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Received 28 April 1986;
in revised form 27 October 1986

Quadratic form schemes and quaternionic schemes

by

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Abstract. Quaternionic schemes, quaternionic structures and abstract Witt rings are known to be equivalent abstract versions of the algebraic theory of quadratic forms. This paper establishes a relationship between quadratic form schemes and the three other axiomatic approaches to quadratic forms. It is shown that cancelative quadratic form schemes coincide with quaternionic schemes.

Introduction. The algebraic theory of quadratic forms focuses on quadratic forms over fields. However, it has become clear that some parts of the theory are best treated by using an appropriate abstract language. Several authors have had ideas of this kind and as a result we are confronted with at least four distinct abstract approaches to quadratic form theory. These are:

(i) Quadratic form schemes ([3], [4], [5], [6], [10], [11], [12]).
(ii) Quaternionic structures ([1], [8], [9]).
(iii) Abstract Witt rings ([1], [8], [9] and earlier papers cited there).
(iv) Quaternionic schemes ([1]).

The relationships among (ii), (iii), and (iv) are fully known. Marshall's book [8] shows that (ii) and (iii) are equivalent and Carson and Marshall [1] prove that (ii) and (iv) are equivalent. It is the aim of this paper to clarify the role of (i) among the abstract theories of quadratic forms.

In Section 1 we exhibit several equivalent sets of axioms for (i) and in Section 2 we do the same for (iv). In both cases we have found that the generally accepted sets of axioms for (i) and (iv) are dependent and we reduce the number of axioms in each case to a pair of independent axioms and even to a single axiom in each case.

Section 3 explains the relationship between two concepts of isometry of forms used in abstract theories (chain isometry and inductively defined isometry following [8] and [9]). The main result, Theorem 3.5, establishes the actual equivalence of quadratic form schemes with cancellation property and quaternionic schemes. A corollary to this result asserts that the classical Witt cancellation theorem for forms of any dimension is a consequence of the cancellation property for 2-dimensional forms.

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In Section 4 we consider two important types of schemes (Pythagorean and Hilbert) and show that for them the cancellation theorem is a consequence of the other (simple) axioms.

1. Quadratic form schemes. Throughout the paper, G will always be an elementary 2-group (i.e., $x^2 = 1$ for every $x \in G$) and $-1 \in G$ will be a distinguished element of G ($-1 = 1$ is not excluded). The product $-1 \cdot a$ will be written $-a$. We write V for a mapping assigning to each $a \in G$ a subgroup $V(a)$ of G. Whenever the triplet $S = (G, -1, V)$ appears, G, -1, V have the above meaning.

DEFINITION 1.1. $S = (G, -1, V)$ is said to be a *quadratic form pre-scheme* if the following two axioms hold for S:

C1. $a \in V(a)$ for every $a \in G$.

C2. $b \in V(-a)$ implies $a \in V(-b)$ for all $a, b \in G$.

DEFINITION 1.2. A *quadratic form f of dimension n* in a pre-scheme $S = (G, -1, V)$ is any n-tuple $f = \langle a_1, \ldots, a_n\rangle$ of elements of G. The set *Df of elements of G represented by f* is defined inductively as follows:

$$D\langle a_1\rangle = \{a_1\},$$

$$D\langle a_1, \ldots, a_n\rangle = \bigcup \{a_1 V(a_1 x) : x \in D\langle a_2, \ldots, a_n\rangle\} \quad \text{for } n \geq 2.$$

In particular, for a binary form $\langle a, b\rangle$, we have $D\langle a, b\rangle = aV(ab)$.

