which satisfies $F\psi = f^J f^{-1}(B^3 \times 2B^3)$. $\psi$ extends to a homeomorphism $\varphi: f^{-1}(U) \to F^{-1}(U)$ defined by $\varphi = f$ on $f^{-1}((B^3 \setminus \frac{1}{2}B^3) \times R^n)$. This ends the proof of Theorem 2.1.

3. Concluding remarks. To prove Theorem 1.1 we now repeat the argument of [C-F]. First we prove a theorem corresponding to the “Main theorem” of [C-F].

For notation, let $B^n$ be a 3-manifold, $3 = m + k$, and let $F: V \to B^3 \times R^n$ be a proper map such that $\partial V = f^{-1}(\partial B^3 \times R^n)$ and $f$ is a homeomorphism over $(B^3 \setminus \frac{1}{2}B^3) \times R^n$.

**Theorem 3.1** (main theorem). Suppose that $V$ contains no fake 3-cells. Then for every $v > 0$ there exists a $\delta > 0$ such that if $f$ is a $b$-equivalence over $B^3 \times 3B^3$ then there exists a proper map $f^J: V \to B^3 \times R^n$ such that:

1. $f$ is an $e$-equivalence over $B^3 \times 2.5B^3$,
2. $f^J$ is a homotopy equivalence over $[B^3 \setminus \frac{1}{2}B^3) \times R^n] \cup [B^3 \times (R^n \setminus 2B^3)]$,
3. $f^J$ is a homeomorphism over $B^3 \times B^3$.

The proof of 3.1 is precisely as in [C-F]. We have only to use the fact that $V$ and subsets of $B^3 \times S^n$ contain no fake 3-cells, and the 3-dimensional “Splitting theorem” of [J]. Having proved Theorem 3.1, we prove 1.1 as in [C-F].

References


Received 10 October 1984; in revised form 30 May 1985

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On indecomposable representations of quivers with zero-relations

by

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Abstract. Let $A$ be a bound quiver algebra $KQ/I$ with zero-relations and $R = KQ/I$ its universal Galois covering. Applying new covering techniques [5], [6] we give a simple description of indecomposable finite dimensional representations of $A$ in case each indecomposable finite dimensional representation of $R$ has a peak [4].

1. Introduction. It is well known that in many cases [4], [10], [12], [14], [16], [18] the representation theory of finite dimensional algebras over an algebraically closed field can be reduced to that for partially ordered sets, shortly posets. In particular, if $A$ is a tree algebra $KQ/I$ of a finite tree $Q$ with zero-relations $I$, then by [4] $A$ is representation-finite, that is admits only finitely many nonisomorphic finite dimensional indecomposable representations, if and only if the partially ordered sets associated to all vertices of $Q$ are representation-finite, and in this case each indecomposable representation of $A$ has a peak. Similarly, by coverings techniques, the classification problem of indecomposables of a representation-finite quiver algebra with zero-relations can be reduced [12], [14] to that for representation-finite tree algebras (with zero-relations), and consequently to posets.

The purpose of this paper is to give a rather simple description of indecomposable finite dimensional representations of an arbitrary quiver algebra with zero-relations for which every indecomposable finite dimensional representation of its universal Galois covering, being a locally bounded tree category with zero-relations, has a peak. Applying the covering techniques developed recently for representation-finite algebras by the second and third author [5], [6], we reduce the classification problem of indecomposable to that for the corresponding posets and to the classification of indecomposable finite dimensional representations over the algebra $K[T, T^{-1}]$ of Laurent polynomials. In particular, we will show that any such algebra is tame if and only if the corresponding posets are tame.

