}(f_{n-1}) \ldots (f_1) \leqslant_1 g^{a+1}(g_{n-1}) \ldots (g_1)$. Consequently, for every $a \in S$,

$$g^{\min S+1}(f_{n-1}) \ldots (f_1)(a) \leqslant I_{n+1}^S(g)(g_{n-1}) \ldots (g_1)(a).$$

Similarly we obtain

$$I_{n+1}^S(g_1')(f_{n-1}) \ldots (f_1)(a) \leqslant I_{n+1}^S(g_1')(g_{n-1}) \ldots (g_1)(a).$$

Using (3) we verify, that the function $I_{n+1}^S(g)(g_{n-1}) \ldots (g_1)(a)$ is increasing and similarly for g_1', g_2' instead of g. We thus obtain $g^{\min S+1} \leqslant_n I_{n+1}^S(g)$ and $I_{n+1}^S(g_1) \leqslant_n I_{n+1}^S(g_2')$.

Moreover, it is easy to verify that the relation $\leqslant_1$ satisfies all the assumptions of Lemma 3.3. Thus the sets $G_1 S, \ldots, G_i S, \ldots$ and the relations $\leqslant_1, \ldots, \leqslant_i, \ldots$ satisfy all the assumptions of Lemma 3.3 including the remark. Hence we have shown (4) and (5).

(6) follows from (4) and (5). ■

Now we are ready to show what we have announced, namely that functions defined by more complicated terms are growing faster.

3.6. COROLLARY (Z_0^-). *Assume that* $3 \leqslant n \in \omega$ *and* $a \in \omega$.

(1) *For every* $S \subseteq \omega$ *such that* $\min S \geqslant 1$

$$I_n^a(I_{n-1}) \dots (I_2)(I_2^2(I_1))^S_i \leqslant_1 I_n^{a+1}(I_{n-1}) \dots (I_2)(I_1)^S.$$

(2) *For every* $S \subseteq \omega$ *and also for* $n = 1, 2$

$$I_n^a(I_{n-1}) \dots (I_1)^S + 1 \leqslant_1 I_n^{a+1}(I_{n-1}) \dots (I_1)^S.$$

Proof. We only prove (1). Assume that $\min S \geqslant 1$. Let $2 \leqslant i < n$, $f \in G_{i+1} S$. By 3.5(3)

$$[(f(I_i)(I_{i-1}))^2]^S \leqslant_{i-1} [(f(I_i)(I_{i-1}))^{\min S+1}]^S.$$

By 3.5(2 and 5) the last functional is $\leqslant_{i-1} I_i(f(I_i)(I_{i-1}))$. By 3.5 (2 and 1),

$$I_i(f(I_i)(I_{i-1}))^S \leqslant_{i-1} f(I_i)(f(I_i)(I_{i-1}))^S.$$

Hence by the transitivity of $\leqslant_{i-1}$,

$$[(f(I_i(I_{i-1}))^2]^S \leqslant_{i-1} (f(I_i))^2(I_{i-1})^S.$$

Since

$$[(I_n^m(I_{n-1}))]^S \leqslant_{n-1} I_n^{m+1}(I_{n-1})^S,$$

we infer from the above that

$$(I_n^m(I_{n-1}) \dots (I_2))^2(I_1)^S \leqslant_1 I_n^{m+1}(I_{n-1}) \dots (I_1)^S.$$

Denote $I_n^{m+1}(I_{n-1}) \dots (I_2)^S$ by g.

We have $I_2^2(I_1)^S \leqslant_1 I_3(I_2)(I_1)^S \leqslant_1 g(I_1)^S$. Thus $g(I_2^2(I_1)^S) \leqslant_1 g^2(I_1^S)$ and this completes the proof. ■

Now we proceed to the proof of the announced simple combinatorial result. In the theory Z_0^- we define for $S \subseteq \omega$

$$g(a) = \min_{b \in S} 2^a \leqslant b \quad \text{for } a \in S, \; S' = \{g^n(\min S) \colon n \in \omega\}.$$

Thus $S' \subseteq S$ and $\forall a, b \in S'$ $(a < b \to 2^a \leqslant b)$. Of course, $\min S = \min S'$. In some sense, the set S' can be regarded as rare.

First, we define $f_1 \leqslant_1 f_2 \Leftrightarrow f_1 \in F_1 S' \wedge f_2 \in F_1 S \wedge f_1, f_2$ are increasing $\wedge$ $\forall a \in S' (f_1(a) \leqslant f_2(a))$.

3.7. Remark. If $f_1 \leqslant_1 f_2$, then $f_2(\min S) \downarrow$ implies $f_1(\min S) \downarrow$.

We verify that $I_1^{S'} \leqslant_1 I_2^2(I_1)^S$. Let $a \in S'$. For every $b \in S$, if $I_2(I_1)^S(b)\downarrow$, then $I_2(I_1)^S(b) = (I_1^S)^{b+1}(b) \geqslant 2b+1 > 2b$. Thus, if $I_2^2(I_1)^S(a)\downarrow$, then $I_2^2(I_1)^S(a) = (I_2(I_1)^S)^{a+1}(a) > 2^{a+1} \cdot a$, i.e., $I_2^2(I_1)^S(a) \geqslant 2^a$. Therefore $I_2^2(I_1)^S(a) \geqslant g(a)$ whence $I_1^{S'}(a) \leqslant I^2(I_1)(a)$. ■

Next, we define

$$f_1 \leqslant_{i+1} f_2 \Leftrightarrow f_1 \in F_{i+1} S' \wedge f_2 \in F_{i+1} S \wedge \forall g_1, g_2 \; (g_1 \leqslant_i g_2 \to f_1(g_1) \leqslant_i f_2(g_2))$$

for $i = 1, 2, \dots$ It is easy to verify that the relations $\leqslant_1, \leqslant_2, \dots$ satisfy the assumptions of Lemma 3.3.

Hence $f_1 \leqslant_i f_2 \wedge g_1 \leqslant_i g_2$ implies that $f_1 \circ f_2 \leqslant_i g_1 \circ g_2$ and $I_i^{S'} \leqslant_i I_i^S$ for $i = 1, 2, \dots$

Since $I_1^{S'} \leqslant_1 I_2^2(I_1)^S$, we infer from the above that for $n \geqslant 1$

$$I_{n+1}^m(I_n) \dots (I_1)^{S'} \leqslant_1 I_{n+1}^m(I_n) \dots (I_2)(I_2^2(I_1))^S.$$

This, together with 3.6(1) and 3.7, implies the announced lemma:

3.8. LEMMA (Z_0^-). *Let* $n \geqslant 1$. *For every* $S \subseteq \omega$, *if* $\min S \geqslant 1$, *then*

$$I_{n+1}^{m+2}(I_n) \dots (I_1)^S(\min S)\downarrow \; \to I_{n+1}^m(I_n) \dots (I_1)^{S'}(\min S')\downarrow. \; ■$$

This result says there exists a sufficiently large and rare set $S' \subseteq S$ if S is sufficiently large.

Assume that t is a term of type 1. By 3.1, $\bar{t} = \{(a, b) \colon t^{[a,b]}(a) = b\}$. Since the image of the formula $t^{[n,b]}(a) = b$ under the interpretation J is of class Σ_1, $\bar{t}$ is a recursive function. The following lemma shows that for $t \in T_1^{n+1}$, $n > 0$, the function $\bar{t}$ is provably recursive in $I\Sigma_n$. It also shows that $(t^{[a,b]}(a) = b)^J$ is a formalization of $\bar{t}$.