All these have natural meaning in case of the pre-scheme of a field F of characteristic $\neq 2$. Here G is the group of square classes $F^{\cdot}/F^{\cdot 2}$, -1 is the coset $(-1)F^{\cdot 2}$ and $V(aF^{\cdot 2})$ is the value group of the quadratic form $X^2 + aY^2$ viewed as a subgroup of G. However, it turns out that in abstract situation the value set $D\langle a_1, \ldots, a_n\rangle$ depends in general on the order of diagonal entries. To rectify this we introduce

DEFINITION 1.3. A pre-scheme $S = (G, -1, V)$ is said to be a *quadratic form scheme* if it satisfies the following axiom:

C3. $$D\langle a, b, c\rangle = D\langle b, a, c\rangle \quad \text{for all } a, b, c \in G.$$

In a series of papers all the schemes on groups of order ≤ 32 have been classified (see [2], [13], [14], [4], [5], [6], [7] for $|G| \leq 8$, [11], [12] for $|G| \leq 16$ and [1] for $|G| \leq 32$). It turns out that all these schemes come from fields.

J. Pękała [10] observed that C1 and the following axiom P2 also define a quadratic form scheme:

P2. $$b \in V(-a)V(-ac) \Rightarrow a \in V(-b)V(-bc).$$

It is convenient to state this axiom in the following equivalent form:

I2. $$bV(-a) \cap V(-ac) \neq \emptyset \Rightarrow aV(-b) \cap V(-bc) \neq \emptyset.$$

Finally, we wish to turn attention to the following consequence of C1 and P2:

QFS. $$b \in V(-a)V(-ac) \Rightarrow -ab \in V(-b)V(-bc).$$

The following theorem determines the strength of the various sets of axioms.

THEOREM 1.4. *For a triplet $S = (G, -1, V)$ the following are equivalent*:

(i) C1, C2 *and* C3 (i.e. S *is a quadratic form scheme*).

(ii) C2 *and* C3.

(iii) C1 *and* P2.

(iv) C1 *and* I2.

(v) QFS.

Proof. (i) $\Rightarrow$ (ii) is trivial and to prove (ii) $\Rightarrow$ (iii) we first prove that (ii) $\Rightarrow$ C1. Observe that without assuming anything on S we have $a \in D\langle a, b\rangle$ and $a \in D\langle a, b, c\rangle$ for all $a, b, c \in G$. Thus $a \in D\langle a, 1, -1\rangle$ and by C3, $a \in D\langle 1, a, -1\rangle$. It follows $a \in V(x)$, where $x \in D\langle a, -1\rangle = aV(-a)$. Thus $x = ay$ with $y \in V(-a)$ and $a \in V(ay)$. By C2, $-ay \in V(-a)$ and so $-a = -ay \cdot y \in V(-a)$. By C2, $a \in V(a)$. This proves C1 for S. Now P2 follows as shown in [10].

(iii) $\Leftrightarrow$ (iv) follows easily via the general fact: $b \in HK$ iff $Hb \cap K \neq \emptyset$ where H and K are subgroups of an arbitrary group. That (iii) $\Rightarrow$ (i) is proved in [10]. Since (iii) $\Rightarrow$ (v) is trivial, it remains to prove (v) $\Rightarrow$ (iii). First observe that applying QFS twice, we get

(1.4.1) $$b \in V(-a)V(-ac) \Rightarrow a \in V(ab)V(abc).$$

Now take $b = c = 1$ and use (1.4.1) to get $a \in V(a)$. This proves QFS $\Rightarrow$ C1. But QFS and C1 obviously imply P2. Hence (v) $\Rightarrow$ (iii), as required.

Remark 1.5. Looking at appropriate examples of triplets $S = (G, -1, V)$, one can conclude that the axioms in (ii), (iii) and (iv) are independent.

2. Quaternionic schemes. In this section we will discuss the pre-schemes introduced by Carson and Marshall in [1]. Given a standard triplet $S = (G, -1, V)$ we write $V(a, b; c, d)$ for $bV(-a) \cap V(-ac) \cap dV(-c)$.

DEFINITION 2.1 ([1]). A quadratic form pre-scheme S is said to be a *quaternionic scheme* if it satisfies the following axiom.