1. Notation and conventions. Throughout this paper, we denote by $K$ an algebraically closed field. By an observation of Gabriel [3], [11] a basic connected finite dimensional $K$-algebra $A$ can be written as $A = KQ/I$, where $Q$ is a finite connected
Following Gabriel [13], for each finite poset $S$, an $S$-space ($K$-representation of $S$) is a system $V = (V, V_s)_{s \in S}$, where $V$ is a $K$-vector space, $V_s$ are subspaces of $V$ and $V_s \leq V$ provided $x \leq y$. A map $f: V \to V'$ between $S$-spaces is a $K$-linear map $f: V \to V'$ such that $f(V_s) \subseteq V'_s$ for all $x \in S$. The coordinate-vector of an $S$-space $V = (V, V_s)_{s \in S}$ is the family $\text{cdn}(V) = (c_v, c_s; x \in S)$ where $c = (V, K)$ and $c_s = (V_s, K_s), x \in S$, and $\text{cdn}(V)$ is finite if $c$ is finite. We will denote by $S-\text{SP}$ the category of all $S$-spaces and by $S$-sp its full subcategory formed by all $V \in S-\text{SP}$ with $\text{cdn}(V)$ finite. A finite poset $S$ is called tame if for any finite coordinate-vector $c = (c_v, c_s; x \in S)$ there is a finite (parametrizing) family of representations $(U^i, U^j \rightarrow U, i = 1, \ldots, n)$ of $S$ in the category of finitely generated free right $K[T]$-modules such that all but a finite number of indecomposable $S$-spaces (over $K$) of coordinate-vector $c$ are of the form $(U^i \otimes N, \text{Im}(U^i \otimes N \rightarrow U^i \otimes N), S)$ where $i \subseteq U^i \rightarrow U^i$ is the canonical injection and $N$ is a simple $K[T]$-module. L. A. Nazarova proved in [17] that a finite poset $S$ is tame if and only if $S$ does not contain full subposet whose Hasse-diagram has one of the forms...

\[ \begin{array}{c}
\text{Indecomposable finite dimensional representations of tame posets are described in} \\
[1], [21], [17], [19].
\end{array} \]

Following Bongartz and Ringel [4] an $R$-module $M$ has a peak $x \in R$, if for each arrow $a$ leading to $x$, $M(a)$ is an injection and for each arrow $b$ going away from $x, M(b)$ is a surjection, where an arrow $a: y \rightarrow z$ is said to be lead to $x$ provided $y$ and $x$ belong to the same connected component of $\mathcal{S}(y, y')$, otherwise $y$ is said to be go away from $x$. For each object $x$ of $R$, we denote by $\mathcal{P}(x)$ the full subcategory of $\text{mod}_R$ consisting of all representations having peak $x$. Moreover, for a connected convex subcategory $A$ of $R$, we will denote by $\mathcal{P}(A)$ the full subcategory of $\mathcal{P}_R$ formed by all objects with support in $A$. Recall that a full subcategory $C = K[T]/K[T] \cap T$ of $R$ is called convex if all vertices of any path in $\mathcal{Q}$ connecting two points of $T$ belong to $T$. It is shown in [4] that the set $S_a$ (resp. $S_a(A)$) of all walks in $R$ (resp. in $A$) with end point $x$, having no subsequence of the form $w_i, x_i^-, w_i$ or $w_i^-$ with $w \in T$, admits a partial ordering and there are two canonical functors $G_a(A), \mathcal{P}_a(A) \rightarrow S_a(A)_{-a}$.

\begin{align*}
G_a(A): & \mathcal{P}_a(A) \rightarrow S_a(A)_{-a}, \\
\mathcal{P}_a(A): & S_a(A)_{-a} \rightarrow \mathcal{P}_a(A)
\end{align*}

which yield an equivalence...
of $\mathcal{S}_a(A)$-sp. Moreover, every $S_x(A)$ is a full subposet of $S_x$ and $S_x = \cup S_x(A)$ where the sum is taken over all finite convex subcategories $A$ of $R$. Finally, observe that for $x \in R$, $g \in G$, $S_x = S_{xG}$, and hence we can associate with each vertex $Gx$ of $Q$ a partially ordered set $S_{Gx} = S_x$.

2. Main results. In this section we formulate the main results of the paper. Let us start with some definitions (see [6]). A full convex subcategory $L$ of $R$ is called a line provided $L$ is isomorphic to the quiver algebra $KQ_L$, where $Q_L$ is a linear quiver of type $A_n$, $A_m$ or $A_2$. A line $L$ is called $G$-periodic if $G_L = \{y \in G ; gy = L\}$ is nontrivial. Since $G$ acts freely on the objects of $R$, for any $G$-periodic line $L$, $G_L$ is an infinite cyclic subgroup of $G$. The group $G$ acts on the set $L$ of all $G$-periodic lines of $R$ and denote by $L_0$ some fixed set of representatives of $G$-orbits in $L$.