3.9. LEMMA. *Let* $n > 0$. *For every* $t \in T_1^{n+1}$

$$(*) \qquad I\Sigma_n \vdash \forall a \exists b \; t^{[a,b]}(a) = b.$$

Proof. Let $A_1(t) \colon = t \in T_1^{n+1} \wedge \forall a \exists b \; t^{[a,b]}(a) = b$. By 2.5 the formula $A_1(t)$ is of class Π_2 in $I\Sigma_n$. We define

$$A_{i+1}(s) = \forall t \left(A_i(t) \to A_i(s(t))\right) \wedge s \in T_{i+1}^{n+1} \quad \text{for } i = 1, \dots, n-1.$$

Let

$$B_i = \{t \in T_i^{n+1} \colon I\Sigma_n \vdash A_i(t)\} \quad \text{for } i = 1, \dots, n,$$

$$B_{n+1} = \{s \in T_{n+1}^{n+1} \colon \forall t \in B_n \, s(t) \in B_n\}.$$

The lemma is of course equivalent to the equality $B_1 = T_1^{n+1}$. To prove this equality it suffices to show, in view of the definition of T_i^{n+1}, that

(1) $s \in B_{i+1} \wedge t \in B_i \to s(t) \in B_i$ for $i = 1, \dots, n$,

(2) $s, t \in B_i \to (s \circ t) \in B_i$ for $i = 1, \dots, n+1$,

(3) $I_i \in B_i$ for $i = 1, \dots, n+1$.

If $i < n$ and $s \in B_{i+1}$, $t \in B_i$, then $I\Sigma_n \vdash A_{i-1}(s) \wedge A_i(t)$. Thus $I\Sigma_n \vdash A_i(s(t))$, i.e. $s(t) \in B_i$. If $i = n$, then (1) results directly from the definition of B_{n+1}.

(2) Case (a). $i = 1$. Assume that $s, t \in B_1$. We carry out the proof in $I\Sigma_n$. Let a be an arbitrary natural number, b_1 a number such that $t^{[a,b_1]}(a) = b_1$, and b_2 a number such that $s^{[b_1,b_2]}(b_1) = b_2$. By 3.3(1), $t^{[a,b_2]}(a) = b_1$ and $s^{[a,b_2]}(b_1) = b_2$. Hence $(s \circ t)^{[a,b_2]}(a) = b_2$. Thus $(s \circ t) \in B_1$.

Case (b). Let $1 < i \leqslant n$ and assume that $s_1, s_2 \in B_i$, i.e. $I\Sigma_n \vdash A_i(s_1) \wedge A_i(s_2)$. It is easy to show that $I\Sigma_n \vdash \forall t \left[A_{i-1}(t) \to A_{i-1}\left(s_1(s_2(t))\right)\right]$. If $i = n+1$, $s_1, s_2 \in B_{n+1}$,

$t \in B_n$, then $s_1(s_2(t)) \in B_n$, i.e. $I\Sigma_n \vdash A_n(s_1(s_2(t)))$. It suffices to show that in the above formulas the term $s_1(s_2(t))$ can be replaced by $(s_1 \circ s_2)(t)$, i.e. it suffices to show that

$$I\Sigma_n \vdash A_i(s_1(s_2(t))) \to A_i((s_1 \circ s_2)(t)) \quad \text{for } i = 1, \ldots, n.$$

We carry out the proof in $I\Sigma_n$. Let $A_i(s_1(s_2(t)))$. Assume that $t_{i+1}, \ldots, t_1$ are terms such that $A_{i-1}(t_{i-1}), \ldots, A_1(t_1)$. Therefore $A_1(s_1(s_2(t)))(t_{i-1}) \ldots (t_1))$. Take any a. Hence there exists a b such that $s_1(s_2(t))(t_{i-1}) \ldots (t_1)^{[a,b]}(a) = b$. Since $s_1(s_2(t))^{[a,b]} \simeq (s_1 \circ s_2)(t)^{[a,b]}$, we have $A_1((s_1 \circ s_2)(t)(t_{i-1}) \ldots (t_1))$, which proves that $A_i((s_1 \circ s_2)(t))$.

(3) It suffices to prove that

$$I\Sigma_n \vdash A_i(t_i) \wedge \ldots \wedge A_1(t_1) \to A_1(I_{i+1}(t_i) \ldots (t_1))$$

for $i = 1, \ldots, n$. We first prove that $I\Sigma_n \vdash A_i(s) \to \forall c\, A_i(s^{c+1})$.

Case (α): $i = 1$. We carry out the proof in $I\Sigma_1$.

Let $s \in T_1^{n+1}$ be a term such that $A_1(s)$. The formula $\forall c\ A_1(s^{c+1})$ is equivalent to $\forall a\ \forall c\ \exists b\ (s^{c+1})^{[a,b]}(a) = b$. Fix a. From the proof of (2) we obtain $A_1(s) \wedge \exists b\, (s^{c+1})^{[a,b]}(a) = b \to \exists b\, (s^{c+2})^{[a,b]}(a) = b$, and the proof is completed by Σ_1-induction.

Case (β): $1 < i \leqslant n$. Observe that A_{i-1} is of class Π_n in $I\Sigma_n$. We carry out the proof in $I\Sigma_n$.

Assume that $A_i(s)$. To prove $A_i(s^{c+1})$ it suffices to show that $\forall t\, \big(A_{i-1}(t) \to A_{i-1}(s^{c+1}(t))\big)$.

Take a t such that $A_{i-1}(t)$. The initial step and the inductive step in proving $\forall c\, A_{i-1}(s^{c+1}(t))$ result from the implication $A_{i-1}(t) \to A_{i-1}(s(t_1))$. Thus, by Π_n-induction, $\forall c\ A_{i-1}(s^{c+1}(t))$. This completes the proof because Π_n-induction follows from Σ_n-induction.

Completion of the proof of (3). We carry out the proof in $I\Sigma_n$. Assume that $A_i(t_i) \wedge \ldots \wedge A_1(t_1)$. Take any a. Thus $A_i(t_i^{a+1})$, and so $A_1(t_i^{a+1}(t_{i-1}) \ldots (t_1))$. Consequently, $\exists b\ t_i^{a+1}(t_{i-1}) \ldots (t_1)^{[a,b]}(a) = b$, which proves that $A_1(I_{i+1}(t_i) \ldots (t_1))$. ■

§ 4. Combinatorial properties and models of arithmetic. One of main aims of this section is to prove a theorem stating that a formal equivalent $Y_n(a, b)$ of the function

$$\overline{Y}_n(a, b) = \max_m I_{n+1}^m(I_n) \ldots (I_1)(a) \leqslant b$$

is in models for $I\Sigma_n$ an indicator for segments satisfyng $I\Sigma_n$, where $n>0$ (Theorem 4.16).

To do this we define a combinatorial property $\mathrm{FCP}_t^j(S)$ (Definition 4.11) basing on a property of approximation (Definition 4.1). We use a combinatorial property $t \xrightarrow{f} (t_1, t_2)$ as a tool in the study of the properties of $\mathrm{FCP}_t^j(S)$ (Definition 4.4).

Basing on the central Lemma 4.9 about the property $t \xrightarrow{f} (t_1, t_2)$ we derive a connection between the function $Y_n(a, b)$ and the property $\mathrm{FCP}_m^n([a, b])$ (Lemma 4.13).

Since the condition: $M \models \mathrm{FCP}_c^n([a, b])$ and $c > \omega$ is sufficient for the existence of an initial cut $I \subset_e M$ such that $I \models I\Sigma_n$ (Lemma 4.15), 4.16 immediately follows, as we show, from 4.13 and 4.9.