CM. $$V(a, b; c, d) \neq \emptyset \Rightarrow V(b, a; d, c) \neq \emptyset.$$

Quaternionic schemes are closely related to quaternionic structures considered in [8] and [9] (see [1] for the proof that the two concepts are equivalent). In order to relate quaternionic schemes to quadratic form schemes considered in Section 1, we introduce the following intersection property I3 and transitivity property T (the latter coming from [1]):

I3. $$V(a, b; c, d) \neq \emptyset \Rightarrow V(b, a; c, d) \neq \emptyset.$$

T. $$V(a, b; c, d) \neq \emptyset \quad \text{and} \quad V(c, d; e, f) \neq \emptyset \Rightarrow V(a, b; e, f) \neq \emptyset.$$

We also consider the following consequence of C1 and I3:

Q. $V(a,b;c,d)\neq\emptyset\Rightarrow V(b,-ab;c,d)\neq\emptyset$.

The interrelations among the four properties are recorded below.

THEOREM 2.2. *For a triplet* $S=(G,-1,V)$ *the following are equivalent*:

(i) C1, C2 *and* CM (*i.e.*, S *is a quaternionic scheme*).
(ii) C1 *and* CM.
(iii) C1 *and* I3.
(iv) C1 *and* T.
(v) Q.

Proof. (i) $\Leftrightarrow$ (ii) will follow if we show CM $\Rightarrow$ C2. Assume CM holds for S and let $b\in V(-a)$. Then $b\in V(a,b;1,b)$ and by CM, we have $V(b,a;b,1)\neq\emptyset$. It follows $aV(-b)\cap V(-b)\neq\emptyset$ and so $a\in V(-b)$, as required.

The second part of the proof will show (iii) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii). Observe that always $V(a,b;c,d)=V(c,d;a,b)$. Now assume I3 holds for S. Then

$$\begin{aligned}V(a,b;c,d)\neq\emptyset&\Rightarrow V(b,a;c,d)\neq\emptyset, &&\text{by I3},\\&\Rightarrow V(c,d;b,a)\neq\emptyset\\&\Rightarrow V(d,c;b,a)\neq\emptyset, &&\text{by I3},\\&\Rightarrow V(b,a;d,c)\neq\emptyset.\end{aligned}$$

This proves I3 $\Rightarrow$ CM and so (iii) $\Rightarrow$ (ii). That (ii) $\Rightarrow$ (iv) is proved explicitly in [1]' proof of Theorem 1.3. Carson and Marshall use in their proof also C2, but this is a consequence of CM as shown above.

(iv) $\Rightarrow$ (iii). Assume $x\in V(a,b;c,d)$. Then $x\in bV(-a)$ and so $-ab\in xV(-a)$, by C1. Again by C1,

$$-ab\in aV(-b)\cap V(-ab)\cap xV(-a)=V(b,a;a,x).$$

Further, $x\in bV(-a)\cap dV(-c)$ implies $bV(-a)=xV(-a)$ and $dV(-c)=xV(-c)$. Hence

$$V(a,b;c,d)=xV(-a)\cap V(-ac)\cap xV(-c)=V(a,x;c,x).$$

Thus $x\in V(a,b;c,d)$ implies

$$V(b,a;a,x)\neq\emptyset \quad\text{and}\quad V(a,x;c,x)\neq\emptyset$$

and using T we conclude

$$V(b,a;c,x)\neq\emptyset.$$

It follows

$$V(b,a;c,d)=aV(-b)\cap V(-bc)\cap xV(-c)=V(b,c;c,x)\neq\emptyset.$$

This proves I3 for S and completes the second part of proof. It remains to prove (v) $\Rightarrow$ (iii) since the converse is obvious. Applying Q twice, we get

$$V(a,b;c,d)\neq\emptyset\Rightarrow V(-ab,a;c,d)\neq\emptyset. \tag{2.2.1}$$

Now take $b=c=d=1$ and observe that $1\in V(a,1;1,1)$. Hence by (2.2.1), $aV(a)\cap V(a)\cap V(-1)=V(-a,a;1,1)\neq\emptyset$. It follows $aV(a)\cap V(a)\neq\emptyset$ and so $a\in V(a)$. This proves $Q\Rightarrow$ C1. But Q and C1 imply I3, since $-abV(-b)=aV(-b)$ follows from C1. Hence (v) $\Rightarrow$ (iii), as required.