For each $L \in L_0$, denote by $B_L$ the indecomposable locally finite dimensional $R$-module with $\text{supp} B_L = L$ defined by setting $B_L(x) = K$ for any object $x$ of $L$ and $B_L(\alpha) = id_K$ for any arrow of $Q_L$. Then $G_L$ acts on $L_0$ by $G_L \leq B_L \cong B_L$. The group algebra $K[G_L]$ is the algebra of Laurent polynomials $K[T, T^{-1}]$, $F_1 B_L$ has a structure of an $A-K[T, T^{-1}]$-bimodule and we have a functor

$$\phi^L : F_1 B_L \otimes_{K[T, T^{-1}]} \cdot \mod K[T, T^{-1}] \rightarrow \mod A$$

where $mod K[T, T^{-1}]$ denotes the category of all finite dimensional (over $K$) left $K[T, T^{-1}]$-modules.

Moreover, denote by $G$ the family of all finite convex connected subcategories of $R$ and by $G_0$ a fixed set of representatives of the $G$-orbits in $G$. For any $G \in G_0$, $x \in A$, consider the composed functor

$$\psi_x(A) : S_x(A)-\text{sp} \rightarrow \mathcal{P}_a(A) \rightarrow \pi F_1 \mod R \rightarrow \mod A$$

We can now formulate our main result.

**Theorem 1.** Assume that each indecomposable finite dimensional $R$-module has a peak. Then in our notation the following statements hold:

(i) Every finite dimensional indecomposable $A$-module is isomorphic either to $\psi_x(A)(V)$ for some $x \in R$, $A \in G_0$ containing $x$ and an indecomposable finite dimensional $S_x(A)$-space $V$, or to $\phi^L(W)$ for some $L \in L_0$ and an indecomposable finite dimensional $K[T, T^{-1}]$-module $W$.

(ii) $A$ is tame if and only if for each poset $S_x$ associated with each vertex $a$ of $Q$, contains no full subposet whose Hasse-diagram has one of the forms $(1, 1, 1, 1)$, $(1, 1, 1, 2)$, $(2, 2, 3)$, $(1, 3, 4)$, $(1, 2, 6)$ or $(N, 5)$.

For each $x \in R$, let $R_x$ be the full subcategory of $R$ formed by all objects of $supp M$, for all $M \in \text{ind} R$ such that $M(x) \neq 0$. Following [5] $R$ is called locally support-finite if $R_x$ is finite (has a finite number of objects) for each object $x$ of $R$. Recall also that $G$ acts freely on $\text{Ind} R$ if $\text{supp} M$ $\neq M$ for any $M \in \text{Ind} R$ and each $1 \neq g \in G$. From a general fact [5] it is known that if $R$ is locally support-finite then every $x \in \text{Ind} R$ is of the form $F_1 M$ for some $M \in \text{ind} R$. The following theorem shows that in our case the converse is also true.

**Theorem 2.** Assume that each indecomposable finite dimensional $R$-module has a peak. Then the following statements are true:

(i) Every finite dimensional indecomposable $A$-module is isomorphic to $\psi_x(A)(V)$ for some $x \in R$, $A \in G_0$ containing $x$, and an indecomposable finite dimensional $S_x(A)$-space $V$.

(ii) $R$ is locally support-finite.

(iii) For each vertex $a$ of $Q$, $S_a$ is finite.

(iv) $G$ acts freely on $\text{Ind} R$.

(v) $\text{Ind} R = \text{Ind} R$.

3. Tameness of a tree category. The main aim of this section is to prove the following proposition which we shall use in the proof of Theorem 1.

**Proposition 1.** Let $A$ be a finite locally bounded tree category such that each indecomposable finite dimensional $A$-module has a peak. Then $A$ is tame if and only if for all posets $S_x(A)$, $x \in A$, are tame.

For the proof of this proposition we need the following lemma.

**Lemma 1.** (a) Let $\Gamma$ be an integral domain and $\phi : \Gamma^* \rightarrow \Gamma^*$ be a monomorphism of left free $\Gamma$-modules where $m, n \in N$. Then there exists an element $h \in \Gamma$ such that $1 \otimes \phi : (\Gamma_h \otimes \Gamma^*) \rightarrow (\Gamma_h \otimes \Gamma^*)$ is a split monomorphism of left free $\Gamma$-modules.