4.1. Definition ($I\Delta_0$+exp). The pair (S_1, S_2) is an *approximation* to f if

(1) S_1, S_2 are codable restricted subsets of ω ($S_1, S_2 \subseteq^J \omega^J$, i.e. $S_1, S_2 \in \mathrm{Fin}$),

(2) f is a codable function with a bounded domain included in ω. We write briefly $f \in \omega^{<\omega}$,

(3) $\max S_1 = \min S_2$ and for every $a \in S_1 - \{\max S_1\}$ we have

$$\forall x < a-1 [f(x)\downarrow \to (f(x) < I^{S_1}(a) \vee f(x) \geqslant \max S_2)].$$

S is an *approximation* to f if, as above, S, f satisfy (1), (2) and

$$\forall a \in S\ \forall x < a-1\ \ [f(x)\downarrow \to (f(x) < I^S(a) \vee f(x) \geqslant \max S)].$$

4.2. Fact ($I\Delta_0$+exp). If the pairs (S_1, S_2), (S_3, S_4) are approximations for $f \in \omega^{<\omega}$ and $S_3 \cup S_4 \subseteq S_2$, then the pair $(S_1 \cup S_3, S_4)$ is also an approximation for f.

4.3. Definition ($I\Sigma_1$). Let $t \in T_1^n$, $n > 0$. We say that a finite set S of natural numbers is *t-large* if $t^S(\min S)\downarrow$ (this notion corresponds to the notion of α-large set in [2]).

The pair (S_1, S_2) is *(t_1, t_2)-large* iff S is t_1-large, S_2 is t_2-large and $\max S_1 = \min S_2$.

4.4. Definition ($I\Sigma_1$). For $t, t_1, t_2 \in T_1^n$ the symbol $t \xrightarrow{f} (t_1, t_2)$ denotes that for every t-large set S there exists a (t_1, t_2)-large pair (S_1, S_2) which is an approximation to f such that $S_1 \cup S_2 \subseteq S$, $\min S_1 = \min S$.

The definition directly implies the following fact.

4.5. Fact ($I\Sigma_1$). If $t \xrightarrow{f} (t_1, t_2)$ and

$$\forall S \in \mathrm{Fin}\ [t^S = t'^S \wedge t_1^S = t_1'^S \wedge t_2^S = t_2'^S],$$

then $t' \xrightarrow{f} (t_1', t_2')$.

In Lemmas 4.6–4.9, we present results related to the property $t \xrightarrow{f} (t_1, t_2)$.

4.6. Lemma ($I\Sigma_1$). *If* $t, t_1, t_2, t_3, t_4 \in T_1^n$, $n > 0$, *and also* $t \xrightarrow{f} (t_1, t_2)$ *and* $t_2 \xrightarrow{f} (t_3, t_4)$, *then* $t \xrightarrow{f} ((t_3 \circ t_1), t_4)$.

Proof. Assume that $t \xrightarrow{f} (t_1, t_2)$, $t_2 \xrightarrow{f} (t_3, t_4)$ and S is t-large. Consequently, there exists a (t_1, t_2)-large pair (S_1, S_2) which is an approximation to f such that $\min S_3 = \min S_2$ and $S_3 \cup S_4 \subseteq S_2$. By 4.2 the pair $(S_1 \cup S_3, S_4)$ is also an approximation to f. Moreover, $\min(S_1 \cup S_3) = \min S_1 = \min S$. Thus it remains to verify that $S_1 \cup S_3$ is $(t_3 \circ t_1)$-large. Write $a = \min S_1$, $b = \max S_1 = \min S_2$, $c = \max S_3$.

Since S_1 is t_1-large, i.e. $t_1^{S_1}(a)\downarrow$ and $(S_1 \cup S_3) \cap [a, b] = S_1$, by 3.2(1) we have $t_1^{S_1}(a) = t_1^{S_1 \cup S_3}(a) \leqslant \max S_1 = b$.

In view of $t_3^{S_3}(b)\downarrow$ and $(S_1\cup S_3)\cap[b,c]=S_3$ we have $t_3^{S_1\cup S_3}(b)\downarrow$. Since $t_1^{S_1}(a)\leqslant b$, we infer by 3.5(6) that $t_3^{S_1\cup S_3}(t_1^{S_1}(a))\downarrow$. Consequently, $t_3^{S_1\cup S_3}(t_1^{S_1\cup S_3}(a))\downarrow$, and so $S_1\cup S_3$ is $(t_3\circ t_1)$-large. ■

We wish to prove that if S is sufficiently large and $f\in\omega^{<\omega}$ then there exists a sufficiently large $S'\subseteq S$ which is an approximation to f. The first step is the following:

4.7. Lemma ($I\Sigma_1$). *For every $t\in T_1^n$, $n>0$, the following combinatorial property is true*:

$$\forall f\in\omega^{<\omega}\ I_2(t)\xrightarrow{f}(I_1,t).$$

Proof. Take an $I_2(t)$-large set S. Denote $\min S$ by a_0. Thus $(t^S)^{a_0+1}(a_0)\downarrow$. Let $a_j=(t^S)^j(a_0)$ for $j=1,\ldots,a_0+1$. Hence $t^S(a_j)=a_{j+1}$ for $j=0,1,\ldots,a_0$, which implies by 3.5(6) that $a_0<a_1\ldots<a_{a_0+1}$. The function f assumes for $x<a_0-1$ at most a_0-1 values, and so by the pigeon-hole principle there exists a j_0, $1\leqslant j_0\leqslant a_0$, such that there is no value of f in $[a_{j_0},a_{j_0+1})$.

Let $S_1=\{a_0,a_{j_0}\}$, $S_2=[a_{j_0},a_{j_0+1}]\cap S$. Thus $I_1^{S_1}(a_0)\downarrow$ and $t^{S_2}(a_{j_0})\downarrow$ by 3.2(1) in view of $t^S(a_{j_0})=a_{j_0+1}$. The pair (S_1,S_2) is thus (I_1,t)-large, $\min S=\min S_1$ and, for every $x<a_0-1$, $f(x)<a_{j_0}$ or $f(x)\geqslant a_{j_0+1}=\max S_2$. Consequently, (S_1,S_2) is an approximation to f. ■

To make the second step we need a mapping h from $T_2^n\cup\ldots\cup T_n^n$ into $T_1^n\cup\ldots\cup T_{n-1}^n$ defined as follows, where t^* denote $h(t)$:

4.8. Definition. The term t^* is defined by the following inductive conditions:

(1) $I_{i+1}^*=I_i$ for $i=1,\ldots,n-1$,

(2) $(t_1\circ t_2)^*=(t_2^*\circ t_1^*)$ for $t_1,t_2\in T_2^n$,

(3) $(t_1\circ t_2)^*=(t_1^*\circ t_2^*)$ for $t_1,t_2\in T_i^n$ where $2<i\leqslant n$,

(4) $t(s)^*=t^*(s^*)$ for $s\in T_i^n$, $t\in T_{i+1}^n$, $2\leqslant i<n$.

Our main and final result concerning the property $t\xrightarrow{f}(t_1,t_2)$ is the following:

4.9. Lemma ($I\Sigma_n$, where $n>0$).

$$\forall t\in T_2^{n+1}\ \forall t_1\in T_1^{n+1}\ \forall f\in\omega^{<\omega}t\ (t_1)\xrightarrow{f}(t^*,t_1).$$

Proof. We define:

$$A_2(t)\Leftrightarrow t\in T_2^{n+1}\wedge\forall t_1\in T_1^{n+1}\ \forall f\in\omega^{<\omega}t\ (t_1)\xrightarrow{f}(t^*,t_1),$$
$$A_{i+1}(s)\Leftrightarrow\forall t\,\big(A_i(t)\to A_i(s(t))\big)\quad\text{for } i=2,\ldots,n.$$

It is easy to observe that A_{i+1} is a formula of class Π_i for $i=1,\ldots,n$. Thus all the formulas under consideration are of class Π_n.