COROLLARY 2.3. *Every quaternionic scheme is also a quadratic form scheme.*

Proof. According to 1.4 and 2.2 it suffices to prove I3 $\Rightarrow$ I2 which is obvious. Another proof is by observing that Q $\Rightarrow$ QFS which becomes particularly easy if QFS is written as the intersection property

$$bV(-a)\cap V(-ac)\neq\emptyset\Rightarrow -abV(-b)\cap V(-bc)\neq\emptyset.$$

Remark 2.4. We point out that the converse of 2.3 is not known to hold. We will discuss this problem in the next sections. We can prove that the axioms in (ii), (iii) and (iv) of 2.2 are independent. Also one can show that T alone is weaker than I3 or CM since it does not imply C2 while both I3 and CM do. We omit the details.

3. Cancellation property. The notion of isometry of quadratic forms can be introduced in abstract pre-schemes in two different ways. The first uses the chain equivalence theorem of Witt ([15], Satz 7) and the second an inductive description of isometry relation for forms over fields (cf. [8]).

DEFINITION 3.1. Two forms $f=\langle a_1,\dots,a_n\rangle$ and $g=\langle b_1,\dots,b_n\rangle$ in a pre-scheme S are said to be *chain isometric*, written $f\sim g$, if

(i) $a_1=b_1$, when $n=1$.
(ii) $a_1a_2=b_1b_2$ and $D\langle a_1,a_2\rangle=D\langle b_1,b_2\rangle$, when $n=2$.
(iii) For $n\geqslant 3$, there exists a chain of forms $f_0=f,f_1,\dots,f_k=g$, $k\geqslant 0$, such that for each $i=1,\dots,k$, the form f_i is obtained from f_{i-1} by replacing two entries a, a' by b, b', respectively, where $\langle a,a'\rangle\sim\langle b,b'\rangle$.

DEFINITION 3.2. Two forms f and g as above are said to be *strongly isometric*, written $f\cong g$, if (i) and (ii) of Definition 3.1 hold when $n=1,2$, and when $n\geqslant 3$, there exist $a,b,c_3,\dots,c_n\in G$ such that $\langle a_1,a\rangle\cong\langle b_1,b\rangle$, $\langle a_2,\dots,a_n\rangle\cong\langle a,c_3,\dots,c_n\rangle$ and $\langle b_2,\dots,b_n\rangle\cong\langle b,c_3,\dots,c_n\rangle$.

It is obvious that chain isometry is an equivalence relation but for strong isometry this is not so. The point is that transitivity of $\cong$ requires some extra assumptions on the pre-scheme S.

THEOREM 3.3. *For a pre-scheme* S *the following are equivalent*:

(i) *Strong isometry* $\cong$ *is transitive.*

(ii) *Strong isometry* $\cong$ *is transitive on* 3-*dimensional forms.*
(iii) *The pre-scheme* S *is a quaternionic scheme.*

Proof. (i) $\Leftrightarrow$ (ii). Marshall's proof of (i) for quaternionic structures ([8], p. 34) includes the proof of (i) $\Leftrightarrow$ (ii). Observe that (ii) holds iff it holds for 3-dimensional forms of determinant 1 (here $\det f = a_1 \cdot \ldots \cdot a_n$ for $f = \langle a_1, \ldots, a_n\rangle$). Indeed, a simple induction argument shows $f \cong g \Rightarrow \det f = \det g$ and it is easy to see that $f \cong g \Rightarrow xf \cong xg$ for any $x \in G$ (here $xf = \langle xa_1, \ldots, xa_n\rangle$).

Thus for the remaining part of the proof we need the following result.

LEMMA 3.4. $\langle -a, -b, ab\rangle \cong \langle -c, -d, cd\rangle \Leftrightarrow V(a, b; c, d) \neq \emptyset$.