(b) Let $\Gamma = K[T_{mn}]$, for some $h \in K[T]$ and $\phi : \Gamma^* \rightarrow \Gamma^*$ be a homomorphism of free left $\Gamma$-modules given by $(m \times n)$-matrix $[f_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$ with coefficients in $\Gamma$. If $\phi$ is a homomorphism of $K$-vector spaces $\phi(i) : K^n \rightarrow K^m$, defined by the scalar matrix $[f_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$, is a monomorphism (resp. epimorphism) for some $i \in K$ which is not a root of $h$, then so is $\phi$ (resp. $1 \otimes \Gamma_h$, for some $h \in A$).

Proof. We start with the following remark. Let $\Gamma$ be a commutative ring and $\phi : \Gamma^* \rightarrow \Gamma^*$ be a homomorphism given by $m \times n$-matrix $m^{ij}_{1 \leq i \leq m, 1 \leq j \leq n}$ having a nonzero minor of order $p$, where $p = \min(n, m)$; then $1 \otimes \phi$ is a split monomorphism (resp. epimorphism) if $m \geq n$ (resp. $m \leq n$). Indeed, if $m \geq n$ and, for example, the matrix $F = [F_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$ has nonzero determinant $h$, then the matrix $F$ considered as a matrix with coefficients in $\Gamma_h$ is invertible and a homomorphism $\phi : (\Gamma_h)^m \rightarrow (\Gamma_h)^n$ defined by matrix $(F)^{-1}$ furnishes a retraction for $1 \otimes \phi$. Dual case can be proved analogously.

Now we will prove (a). Let $\Gamma$ and $\phi$ satisfy the assumption of (a). Observe that $\phi$ produces a monomorphism of $\Gamma(q)$ vector spaces $\phi(q)^m \rightarrow \phi(q)^n$. Then there exists a nonzero minor $h$ of order $n$ and (a) is a consequence of the remark.
are finite, by Lemma 1 we can assume that all inclusions in \( U_i^p \) are splittable monomorphism. Hence, by the definition of \( G_i(A), G_i(A)(U_i^p \otimes N) = G_i(A)(U_i^p) \otimes N \) for any \( i \) and \( N \). Thus the family of bimodules \( G_i(A)(U_i^p) \) for \( x \in A \) and \( i = 1, \ldots, n_x \) parametrizes all but a finite set of isoclasses of indecomposable \( A \)-modules of dimension-vector \( d \). Consequently by [7, Proposition 1] \( A \) is tame.

4. Proof of Theorem 1. Assume that each indecomposable finite dimensional \( R \)-module has a peak. Let \( A = KT_r \), where \( T \) is a connected subcategory of \( \mathcal{O} \) and let \( I = KT_r \mathcal{O} \), be a convex subcategory of \( R \). Denote by \( T' \) the set of vertices of \( T \) and by \( T' \) the set of its arrows. The degree \( deg_\mathcal{O}(x) \) of a vertex \( x \in T' \) is the number of times \( x \) is used as an endpoint or a startpoint of the arrows in \( T \). A vertex \( x \in T' \) degs_\mathcal{O}(x) = 1 is called a tip of \( T \). Denote by \( A(T) \) the maximum of degs_\mathcal{O}(x) for \( x \in T' \) and by \( \Omega(T) \) the number of tips of \( T \) if it is finite or infinite otherwise. Recall that \( M \in \text{Ind}A \) is sincere if \( M(x) \neq 0 \) for all \( x \in T' \). For each sincere \( M \in \text{Ind}A \) we will denote by \( A_M = KT_r M \) the shrunk algebra of \( A \) obtained by shrinking (in the sense of [4, §3]) of all arrows \( x \rightarrow y \) for which \( M(y) \) is isomorphic and \( x, y \) are not tips of \( T \), and by \( M \) the \( A_M \)-module corresponding to \( M \). Then \( \text{mod}A_M \) can be interpreted as the full subcategory of \( \text{mod}A \) containing only modules for which all these shrunk arrows \( \beta \) are represented by isomorphisms.