Claim 1. $A_2(s)\wedge A_2(t)\to A_2((s\circ t))$.

Let $A_2(s)$, $A_2(t)$. Take an arbitrary $t_1\in T_1^{n+1}$ and an arbitrary $f\in\omega^{<\omega}$. Therefore $t(t_1)\xrightarrow{f}(t^*,t_1)$, and since $t(t_1)\in T_1^{n+1}$, we also have $s(t(t_1))\xrightarrow{f}(s^*,t(t_1))$. By 4.6 and the last two corollaries we have $s(t(t_1))\xrightarrow{f}((t^*\circ s^*),t_1)$. Since

$$\forall S\ (S \text{ is } s(t(t_1))\text{-large}\Leftrightarrow S \text{ is } (s\circ t)(t_1)\text{-large})$$

and $(s\circ t)^*=(t^*\circ s^*)$, we have $(s\circ t)(t_1)\xrightarrow{f}((s\circ t)^*,t_1)$. Hence $A_2((s\circ t))$.

Claim 2. $A_{i+1}(s)\wedge A_{i+1}(t)\to A_{i+1}((s\circ t))$ *for* $i=2,\ldots,n$.

Take an arbitrary t_i such that $A_i(t_i)$. Then $A_i(t(t_i))$ and for arbitrary $t_{i-1},\ldots,t_1$ such that $A_{i-1}(t_{i-1}),\ldots,A_2(t_2)$, $t_1\in T_1^{n+1}$ and an arbitrary $f\in\omega^{<\omega}$ we have

$$s(t(t_i))(t_{i-1})\ldots(t_2)(t_1)\xrightarrow{f}\big(s(t(t_i))(t_{i-1})\ldots(t_2)^*,t_1\big).$$

By 4.5 and the definition of $*$ we obtain

$$(s\circ t)(t_i)\ldots(t_2)(t_1)\xrightarrow{f}((s\circ t)(t_i)\ldots(t_2)^*,t_1),$$

which proves that $A_{i+1}((s\circ t))$.

Claim 3. $A_{i+1}(s)\wedge A_i(t)\to A_i(s(t))$ *for* $i=2,3,\ldots,n$.

This results immediately from the definition of the formulas A_i.

Claim 4. $A_{i+1}(I_{i+1})$ *for* $i=1,\ldots,n$.

For $i=1$, Claim 4 coincides with 4.7. Thus we can assume that $i\geqslant 2$. Take arbitrary $t_i,t_{i-1},\ldots,t_1$ such that $A_i(t_i),\ldots,A_2(t_2)$, $t_1\in T_1^{n+1}$, $f\in\omega^{<\omega}$, and an arbitrary S which is $I_{i+1}(t_i)\ldots(t_2)(t_1)$-large. Let $a=\min S$. By Claims 1 and 2 we deduce from $A_i(t_i)$, by Π_n-induction, that $A_i(t_i^{a+1})$. Hence

$$t_i^{a+1}(t_{i-1})\ldots(t_2)(t_1)\xrightarrow{f}(t_i^{a+1}(t_{i-1})\ldots(t_2)^*,t_1).$$

Since $I_{i+1}(t_i)\ldots(t_2)(t_1)(a)\simeq t_i^{a+1}(t_{i-1})\ldots(t_2)(t_1)(a)$ the set S is

$$t_i^{a+1}(t_{i-1})\ldots(t_2)(t_1)\text{-large}.$$

Thus there exists a $(t_i^{*a+1}(t_{i-1}^*)\ldots(t_2^*),t_1)$-large pair (S_1,S_2) which is an approximation to f and $\min S_1=\min S=a$ and $S_1\cup S_2\subseteq S$. Since $\min S_1=a$, the set S_1 is $I_{i-1}(t_i^*)(t_{i-1}^*)\ldots(t_2^*)$-large, i.e. $I_i(t_i)\ldots(t_2)^*$-large. We have shown that for arbitrary $t_i,t_{i-1},\ldots,t_2,t_1,f$ as above we have $I_{i-1}(t_i)\ldots(t_2)(t_1)\xrightarrow{f}(I_{i+1}(t_i)\ldots(t_2)^*,t_1)$. Hence $A_{i+1}(I_{i+1})$.

Using Claims 1–4 we now prove by Π_n-induction that $\forall t\in T_{n+1}^{n+1}\ t\in A_{n+1}$, which implies by Π_n-induction that $\forall t\in T_n^{n+1}\ t\in A_n$, etc. In the n-th step we show that $\forall t\in T_2^{n+1}\ t\in A_2$, and this completes the proof. ■

By 3.6(2) $I\Delta_0+\exp \vdash I^c_{n+1}\ (I_n)\dots(I_1)^{[a,b]}(a)\uparrow \to I^{c+1}_{n+1}(I_n)\dots(I_1)^{[a,b]}(a)\uparrow$. Thus the formula

$$Y(a,b,c)\colon\ I_{n+1}(I_n)\dots(I_1)^{[a,b]}(a)\uparrow\ \wedge c=0\ \vee$$
$$I^c_{n+1}(I_n)\dots(I_1)^{[a,b]}(a)\downarrow\ \wedge I^{c+1}_{n+1}(I_n)\dots(I_1)^{[a,b]}(a)\uparrow$$

defines in $I\Delta_0+\exp$ a certain mapping, denoted in the sequel by $Y_n(a,b)$, which by Theorem 3.1 can be regarded as a formal counterpart of the function $\overline{Y}_n(a,b)$, recalled at the beginning of this section. Before we prove that $Y_n(a,b)$ is an indicator (Theorem 4.16) let us first check that it has some of the required properties.

4.10. Lemma. *Let* $n>0$.

(1) $I\Delta_0+\exp\vdash b_1\leqslant b\to Y_n(a,b_1)\leqslant Y_n(a,b)$,

(2) $I\Sigma_n\vdash\forall a\,\exists b\ Y_n(a,b)\geqslant \underline{m}$ *for all* $m\in\omega$,

(3) *If* $I\subset_e M$ *and* I, M *are models for* $I\Sigma_n$, *then*

$$\forall a,b\in M\ \big(a\in I<b\to Y_n^M(a,b)\big)>\omega\big).$$

Proof. (1) We work in $I\Delta_0+\exp$. Let $b_1\leqslant b$ and $c=Y_n(a,b_1)>0$. Thus $I^c_{n+1}(I_n)\dots(I_1)^{[a,b_1]}(a)\downarrow$. By 3.2(1) we also have $I^c_{n+1}(I_n)\dots(I_1)^{[a,b]}(a)\downarrow$. Hence $Y_n(a,b)\geqslant c$.

(2) For $m\in\omega$ let t_m denote the term $I^m_{n+1}(I_n)\dots(I_1)$. Fix $m>0$. By Lemma 3.9, $I\Sigma_n\vdash\forall a\,\exists b\ t_m^{[a,b]}(a)=b$. Thus it suffices to observe that $I\Sigma_n\vdash t_m^{[a,b]}(a)=b\to$ $\to Y_n(a,b)\geqslant\underline{m}$.