Using 3.4 we conclude that (ii) holds iff S satisfies the transitivity axiom T from Theorem 2.2 and this holds iff S is a quaternionic scheme, by Theorem 2.2.

Thus it remains to prove Lemma 3.4. The two forms are strongly isometric iff there are $e, f, g \in G$ such that $\langle -a, e\rangle \cong \langle -c, f\rangle$, $\langle -b, ab\rangle \cong \langle e, g\rangle$ and $\langle -d, cd\rangle \cong \langle f, g\rangle$. Comparing determinants, $-a = eg$, $-c = fg$ hence $e = -ag = ax$, $f = -cg = cx$, where $x := -g$. Thus the forms are isometric iff $\langle -a, ax\rangle \cong \langle -c, cx\rangle$, $\langle -b, ab\rangle \cong \langle -x, ax\rangle$, $\langle -d, cd\rangle \cong \langle -x, cx\rangle$ iff $ac \in V(-x)$, $bx \in V(-a)$, $dx \in V(-c)$ iff

$$x \in bV(-a) \cap V(-ac) \cap dV(-c) = V(a, b; c, d).$$

This finishes the proof of 3.3.

Let $\simeq$ denote an isometry relation for forms in a pre-scheme S and let us agree that $\langle a_1, \ldots, a_n\rangle + \langle b_1, \ldots, b_m\rangle := \langle a_1, \ldots, b_m\rangle$.

DEFINITION 3.4. We say the pre-scheme S is *n-cancelative* for the isometry $\simeq$ if for any two forms f and g of dimension n and for every $a \in G$,

$$\langle a\rangle + f \simeq \langle a\rangle + g \Rightarrow f \simeq g.$$

S is said to be *cancelative* for $\simeq$ if it is n-cancelative for every $n \geqslant 1$.

The cancellation property behaves quite differently in the two types of schemes considered here. On the one hand, if we work with strong isometry $\cong$, we have to assume S is a quaternionic scheme (cf. 3.3) and then it is quite easy to prove that S is cancelative for $\cong$ (cf. [8], p. 35) and that strong isometry $\cong$ and chain isometry $\sim$ coincide ([9], Cor. 2.4). On the other hand, if S is assumed to be a quadratic form scheme and the isometry chosen is chain isometry it appears to be a hard problem to decide whether S is cancelative for $\sim$. Notice that all the quadratic form schemes on groups of order $\leqslant 32$ come from fields and so are cancelative. Hence, if a non-cancelative scheme exists, it lives on a group of order at least 64.

In order to develop a meaningful abstract theory of forms one has to add one more axiom to C1, C2 and C3, and this is cancellation for chain isometry (cf. [11], [12]). The question now arises: how do quaternionic schemes compare to these cancelative quadratic form schemes?

THEOREM 3.5. *For a pre-scheme* S *the following are equivalent*:
(i) S *is a quaternionic scheme.*
(ii) S *is a quadratic form scheme and* S *is* 2-*cancelative for chain isometry.*

Proof. (i) $\Rightarrow$ (ii). As mentioned above, (i) implies that chain isometry coincides with strong isometry and the latter is cancelative. Also C3 follows from [8], Prop. 2.1 on p. 31.

(ii) $\Rightarrow$ (i). We need a lemma analogous to 3.4.

LEMMA 3.6. *Assume* S *satisfies* (ii). *Then for any* $a, b, c, d \in G$,

$$\langle -a, -b, ab\rangle \sim \langle -c, -d, cd\rangle \Leftrightarrow V(a, b; c, d) \neq \emptyset.$$