Lemma 2. Assume \( T \) is finite and \( A(T) \geq 2 \). Then, for any sincere indecomposable finite dimensional \( A \)-module \( M \), \( A(T) \leq 2d(T) - 2 \).

Proof. Obviously, for \( A(T) = 2 \), \( A(T) = A(T) = 2d(T) - 2 \). Assume \( A(T) > 3 \), since any quiver algebra of a Dynkin type \( D_n, n \geq 5 \), has an indecomposable module without peak and \( M \) is sincere, \( T \) has no subtree of the form

\[ \begin{array}{cccccc}
  &  &  &  & a_i & \\
  & a_1 & \cdots & a_n & & \\
  & & & & & \\
 & & & & & \\
 & & & & & \\
\end{array} \]

where \( n \geq 1 \) and \( M(a_1), \ldots, M(a_n) \) are isomorphisms. Then a simple analysis yields the required inequality.

Lemma 3. Assume \( T \) is finite and \( 2 \leq \text{ord}(A) \leq e \). Then for any \( M \in \text{Ind}A \) with \( (M(x); K) \leq d \) for all \( x \in T' \), holds \( \Omega(\text{supp}M) \leq (2e - 2j)^2 \).

Proof. First observe that \( \Omega(\text{supp}M) = \Omega(\text{supp}M') \) and for each arrow \( y \rightarrow z \) in \( T_M \) such that \( y \neq z \) are not tips, \( M(x) \) is a proper monomorphism or a proper epimorphism. Let \( x \) be a peak of \( M \) in \( A \). Then \( x \) is a peak of \( M \) in \( A_M \) and from the above remark, the lengths of the walks

\[ \begin{array}{cccccc}
  & x_1 & \cdots & x_r & \\
  & & & & & \\
 & & & & & \\
\end{array} \]

are bounded by \( (2e - 2j)^2 \)
in \( T_y \) are bounded by \( d \). Since by Lemma 2, \( A(T_y) \leq 2d - 2 \), we get \( \Omega(\text{supp} M) \leq (2e - 2)^d \).

**Proof of Theorem 1.** In order to prove (i), it is enough by [6, Theorem 3.6] to show that the support of any \( Y \in \text{Ind} R \) with \( G_Y = \{ g \in G; Y \cong Y \} \) nontrivial and \( \text{supp} Y / G_Y \) finite is a line. Let \( Y \) be such an \( R \)-module. We will apply fundamental sequences introduced in [6, §4]. For a full subcategory \( C \) of \( R \) we will denote by \( \tilde{C} \) the full subcategory of \( R \) formed by all objects \( x \) such that \( R(x, y) \neq 0 \) or \( R(y, x) \neq 0 \) for some \( y \in C \). For \( X \) and \( Y \in \text{MOD} C \), \( X \cong Y \) denotes that \( X \) is isomorphic to a direct summand of \( Y \). Moreover, if \( D \) is a full subcategory of \( C \) and \( X \in \text{MOD} C \), then \( X[D] \) is the restriction of \( X \) to \( D \). Denote by \( C_n, n \in \mathbb{N} \), a fixed family of finite full subcategories of \( R \) defined by setting \( C_0 \) is a subcategory given by a fixed object \( x \in R \) and \( C_{n+1} = C_n \), for \( n \in \mathbb{N} \). Observe that \( R \) is the union of \( C_n, n \in \mathbb{N} \). Then \( Y \) produces a sequence \( (Y_n, u_n)_{n \in \mathbb{N}} \) where \( Y_n = \text{supp} C_n \) and \( u_n : Y_n \to Y_{n+1} \), is a \( C_{n+1} \)-homomorphism for all \( n \in \mathbb{N} \), satisfying the following conditions:

(a) For each \( n \in \mathbb{N} \), \( Y_n \in \text{ind} C_n \) or \( Y_n = 0 \).
(b) \( Y_n \neq 0 \) for some \( n \in \mathbb{N} \).
(c) For each \( n \in \mathbb{N} \), \( u_n \) is a splittable monomorphism.
(d) For each \( n \in \mathbb{N} \), \( Y_n \in \text{supp} C_n \).