(3) Assume that $M\vDash I\Sigma_n$, $I\subset_e M$, $I\vDash I\Sigma_n$. Take arbitrary a,b such that $a\in I<b$. By (2) $\forall m\in\omega\ \exists b_1\in I\,Y_n^I\ (a,b_1)\geqslant m$. Since the formula $Y_n(x,y)=z$ is of class $\Delta_0(\exp)$, it is absolute and hence $\forall m\in\omega\ \exists b_1\leqslant b\ \ Y_n^M(a,b_1)\geqslant m$. By (1) we thus have $Y_n^M(a,b)>\omega$. ■

Below we give a definition of certain weaker variants of the combinatorial property FCP (finite combinatorial principle) studied by Smoryński in [5] whose independence from PA was discovered by Paris and Pudlák. The variants considered here are applicable in constructing segments which are models for $I\Sigma_n$. Assume that $t\in T_1^{n+1}$ where $n\geqslant 1$.

4.11. Definition ($I\Delta_0+\exp$). The symbol $\mathrm{FCP}^0_t(S)$ denotes that S is t-large and $\forall c,d\in S\ (c<d\to 2^c<d)$. By $\mathrm{FCP}^j_t(S)$ where $j\geqslant 1$ we denote the following property of the finite set S:

$$\forall f_1\in\omega^{<\omega}\ \exists S_1\subseteq S\dots\forall f_{i+1}\in\omega^{<\omega}\exists S_{i+1}\subseteq S_i\dots\forall f_j\ \exists S_j\subseteq S_{j-1}$$

$\big[$"$\bigwedge_{k=1}^{j} S_k$ is an approximation to f_k" $\wedge$ "S_j is t-large" $\wedge\ \forall c,d\in S_j(c<d\to 2^c<d)\big]$.

For $t=I_1^m$, S_j is t-large $\Leftrightarrow |S_j|>m$.

Instead of $\mathrm{FCP}^j_t(S)$ we write in this case $\mathrm{FCP}^j_m(S)$. The definition of $\mathrm{FCP}^j_t(S)$ and Lemma 4.9 immediately imply that the following implication is provable for every $j>0$.

4.12. Lemma ($I\Sigma_n$). $\forall t\in T_2^{n+1}(\mathrm{FCP}^j_{t(I_1)}(S)\to\mathrm{FCP}^{j+1}_{t^*}(S))$. ■

We have the following connection between $Y_n(a,b)$ and $\mathrm{FCP}^n_m([a,b])$.

4.13. Lemma. *For arbitrary* $1\leqslant m,n\in\omega$

$$I\Sigma_n\vdash\forall a\geqslant 2\forall b[Y_n(a,b)\geqslant m+2\to\mathrm{FCP}^n_m([a,b])].$$

Proof. We work in $I\Sigma_n$. Let $Y_n(a,b)\geqslant\underline{m}+2$, $a\geqslant 2$. Thus

$$I^{m+2}_{n+1}(I_n)\dots(I_1)^{[a,b]}(a)\downarrow.$$

For $0\leqslant j\leqslant n$, $1\leqslant k\leqslant m+2$, let t^k_j denote the term $I^k_{j+1}(I_j)\dots(I_1)$; we can thus write that $[a,b]$ is t^{m+2}_{n+1}-large. By Lemma 3.8 there exists a set $S\subseteq[a,b]$ such that $\forall c,d\in S[c<d\to 2^c<d]$ and $t^m_{n+1}(\min S)\downarrow$. Hence S has the combinatorial property $\mathrm{FCP}^0_{t^m_{n+1}}(S)$. Let $1\leqslant j\leqslant n$. Since there exists a $t\in T_2^n$ such that $t^m_{j+1}=t\ (I_1)$ where $t^*=t^m_j$, we infer by 4.12 that $\forall S'[\mathrm{FCP}^{n-j}_{t^m_{j+1}}(S')\to\mathrm{FCP}^{n-(j-1)}_{t^m_j}(S')]$. And so, on account of the initial condition $\mathrm{FCP}^0_{t^m_{n+1}}(S)$, we obtain $\mathrm{FCP}^n_{t^m_1}(S)$, i.e. $\mathrm{FCP}^n_m(S)$. It follows that also $\mathrm{FCP}^n_m([a,b])$. ■

4.14. Remark. Essentially, we have proved that for arbitrary $1\leqslant m,n\in\omega$

$$I\Sigma_n\vdash\forall S\in\mathrm{Fin}(I^{m+2}_{n+1}(I_n)\dots(I_1)^S(\min S)\downarrow\ \wedge\ \min S\geqslant 2\to\mathrm{FCP}^n_m(S)).$$

We now prove a key lemma on the existence of initial segments for $I\Sigma_n$.

4.15. Lemma. *Assume that* $M\vDash I\Delta_0+\exp$ *and that* M *is nonstandard.*

(1) *For arbitrary* $a,b\in M$ *such that for some* $c>\omega$, $M\vDash\mathrm{FCP}^n_c([a,b])$, *there exists an* $I\subset_e M$ *such that* $a\in I<b$ *and* $I\vDash I\Sigma_n$.

(2) *For every* $X\in\mathrm{Fin}^M$ *and every* $S\in\mathrm{Fin}^M$ *such that for some* $c>\omega$, $M\vDash\mathrm{FCP}^n_c(S)$ *there exists an* $I\subset_e M$ *such that* $S\cap I$ *is cofinal in* I *and* $(I,X\cap I)\vDash I\Sigma_n(R)$, *where* R *denotes the set* $X\cap I$, $\Sigma_n(R)$ *is the class of* Σ_n-*formulas in the language* $L_A\cup\{R\}$, *and* $I\Sigma_n(R)$ *denotes the induction scheme for such formulas plus the axioms* PA^-.

Proof. (1) is, of course, a particular case of (2). Observe first that, because of the existence in the theory $I\Sigma_n(R)$ of a universal formula for $\Pi_n(R)$-formulas, $I\Sigma_n(R)$ is equivalent to the theory PA^- plus the minimum principle for some $\Pi_n(R)$-formula $\chi(x,y)$:

$$\forall x[\exists y\ \chi(x,y)\to\exists y(\chi(x,y)\wedge\forall z<y\ \neg\chi(x,y))].$$

Assume now that M is a nonstandard model for $I\Sigma_n$. For every $i\in\omega$ denote by Π_i^* the class Π_i of formulas in the language L_A with parameters belonging to M. If $I\subset_e M$, then we define a relation $I\vDash^*\varphi$ on the set of sentences belonging to Π_i^* by $I\vDash^*\varphi\Leftrightarrow M\vDash\varphi$ for $\varphi\in\Delta_0^*=\Pi_0^*$ and by Tarski's conditions:

$$(I\vDash^*\forall x\varphi(x))\Leftrightarrow\forall a\in I\ I\vDash^*\varphi(a),\ I\vDash^*(\varphi\to\psi)\Leftrightarrow[(I\vDash^*\varphi)\to(I\vDash^*\psi)].$$

Let $c\in M$, $c>\omega$.

We shall prove by induction with respect to $i=0,\dots,n$ the following claim.

CLAIM. *For every $S \in \mathrm{Fin}^M$ such that $M \models \mathrm{FCP}^i_c(S)$ and every formula $\varphi(\bar{x}) \in \Pi^*_i$, there exists an initial segment $I \subset_e M$ such that*

(α) *$\forall a \in I\ 2^a \in I$, $S \cap I$ is cofinal in I,*

(β) *there exists a $\psi(\bar{x}) \in \Delta^*_0$ such that*

$$\forall \bar{a} \in I\ I \models^* \varphi(\bar{a}) \Leftrightarrow M \models \psi(\bar{a}) .$$

Assume that $i = 0$ and $M \models \mathrm{FCP}^0_c(S)$. Then $|S|^M \geqslant c > \omega$ and the initial segment determined by ω initial elements of S satisfies of course (α). Property (β) does not require a proof.