Proof of 3.6. If $V(a, b; c, d) \neq \emptyset$, the two forms are strongly isometric by Lemma 3.4, hence also chain isometric. Conversely, assume $f := \langle -a, -b, ab\rangle \sim \langle -c, -d, cd\rangle =: g$. Then $Df = Dg$ and so $-c \in Df$. According to Def. 1.2, $c \in D\langle -a, y\rangle$, where $y \in D\langle -b, ab\rangle$. It follows $ac \in V(-ay)$ and $ay \in bV(-a)$. Thus with $x := ay$ we have

$$x \in bV(-a) \cap V(-ac). \tag{3.6.1}$$

It also follows that $\langle -b, ab\rangle \sim \langle -x, ax\rangle$ and $\langle -a, ax\rangle \sim \langle -c, cx\rangle$ and so we get the chain

$$\langle -a, -b, ab\rangle, \quad \langle -a, -x, ax\rangle, \quad \langle -c, -x, cx\rangle$$

showing that $f \sim \langle -c, -x, cx\rangle$. Since $f \sim g$ it follows that

$$\langle -c, -x, cx\rangle \sim \langle -c, -d, cd\rangle.$$

Now use 2-cancellation for $\sim$ and get $\langle -x, cx\rangle \sim \langle -d, cd\rangle$. Hence $x \in dV(-c)$. On combining with (3.6.1) it follows $x \in V(a, b; c, d)$, as required. This proves Lemma 3.6.

Now we are ready to prove (ii) $\Rightarrow$ (i) in 3.5. If (ii) holds and $V(a, b; c, d) \neq \emptyset$, then by Lemma 3.6, $\langle -a, -b, ab\rangle \sim \langle -c, -d, cd\rangle$. It follows trivially that $\langle -b, -a, ba\rangle \sim \langle -d, -c, dc\rangle$, and by 3.6 again, $V(b, a; d, c) \neq \emptyset$. Thus S satisfies CM and so is a quaternionic scheme. This completes the proof of 3.5.

Remarks 3.7. Since quaternionic schemes are cancelative, we arrive at an interesting corollary to 3.5 that for a quadratic form scheme 2-cancellation for chain isometry implies cancellation property in full generality. This observation seems to be new even in the classical context of quadratic forms over fields. Thus, defining cancelative quadratic form schemes, we may require only 2-cancellation instead of n-cancellation for every n. If G is finite, this makes it possible to check the cancellation property for $S = (G, -1, V)$ in a finite number of steps.

3.8. The result in 3.5 asserts that for a pre-scheme S the axiom CM is equivalent to C3 plus 2-cancellation property. It is an open problem whether CM is actually equivalent to C3 alone.

4. Pythagorean and Hilbert schemes. In this section we consider two important special types of pre-schemes and show that for them the cancellation property is a consequence of the other axioms. This solves the problem discussed in 3.8 above for the two types of pre-schemes.

A pre-scheme $S = (G, -1, V)$ is said to be *Pythagorean* if $1 \neq -1$ and $V(1) = \{1\}$. Pythagorean fields produce examples of Pythagorean schemes and every cancelative Pythagorean quadratic form scheme with finite group G comes from a Pythagorean field (cf. [4]). We will now show that for quadratic form schemes (C1, C2 and C3 assumed) with $|G| < \infty$ Pythagoreanity implies cancellation property.

THEOREM 4.1. *Let $S = (G, -1, V)$ be a Pythagorean quadratic form scheme and assume the group G is finite. Then S is cancelative for chain isometry and so S is a quaternionic scheme.*

Before proving the theorem we recall two basic properties of Pythagorean schemes.

LEMMA 4.2. *For any form f in a Pythagorean scheme S and for any positive integer n, we have $D(n \times f) = Df$.*

Here $n \times f := f + \ldots + f$ (n times). This is a simple consequence of the representation criterion (cf. [8], Prop. 2.10, p. 36 and [12]).

LEMMA 4.3. *In a Pythagorean scheme, $a \in V(b) \Leftrightarrow V(a) \subset V(b)$.*

Proof of 4.3. Assume $a \in V(b)$ and $c \in V(a)$. Then $\langle 1, b\rangle \sim \langle a, ab\rangle$ and $\langle 1, a\rangle \sim \langle c, ac\rangle$. Consider $f := \langle 1, b, 1, b\rangle$. The above isometries yield

$$f \sim \langle a, ab, 1, b\rangle \sim \langle b, c, ab, ac\rangle .$$

It follows that $c \in Df = D\langle 1, b\rangle$, the latter by Lemma 4.2. Thus $c \in V(b)$ and so $V(a) \subset V(b)$. The other implication in 4.3 results from C1.