Observe that, since \( \text{supp} Y / G_Y \) is finite and \( Y \) is locally finitely dimensional, there is a common bound \( d \) for \( (Y(x); y \in R, n \in \mathbb{N}) \). Moreover, we know by [6, Corollary 4.4] that \( Y = \lim (Y_n, u_n) \). Let \( D_n = \text{supp} Y_n, n \in \mathbb{N} \), and \( D = \text{supp} Y \). Obviously, \( D = \bigcup D_n \). Moreover \( D = KTP_{DP}, D_n = KTP_{DP}, n \in \mathbb{N} \), for connected sets.

Subtrees \( T_D \) and \( T_{DP} \) of \( \tilde{Q} \), and \( I_D = KTP_{DP} \), \( I_n = KTP_{DP} \), \( D \) is a line if and only if \( A(T_D) \leq 2 \). Suppose \( A(T_D) \geq 3 \) and denote by \( U \) the set of all vertices \( x \) of \( T_D \) with \( \text{deg}_{T_D}(x) \geq 3 \). We claim that \( U \) is finite. Indeed, \( e = A(\tilde{Q}) = A(Q) \) is finite and hence \( 2 \leq A(T_{DP}) \leq e \) for \( D \) containing at least three objects, say for \( n \geq n_0 \). Thus there is \( m \geq n_0 \) such that \( A(T_{DP}) = A(T_{DP}) \) for \( n \geq m \). Hence \( U \) is contained in \( T_{DP} \), and consequently is finite. On the other hand, \( G_U = \{ g \in G; gD = D \} \) contains \( G = \{ g \in G; Y \cong Y \} \) so there is \( 1 \neq g \in G \) such that \( gU = U \). But this is impossible since \( G \) is free and \( U \) is finite. Therefore, \( A(T_D) \leq 2 \) and \( D \) is a line. This finishes the proof of (i).

In order to prove (ii) we observe that for each \( x \in R \) the posets \( S_x(A), \) where \( A \in S \) with \( x \in A \), have a family of full subposets \( S_x \) such that any finite subposet of \( S_x \) is contained in some \( S_x(A) \). Then (ii) is an immediate consequence of Proposition 1 and [7, Theorem 5].

**5. Proof of Theorem 2.** (i) \( \implies \) (iii). Assume that every indecomposable finite dimensional \( A \)-module \( M \) is of the form \( F_k N \) for some \( N \in \text{Ind} R \). We will show that \( S_x \) is a finite poset for any vertex \( x \) of \( Q \). Suppose that \( S_x \) is finite for some \( x \in \tilde{Q} \). Then we claim that \( R \) contains a \( G \)-periodic line. Indeed, \( \tilde{Q} \) is a locally finite tree and \( I \) contains the ideal \( J \) of \( \tilde{Q} \), generated by all paths of length \( n, \) for some \( n \in \mathbb{N} \), so there exist finite lines in \( R \) with arbitrary large number of changes of orientation. Thus, since \( Q \) is finite, there exists a finite line \( C \) in \( R \) whose tips \( b \) and \( c \) are both ending points and belong to the same \( G \)-orbit in \( R \). Hence the full subcategory \( L \) of \( R \) formed by the union of subcategories \( g^* C, z \in Z \), where \( gb = c \), is a \( G \)-periodic line. Then, since \( G_{B_n} = \{ g \in G; gB_n \cong B_n \} \) is nontrivial, we get a contradiction with [6, Proposition 2.4].

(ii) \( \implies \) (i). First observe that if an indecomposable \( R \)-module \( M \) has a peak \( x \) and \( M(y) \neq 0 \) then there exists a (unique) finite line in \( R \) with tips \( x \) and \( y \). Assume that all posets \( S_x, x \in \tilde{Q} \), are finite. We will show that the category \( R \) is locally support-finite. For any object \( x \in R \), denote by \( U_x \) the set of all \( y \in R \) such that there exists a finite line in \( R \) with \( y \) as tip and \( x \) as another tip. The sets \( U_x, x \in R \), are finite because the posets \( S_x, x \in \tilde{Q} \), are finite. Take any objects \( x \in R \) and \( x \in \text{ind} R \), with \( x \neq x \). Since by our general assumption \( X \) has a peak, from the above observation \( \text{supp} X \) is contained in the full subcategory \( W_x \) of \( R \) formed by the union of sets \( U_x, y \in U_x \). Consequently \( R \) is locally support-finite because, for each \( x \in R \), \( R_x \) is contained in the finite category \( W_x \).