Assume now the validity of the inductive hypothesis for i and assume that we are given an $S \in \mathrm{Fin}^M$ such that $M \models \mathrm{FCP}^{i+1}_c(S)$ and $\varphi(\bar{x}) \in \Pi^*_{i+1}$. We have

$$\varphi(\bar{x}) \Leftrightarrow \forall y_1\ \exists y_2 \ldots Q_{i+1} y_{i+1}\ \varphi_0(\bar{x}, y_1, \ldots, y_{i+1}) ,$$

where φ_0 is of class Δ^*_0. Let k denote the length of the sequence of variables $\bar{x}$. We denote by $\bar{\varphi}_0$ the formula φ_0 or $\neg\varphi_0$ according as $Q_{i+1} = \exists$ or $Q_{i+1} = \forall$.

We define a function $f \in M$:

$$M \models \forall x \leqslant \max S\ f(x) = \min_{y_{i+1} \leqslant \max S} \bar{\varphi}_0((x)_1, \ldots, (x)_{k+i}, y_{i+1}) ,$$

where $(x)_i$ is the function decoding the polynomial code J for $(k+i)$-element sequences in M. Since $M \models \mathrm{FCP}^{i+1}_c(S)$, there exists an $S_1 \subseteq S$ such that $M \models \mathrm{FCP}^i_c(S_1)$ and $M \models$ "S_1 is an approximation to f".

Let $d = \max S_1$. We define a formula of class Π^*_i:

$$\varphi_1(\bar{x}) \Leftrightarrow \forall y_1\ \exists y_2 \ldots Q_i y_i Q_{i+1} y_{i+1} < d\ \varphi_0(\bar{x}, y_1, \ldots, y_i, y_{i+1}) .$$

Since $M \models \mathrm{FCP}^i_c(S_1)$, by the inductive assumption there exists an $I \subset_e M$ such that $\forall a \in I\ 2^a \in I$, $S_1 \cap I$ is cofinal in I and there exists a $\psi(\bar{x}) \in \Delta^*_0$ such that

$$\forall \bar{a} \in I\ I \models^* \varphi_1(\bar{a}) \Leftrightarrow M \models \psi(\bar{a}) .$$

Moreover, observe that $I < d$.

Thus it suffices to prove that

$$\forall \bar{a} \in I\ I \models^* \varphi(a) \Leftrightarrow I \models^* \varphi_1(a) ,$$

which follows from the fact that

$$\forall \bar{a}, \bar{b} \in I\ I \models^* Q_{i+1} y_{i+1}\ \varphi_0(\bar{a}, \bar{b}, y_{i+1}) \Leftrightarrow$$
$$\Leftrightarrow I \models^* Q_{i+1} y_{i+1} < d\ \varphi_0(\bar{a}, \bar{b}, y_{i+1}) .$$

To prove this fact, take arbitrary $\bar{a}, \bar{b} \in I$. Since I is closed under exponentiation, there exists a $c_1 \in S_1 \cap I$ such that $J(\bar{a}, \bar{b}) < c_1$. Let c_2 denote the immediate successor of c_1 in S_1.

Since S_1 is an approximation to f in M, we infer that

$$f(J(\bar{a}, \bar{b})) < c_2 \Leftrightarrow f(J(\bar{a}, \bar{b})) < \max S_1 = d .$$

Hence, by the definition of f,

$$M \models \exists y_{i+1} < c_2\ \bar{\varphi}_0(\bar{a}, \bar{b}, y_{i+1}) \Leftrightarrow M \models \exists y_{i+1} < d\ \bar{\varphi}_0(\bar{a}, \bar{b}, y_{i+1}) .$$

Since $c_2 \in I < d$, we also have

$$I \models^* \exists y_{i+1}\ \bar{\varphi}_0(\bar{a}, \bar{b}, y_{i+1}) \Leftrightarrow I \models^* \exists y_{i+1} < d\ \bar{\varphi}_0(\bar{a}, \bar{b}, y_{i+1})$$

and the fact in question follows from the definition of $\bar{\varphi}_0$.

In order to complete the proof of the theorem, assume that $X \in \mathrm{Fin}\, M$, $S \in \mathrm{Fin}\, M$ and for some $c \in M$, $c > \omega$, $M \models \mathrm{FCP}^n_c(S)$. Denote by $\psi_0(x)$ a Δ_0-formula such that $M \models \forall x\ (x \in X \Leftrightarrow \psi_0(x))$, and let $\chi_0(x, y)$ denote the Π^*_0-formula obtained from the formula $\chi(x, y)$ (defined at the beginning of the proof) by substituting $\psi_0(z)$ for R.

By the claim there exists an initial segment $I \subset_e M$ such that $S \cap I$ is cofinal in I and there exists a formula $\psi(x, y) \in \Delta^*_0$ such that

$$\forall a, b \in I\ I \models \chi_0(a, b) \Leftrightarrow M \models \psi(a, b) .$$

By the definition of $\models^*$, $\forall a \in I\ ((I, X \cap I) \models a \in X) \Leftrightarrow (I \models^* \psi_0(a))$. Hence $(I, X \cap I) \models \chi(a, b) \Leftrightarrow I \models^* \chi_0(a, b)$ for $a, b \in I$. Thus, finally,

$$\forall a, b \in I\ ((I, X \cap I) \models \chi(a, b)) \Leftrightarrow (M \models \psi(a, b)) .$$

Hence, in view of the minimum principle for $\psi(a, y)$ in M for any parameter $a \in M$, we get the minimum principle for $\chi(a, y)$ in $(I, X \cap I)$ for any parameter $a \in I$. Therefore $(I, X \cap I) \models I\Sigma_n(R)$, which completes the proof. ■

Now we are ready to prove the main results of this section, related to the functions provably recursive in $I\Sigma_n$.

4.16. THEOREM. *In every model M for $I\Sigma_n$ the formula $Y_n(x, y) = z$ is an indicator for segments which are models for $I\Sigma_n$, i.e.*

$$\forall a, b \in M\ [Y^M_n(a, b) > \omega \Leftrightarrow \exists I\ a \in I < b \wedge I \models I\Sigma_n] .$$

Proof. The implication $\Leftarrow$ is the content of Lemma 4.10(3). To prove the opposite implication, assume that $Y^M_n(a, b) > \omega$. Thus, by Lemma 4.13,

$$\forall m \in \omega\ M \models \mathrm{FCP}^n_m([a, b]) .$$

Hence there exists a $c > \omega$ such that $M \models \mathrm{FCP}^n_c([a, b])$. By Lemma 4.15 there exists an $I \subset_e M$ such that $I \models I\Sigma_n$, $a \in I < b$, which completes the proof. ■

4.17. THEOREM. *The family of functions $\{I^m_{n+1}(I_1) \ldots (I_1):\ m \in \omega\}$ is a cofinal set in the class of provably recursive functions in $I\Sigma_n$.*

Proof. By Lemma 3.9 each of the functions in the above family is provably recursive in $I\Sigma_n$. On the other hand, the fact that $Y_n(x, y)$ is an indicator for $I\Sigma_n$ in every model for $\mathrm{Th}(N)$ where $N = (\omega, <, +, \cdot)$ implies, in a standard manner, that if f is provably recursive in $I\Sigma_n$, then there exists an $m \in \omega$ such that $f(a) < \min_b Y_n^N(a, b) \geqslant m$ for $a \in \omega$ (cf. [3]). We have

$$\min_b \big(Y_n^N(a, b) \geqslant m \big) \leqslant \min_b N \models I_{n+1}^m(I_n) \dots (I_1)^{[a,b]}(a) \leqslant b$$

and the last value is, by 3.1, $\leqslant \bar{I}_{n+1}^m(\bar{I}_n) \dots (\bar{I}_1)(a)$. Thus $f < \bar{I}_{n+1}^m(\bar{I}_n) \dots (\bar{I}_1)$. ■

§ 5. Hardy's functions and functionals. To render the picture complete we give in this section a short proof of a theorem defining the relation of the function $\bar{I}_n^m(\bar{I}_{n-1}) \dots (\bar{I}_1)$ to the sequence of Hardy's functions. Because of this relation, Theorem 4.7 takes the form of Corollary 5.6, which is known as Wainer's theorem about the majorization of the class $\mathrm{Rec}(I\Sigma_n)$ by Hardy's functions.