In a Pythagorean scheme one can introduce a partial ordering on G by putting $x \leqslant y \Leftrightarrow V(x) \subset V(y)$ (checking antisymmetry requires $V(1) = \{1\}$). If we assume G is finite, there are some maximal elements in G with respect to $\leqslant$. This will be used in the proof of Theorem 4.1 which follows.

By Theorem 3.5 it is sufficient to prove that chain isometry has 2-cancellation property and this (by scaling) says that

$$\langle 1\rangle + \langle a, b\rangle \sim \langle 1\rangle + \langle c, d\rangle \Rightarrow \langle a, b\rangle \sim \langle c, d\rangle .$$

So assume $\langle 1, a, b\rangle \sim \langle 1, c, d\rangle$. Comparing determinants we get $ab = cd =: x$ and we wish to show that

(4.1.1) $$\langle 1, a, ax\rangle \sim \langle 1, c, cx\rangle \Rightarrow \langle a, ax\rangle \sim \langle c, cx\rangle .$$

Let $y \in D\langle 1, a, ax\rangle$ be maximal with respect to $\leqslant$. We have $y \in D\langle 1, z\rangle = V(z)$ for some $z \in D\langle a, ax\rangle$. By Lemma 4.3 we have $V(y) \subset V(z)$ and by the maximality of y, we have $y = z$. Hence

(4.1.2) $$\langle a, ax\rangle \sim \langle z, zx\rangle = \langle y, xy\rangle .$$

Since $D\langle 1, a, ax\rangle = D\langle 1, c, cx\rangle$, we have $y \in D\langle 1, c, cx\rangle$ and y is a maximal element in this set. Similarly to the above, there is a $t \in D\langle c, cx\rangle$ with $y \in D\langle 1, t\rangle = V(t)$ and so $y \leqslant t$ and $y = t$ by the maximality of y. It follows that

(4.1.3) $$\langle c, cx\rangle \sim \langle t, tx\rangle = \langle y, xy\rangle .$$

By transitivity, (4.1.2) and (4.1.3) imply $\langle a, ax\rangle \sim \langle c, cx\rangle$ which proves (4.1.1). This completes the proof of Theorem 4.1.

A pre-scheme $S = (G, -1, V)$ is said to be *Hilbert* if each subgroup $V(a)$ has index $|G\colon V(a)| \leqslant 2$ and equality holds for at least one $a \in G$. Classical local fields are Hilbert in the sense that their quadratic form schemes are Hilbert. We will show that Hilbert pre-schemes are cancelative for chain isometry and also satisfy C3.

THEOREM 4.4. *Every Hilbert pre-scheme is a quaternionic scheme.*

Proof. According to Theorem 2.2, it is sufficient to prove that every Hilbert pre-scheme S satisfies I3. Our proof below does not use axiom C1. Write

$$L := bV(-a) \cap V(-ac) \cap dV(-c) \quad \text{and} \quad R := aV(-b) \cap V(-bc) \cap dV(-c)$$

and assume $L \neq \emptyset$. We want $R \neq \emptyset$. We begin with

LEMMA 4.5. *If S is Hilbert and $L \neq \emptyset$, then*

$$b \in V(-a) \Leftrightarrow d \in V(-c) .$$

Proof of 4.5. Assume $b \in V(-a)$. By C2, we have $V(-a) \cap V(-ac) \subset V(-c)$ and on the other hand $V(-a) \cap V(-ac) \cap dV(-c) \neq \emptyset$ since $bV(-a) = V(-a)$ and $L \neq \emptyset$. It follows that $V(-c) \cap dV(-c) \neq \emptyset$ and so $d \in V(-c)$. Thus $b \in V(-a) \Rightarrow d \in V(-c)$. Since $L = V(a, b; c, d) = V(c, d; a, b)$, using the same argument yields $d \in V(-c) \Rightarrow b \in V(-a)$. This proves 4.5.