(iii) \( \implies \) (iv). Follows from [6, Proposition 2.5].

(iv) \( \implies \) (i). Follows directly from the fact that \( G \) is free.

(i) \( \implies \) (ii). This is a consequence of [6, Corollary 2.3].

**6. Examples.** We end the paper with examples illustrating Theorem 1. Let \( A \) be the bound quiver algebra \( KQ/I \) where \( Q \) is the quiver

and \( I \) is the ideal in \( KQ \) generated by the elements \( \beta_x, \sigma_v, \eta_y, \mu \beta, \mu \sigma, \mu \eta, \mu \nu \) and \( \xi_y \). Similarly as in [6, 3.3], one proves that the support of any indecomposable \( R \)-module, where \( R = KQ/I \), is the universal cover of \( A \), either a finite line or a subquiver of the quiver.
Thus every indecomposable finite dimensional $R$-module has a peak, by well-known classification of indecomposable representations of the above quiver of extended Dynkin type $D_n$. Moreover, the partially ordered sets associated with the vertices of $Q$ have the following form:

\[
\begin{array}{c}
\begin{array}{c}
S_1 \\
S_2 \\
S_3 \\
S_4
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
S_1 \cap S_2 \\
S_3 \\
S_4 \cap S_5
\end{array}
\end{array}
\]

Hence by Theorem 1(ii) $A$ is tame.

Finally, we shall present an example showing that the assumption in Theorem 1 is for its form essential. Let $A' = KQ'/I'$ be the zero-relation algebra given by the quiver

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{array}
\]

and the ideal $I'$ in $KQ'$ generated by the element $a^2$, and $R'$ the universal cover of $A'$. One can show that the classification problem of indecomposable finite dimensional $R'$-modules contains difficult classification problem for the Ringel's pattern $\left( \frac{B_n}{n-2} \right)$ [18]. Hence there are indecomposable finite dimensional $R'$-modules without peak and thus they are not determined by indecomposable representations of the associated posets. Moreover, there are modules $Y \in \text{Ind } R'$ with nontrivial stabilizers and nonlinear supports which create 1-parameter series of indecomposable finite dimensional $A'$-modules which are not in the image of the push-down functor $F$: $\text{mod } R' \to \text{mod } A'$ (modules of the second kind in the sense of [6]).

References

Quadratic form schemes and quaternionic schemes

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Abstract. Quaternionic schemes, quaternionic structures and abstract Witt rings are known to be equivalent abstract versions of the algebraic theory of quadratic forms. This paper establishes a relationship between quadratic form schemes and the three other axiomatic approaches to quadratic forms. It is shown that cancelative quadratic form schemes coincide with quaternionic schemes.

Introduction. The algebraic theory of quadratic forms focuses on quadratic forms over fields. However, it has become clear that some parts of the theory are best treated by using an appropriate abstract language. Several authors have had ideas of this kind and as a result we are confronted with at least four distinct abstract approaches to quadratic form theory. These are:

(i) Quadratic form schemes ([2], [4], [5], [6], [10], [11], [12]).
(ii) Quaternionic structures ([11], [8], [9]).
(iii) Abstract Witt rings ([1], [8], [9] and earlier papers cited there).
(iv) Quaternionic schemes ([1]).

The relationships among (ii), (iii), and (iv) are fully known. Marshall's book [8] shows that (ii) and (iii) are equivalent and Carson and Marshall [1] prove that (i) and (iv) are equivalent. It is the aim of this paper to clarify the role of (i) among the abstract theories of quadratic forms.

In Section 1 we exhibit several equivalent sets of axioms for (i) and in Section 2 we do the same for (iv). In both cases we have found that the generally accepted sets of axioms for (i) and (iv) are dependent and we reduce the number of axioms in each case to a pair of independent axioms and even to a single axiom in each case.

Section 3 explains the relationship between two concepts of isometry of forms used in abstract theories (chain isometry and inductively defined isometry following [8] and [9]). The main result, Theorem 3.5, establishes the actual equivalence of quadratic form schemes with cancellation property and quaternionic schemes. A corollary to this result asserts that the classical Witt cancellation theorem for forms of any dimension is a consequence of the cancellation property for 2-dimensional forms.

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