Moreover, we present here a sketch of a much shorter proof of Wainer's theorem.

Let us begin by recalling the fundamental concepts necessary for the definition according to Hardy's method.

Let $\omega_0^m = m$, $\omega_{n+1}^m = \omega^{\omega_n^m}$ for $m, n \in \omega$. In particular, we have $\varepsilon_0 = \sup_{n \in \omega} \omega_n^m$ for $m \in \omega$.

For $\alpha, \beta < \varepsilon_0$ we define $\alpha \gg \beta$ if and only if there exist ordinal numbers γ, δ and a number $n \in \omega$ such that $\alpha = \omega^\delta \cdot \gamma$ and $\beta \leqslant \omega^\delta \cdot n$. This definition directly implies the following properties:

5.1. (1) $0 \gg \alpha$ and $\alpha \gg n$ for every $\alpha < \varepsilon_0$ and every $n \in \omega$,

(2) For every number $1 < \gamma < \varepsilon_0$ either γ is of the form ω^α or there exist numbers $0 < \alpha$, $\beta < \varepsilon_0$ such that $\gamma = \alpha + \beta$, $\alpha \gg \beta$.

(3) $\alpha \gg \beta$, $\alpha > 0$ then $\beta < \alpha + \beta$.

(4) If $\gamma = \alpha + \beta$, $\gamma = \alpha' + \beta'$, $\alpha \gg \beta$, $\alpha' \gg \beta'$ and $\alpha > \alpha'$ then $\alpha' \gg \alpha - \alpha' \gg \beta$.

5.2. Fact. There exists exactly one family $(\{\alpha\}(n))_{n \in \omega}$: $0 < \alpha < \varepsilon_0$ of sequences with the following properties:

(1) $(\alpha+1)(n) = \alpha$ for $\alpha < \varepsilon_0$, $n \in \omega$, $\alpha = \sup_{n \in \omega} \{\alpha\}(n)$ for $\alpha \in \mathrm{Lim} \cap \varepsilon_0$.

(2) $\{\alpha+\beta\}(n) = \alpha + \{\beta\}(n)$ for $\alpha \gg \beta$, $\alpha, \beta < \varepsilon_0$, $\beta \neq 0$ and $n \in \omega$.

(3) $\{\omega^\alpha\}(n) = \omega^{\{\alpha\}(n)}$ for $\alpha \in \mathrm{Lim} \cap \varepsilon_0$, $n \in \omega$.

(4) $\{\omega^{\alpha+1}\}(n) = \omega^\alpha n$ for $\alpha < \varepsilon_0$, $n \in \omega$.

The sequence $(\{\alpha\}(n))_{n \in \omega}$ is called the *fundamental sequence* for α.

The sequence of Hardy's functions H_α: $0 < \alpha < \varepsilon_0$ is defined by the following inductive conditions: $H_1(n) = n+1$, $H_{\alpha+1}(n) = H_\alpha(n+1)$ for $0 < \alpha < \varepsilon_0$, $H_\alpha(n) = H_{\{\alpha\}(n+1)}(n)$ for $0 < \alpha \in \varepsilon_0 \cap \mathrm{Lim}$. The above definition differs from that encountered in the literature in changing $H_{\{\alpha\}(n+1)}$ into $H_{\{\alpha\}(n)}$ in the inductive condition for H_α. Now we can formulate the announced basic result on functionals and Hardy's functions.

5.3. THEOREM. *The following equality holds*:

$$\bar{I}_n^m(\bar{I}_{n-1}) \dots (\bar{I}_1) = H_{\omega_{n-1}^m} \quad \textit{for } 0 < m, n \in \omega .$$

Remark. If we modify the definition of $\bar{I}_n$ so as to introduce the equality $\bar{I}_n(f_n) \dots (x) = (f_n)^x \dots (x)$, we shall have a true equivalent of 5.3 for the original definition of Hardy's sequence. All the theorems proved so far will of course remain valid, up to minute details, in the case of modification of the definition of the functionals $\bar{I}_n$.

In order to prove Theorem 5.3 we shall consider certain functionals defined on the pattern of the definition of Hardy's functions.

5.4. DEFINITION. For every natural number $i > 0$ the sequence of Hardy's functionals H_α^i: $0 < \alpha < \varepsilon_0$ from $F_{n-1}\omega$ to $F_{n-1}\omega$ is defined by the following inductive conditions:

(1) $H_1^i = \bar{I}_i$,

(2) $H_{\beta+\gamma}^i = H_\beta^i \circ H_\gamma^i$ for $\beta \gg \gamma$, $0 < \beta$, $\gamma < \varepsilon_0$,

(3) $H_{\omega^\beta}^i(f_{i-1}) \dots (f_1)(n) = H_{\{\omega^\beta\}(n+1)}^i(f_{i-1}) \dots (f_1)(n)$.

The consistency and completeness of the system of cases (1)–(3) results from 5.1, and the inductive character of (2) and (3) follows from 5.1(3) and 5.2(3), (4). However, (2) is not a usual form of an inductive condition since there are numbers $\alpha < \varepsilon_0$ which can be represented in several ways in the form of a sum $\beta + \gamma$, where $\beta \gg \gamma$, $0 < \beta$, $\gamma < \varepsilon_0$. Thus we must show that (2) is unique in the context of the above definition; we prove this by induction. We assume that the definition is correct below α and assume $\alpha = \beta + \gamma = \beta' + \gamma'$, where $\beta \gg \gamma$, $\beta' \gg \gamma'$ and $\beta, \gamma, \beta', \gamma' > 0$. We have $\beta, \beta' < \alpha$. We may assume that $\beta > \beta'$. Let $\beta'' = \beta - \beta'$. By 5.1(4), $\beta' \gg \beta'' \gg \gamma$. Hence $H_\beta^i \circ H_\gamma^i = H_{\beta'+\beta''}^i \circ H_\gamma^i = H_{\beta'}^i \circ H_{\beta''}^i \circ H_\gamma^i$, because $\beta' + \beta'' < \alpha$. Since $\beta'' + \gamma = \gamma' < \alpha$ we also have $H_{\beta''}^i \circ H_\gamma^i = H_{\gamma'}^i$, i.e, finally $H_\beta^i \circ H_\gamma^i = H_{\beta'}^i \circ H_{\gamma'}^i$ which was to be proved.