Now we proceed to the proof of 4.4. So let $L \neq \emptyset$. If $1 \in aV(-b) \cap dV(-c)$, then $1 \in R$ and so $R \neq \emptyset$, as required. So assume $1 \notin aV(-b) \cap dV(-c)$. It follows $a \notin V(-b)$ or $d \notin V(-c)$ and by Lemma 4.5 we actually get $a \notin V(-b)$ *and* $d \notin V(-c)$. By Hilberticity, this means $V(-b)$ and $V(-c)$ are subgroups of index 2 in G and so certainly $aV(-b) \cap dV(-c) \neq \emptyset$. Let $x \in aV(-b) \cap dV(-c)$. Then $ax \in V(-b)$, $dx \in V(-c)$ and so $x \notin V(-b)$ and $x \notin V(-c)$. Hence $b \notin V(-x)$ and $c \notin V(-x)$, by C2, and so $bc \in V(-x)$, by Hilberticity. Hence $x \in V(-bc)$ and so $x \in R$. Thus $R \neq \emptyset$ and we are through.

Remark 4.6. Another proof of 4.4 can be given by using quaternionic structures and their relation to quaternionic schemes ([11]). For a 2-element set $Q = \{0, 1\}$ we define the mapping $q\colon G \times G \to Q$ by putting $q(a, b) = 0$ or 1 according as

$b \in V(-a)$ or $b \notin V(-a)$. It turns out that (G, q, Q) is a quaternionic structure in the sense of [8] and its scheme coincides with S. By [1], CM holds for S, so S is a quaternionic scheme.

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Received 9 July 1986

A combinatorial analysis of functions provably recursive in $I\Sigma_n$

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Abstract. We use certain functionals of finite type to define an indicator for $I\Sigma_n$. We show that this indicator is equivalent in $I\Sigma_n$ to an indicator of combinatorial character. The syntactical-combinatorial part is definitely separated from the model — theoretic part. Finally we obtain a simple proof of the estimation of the growth for recursive functions provably total in $I\Sigma_n$.

§ 1. Introduction. This paper is devoted to an application of a family of selected primitive recursive functionals to the investigation of provably recursive functions in $I\Sigma_n$, where $n \geq 1$. We first define the spaces $\bar{F}_k\omega$ on which the above-mentioned functionals are defined Let $\bar{F}_0\omega = \omega$; then we define by induction:

$$\bar{F}_{k+1}\omega = (\bar{F}_k\omega)^{\bar{F}_k\omega}$$

for $k \in \omega$.

We assume that $\bar{I}_1 \colon \omega \to \omega$ is the function of the immediate successor and we define the subsequent functionals by

$$\bar{I}_k(f^{k-1}) \dots (f^1)(x) = (f^{k-1})^{x+1}(f^{k-2}) \dots (f_1)(x)$$

for all $x \in \omega$, $f^1 \in \bar{F}_1\omega, \dots, f^{k-1} \in \bar{F}_{k-1}\omega$.

The functionals belonging to the space $\bar{F}_k\omega$ will be said to be of *type k*. In particular, for every $k \in \omega$, $k \geq 1$, the functional $\bar{I}_k$ is of type k.

The idea of using functionals like $\bar{I}_2, \dots, \bar{I}_{n+1}$ is not new. In [4] Paris presents, referring to Aczél, a sketch of proof that for every $\alpha < \omega_{n+1}$ the existence of α-large sets is provable in $I\Sigma_n$. That proof is based on the use of the above-mentioned functionals.

Unfortunately, a considerable difficulty in reading that proof arises from problems connected with the formalization of the above functionals in arithmetic. Moreover, all lemmas are sketched and it is not obvious that they can be formalized in $I\Sigma_n$.

In the present paper we only formalize functionals of type 1, strictly speaking only those of them which are formed of $\bar{I}_1, \bar{I}_2, \dots$ by means of application and superposition. In order to reach this objective we use a kind of miniaturization of all functionals. This topic is discussed in § 2 and § 3.