Remark. It is easy to verify that the sequence of functions H_α^1: $0 < \alpha < \varepsilon_0$ satisfies the inductive conditions for Hardy's functions. Thus $H_\alpha^1 = H_\alpha$ for $0 < \alpha < \varepsilon_0$. Let

$$C = \{\omega^\gamma n \colon \gamma < \varepsilon_0,\ 0 < n \in \omega\} .$$

Theorem 5.3 is a direct consequence of the following lemma:

5.5. LEMMA. *For every natural number $i > 0$ and for every $\beta \in C$*

(*) $H_\beta^{i+1}(H_{\omega^\alpha}^i) = H_{\omega^{\alpha+\beta}}^i$ *on condition that* $\alpha \gg \beta$.

Proof. Fix $i > 0$. We use induction with respect to $\beta \in C$.

Case 1. $\beta = 1$. Denote the functional $H_1^{i+1}(H_{\omega^\alpha}^i) = I_{i+1}(H_{\omega^\alpha}^i)$ by H. From the definition of I_{i+1} we infer that

$$H(f_{i-1}) \dots (f_1)(n) = (H_{\omega^\alpha}^i)^{n+1}(f_{i-1}) \dots (f_1)(n) .$$

Thus, by part (2) of Definition 5.4 and part (4) of Fact 5.2, we have

$$H(f_{i-1}) \dots (f_1)(n) = H_{\{\omega^{\alpha+1}\}(n+1)}^i(f_{i-1}) \dots (f_1)(n) .$$

Finally, by 5.4(3)

$$H(f_{i-1}) \dots (f_1)(n) = H^i_{\omega^{\alpha+1}}(f_{i-1}) \dots (f_1)(n), \quad \text{i.e.}$$
$$H = H^i_{\omega^{\alpha+1}}.$$

Case 2. The inductive step $\omega^\gamma \to \omega^\gamma n$. We show it by induction with respect to $n > 0$. Assume that (*) holds for ω^γ and $\omega^\gamma n$. Let $\alpha \gg \omega^\gamma(n+1)$. Therefore $\alpha \gg \omega^\gamma n$ and $\alpha + \omega^\gamma n \gg \omega^\gamma$. We thus have

$$H^{i+1}_{\omega^\gamma(n+1)}(H^i_{\omega^\alpha}) = H^{i+1}_{\omega^\gamma}(H^{i+1}_{\omega^\gamma n}(H^i_{\omega^\alpha})) = H^{i+1}_{\omega^\gamma}(H^i_{\omega^{\alpha+\omega^\gamma n}}) = H^i_{\omega^{\alpha+\omega^\gamma(n+1)}}.$$

Case 3. We assume that (*) is true for numbers in C less than ω^γ where $0 < \gamma < \varepsilon_0$. We shall prove that it is also true for $\beta = \omega^\gamma$. Assume that $\alpha \gg \beta$.

Note first that $\{\beta\}(n+1) \in C$, $\{\beta\}(n+1) < \beta$ and $\alpha \gg \{\beta\}(n+1)$ for every $n \in \omega$. By part (2) of Definition 5.4 applied to $H^{i+1}_{\omega^\gamma}$ we have $H^{i+1}_\beta(H^i_{\omega^\alpha})(f_{i-1}) \dots (f_1)(n) = H^{i+1}_{\{\beta\}(n+1)}(H^i_{\omega^\alpha})(f_{i-1}) \dots (f_1)(n)$ — denote this value by m. Hence, by the inductive assumption, $m = H^i_{\omega^{\alpha+\{\beta\}(n+1)}}(f_{i-1}) \dots (f_i)(n)$. Since by 5.2(2), $\alpha + \{\beta\}(n+1) = \{\alpha+\beta\}(n+1)$ and by 5.2(3), $\{\omega^{\alpha+\beta}\}(n+1) = \omega^{\{\alpha+\beta\}(n+1)}$ because $\alpha+\beta \in \mathrm{Lim}$, we see that $m = H^i_{\{\omega^{\alpha+\beta}\}(n+1)}(f_{i-1}) \dots (f_1)(n)$, i.e. $m = H^i_{\omega^{\beta+\alpha}}(f_{i-1}) \dots (f_1)(n)$. Therefore $H^{i+1}_\beta(H^i_{\omega^\alpha}) = H^i_{\omega^{\alpha+\beta}}$.

By Theorems 4.17 and 5.3 we obtain

5.6. Corollary. *The family of functions* $\{H_{\omega^m_n}: m \in \omega\}$ *is a cofinal set in the class* $\mathrm{Rec}(I\Sigma_n)$ *for every* $n > 0$.

The fact that every function $f \in \mathrm{Rec}(I\Sigma_n)$ is bounded by some H_α, where $\alpha < \omega_{n+1}$ is interesting in its own right. Therefore we shall also give a sketch of a more direct proof, based on a modification of some definitions and lemmas used in this paper.

If $S \subseteq \omega$ then we define $H^S_1 = I^S_1$, $H^S_{\alpha+1}(x) \simeq H^S_\alpha(H^S_1(x))$, $H^S_\lambda(x) \simeq H^S_{\{\lambda\}(x)}(x)$ for $x \in S$.

We say that S is α-*large* iff $H^S_\alpha(\min S)\downarrow$. This notion somewhat differs from the notion considered in [2]. Substituting "S is t-large" by "S is α-large" in Definition 4.4, we obtain

5.7. Definition. The symbol $\alpha \xrightarrow{f} (\beta, \gamma)$ denotes the sentence: for every α-large set S such that $\min S \geq 2$ there exist $S_1, S_2 \subseteq S$ such that $\min S = \min S_1$, (S_1, S_2) is an approximation to f and S_1 *is* β*-large*, S_2 is γ-large.

A counterpart of Lemma 4.9 is the following:

5.8. Lemma. *The combinatorial property* $\forall f\ \omega^{\alpha+\beta} \xrightarrow{f} (\beta, \omega^\alpha)$ *is true (in* N*) for all* $\alpha, \beta < \varepsilon_0$ *such that* $\alpha \gg \beta$ *and* $\alpha \geq 2$.

This can be proved by induction on β. The first step and the nonlimit step are similar to the corresponding steps in the proof of 4.7.

By the lemma we easily infer the following connection just as 4.13 was deduced from 4.9.

5.9. Lemma. *If* S *is* ω^{k+2}_n*-large and* $\min S \geq 2$ *then* $\mathrm{FCP}^n_k(S)$.

By 4.15, we can now infer that if $M \equiv N$ and $\forall m \in \omega\ M \models$ "$[a, b]$ is ω^m_n-large" then there exists $I \subset_e M$ such that $a \in I < b$ and $I \models I\Sigma_n$. From this it follows immediately that each $f \in \mathrm{Rec}(I\Sigma_n)$ is bounded by some $H_{\omega^m_n}$, where $m \in \omega$.

References

[1] K. Gödel, *Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes*, Dialectica 12 (1958), 280–287.

[2] J. Ketonen and R. Solovay, *Rapidly groving Ramsey functions*, Ann. of Math. 113 (1981), 267–314.

[3] J. Paris, *Some independence results for Peano arithmetic*, J. S. L. 43 (1978), 725–731.

[4] — *A hierarchy of cuts in models of arithmetic*, in L. Pacholski, J. Wierzejewski and A. J. Wilkie, eds. *Model Theory of Algebra and Arithmetic*, Springer-Verlag, Heidelberg.

[5] C. Smoryński, *Nonstandard Models of Arithmetic*, Course Notes, Utrecht (1978).

[6] H. Schwichtenberg, *Elimination of higher type levels in definitions of primitive recursive functionals by means of transfinite recursion*, in Logic Colloq. 73, ed. by H. E. Rose and J. C. Sheperdson, pp. 279–303.

[7] S. Wainer, *Ordinal recursion, and a refinement of the extended Grzegorczyk hierarchy*, J. S. L. 37 (1972), 281–292.

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