which satisfies $F\psi = f|f^{-1}(B^k\times 2B^m)$. ψ extends to a homeomorphism $\varphi\colon f^{-1}(U)\to F^{-1}(U)$ defined by $\varphi = f$ on $f^{-1}((B^k\setminus\frac{5}{6}\mathring{B}^k)\times R^m)$. This ends the proof of Theorem 2.1.

3. Concluding remarks. To prove Theorem 1.1 we now repeat the argument of [C–F]. First we prove a theorem corresponding to the "Main theorem" of [C–F].

For notation, let V^3 be a 3-manifold, $3 = m+k$, and let $F\colon V\to B^k\times R^m$ be a proper map such that $\partial V = f^{-1}(\partial B^k\times R^m)$ and f is a homeomorphism over $(B^k\times\frac{1}{2}\mathring{B}^k)\times R^m$.

THEOREM 3.1 (main theorem). *Suppose that V contains no fake 3-cells. Then for every $\varepsilon>0$ there exists a $\delta>0$ such that if f is a δ-equivalence over $B^k\times 3B^m$ then there exists a proper map $\tilde{f}\colon V\to B^k\times R^m$ such that:*

(1) *$\tilde{f}$ is an ε-equivalence over $B^k\times 2.5B^m$,*

(2) *$\tilde{f} = f$ over $[(B^k\setminus\frac{2}{3}\mathring{B}^k)\times R^m]\cup[B^k\times(R^m\setminus 2\mathring{B}^m)]$,*

(3) *$\tilde{f}$ is a homeomorphism over $B^k\times B^m$.*

The proof of 3.1 is precisely as in [C–F]. We have only to use the fact that V and subsets of $B^k\times S^m$ contain no fake 3-cells, and the 3-dimensional "Splitting theorem" of [J]. Having proved Theorem 3.1, we prove 1.1 as in [C–F].

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On indecomposable representations of quivers with zero-relations

by

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Abstract. Let A be a bound quiver algebra KQ/I with zero-relations and $R = KQ/\tilde{I}$ its universal Galois covering. Applying new covering techniques [5], [6] we give a simple description of indecomposable finite dimensional representations of A in case each indecomposable finite dimensional representation of R has a peak [4].

0. Introduction. It is well known that in many cases [4], [10], [12], [14], [16], [18] the representation theory of finite dimensional algebras over an algebraically closed field can be reduced to that for partially ordered sets, shortly posets. In particular, if A is a tree algebra KQ/I of a finite tree Q with zero-relations I, then by [4] A is representation-finite, that is admits only finitely many nonisomorphic finite dimensional indecomposable representations, if and only if the partially ordered sets associated to all vertices of Q are representation-finite, and in this case each indecomposable representation of A has a peak. Similarly, by coverings techniques, the classification problem of indecomposables of a representation-finite quiver algebra with zero-relations can be reduced [12], [14] to that for representation-finite tree algebras (with zero-relations), and consequently to posets.

The purpose of this paper is to give a rather simple description of indecomposable finite dimensional representations of an arbitrary quiver algebra with zero-relations for which every indecomposable finite dimensional representation of its universal Galois covering, being a locally bounded tree category with zero-relations, has a peak. Applying the covering techniques developed recently for representation-infinite algebras by the second and third author [5], [6], we reduce the classification problem of indecomposable to that for the corresponding posets and to the classification of indecomposable finite dimensional representations over the algebra $K[T, T^{-1}]$ of Laurent polynomials. In particular, we will show that any such algebra is tame if and only if the corresponding posets are tame.

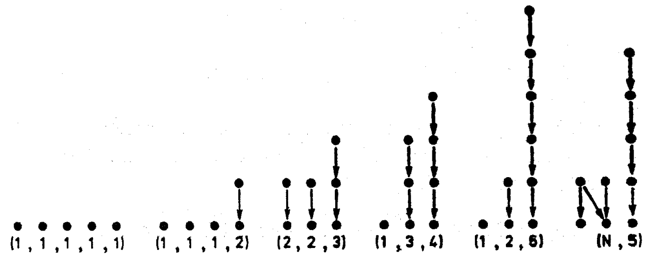
1. Notation and conventions. Throughout this paper, we denote by K an algebraically closed field. By an observation of Gabriel [3], [11] a basic connected finite dimensional K-algebra A can be written as $A = KQ/I$, where Q is a finite connected quiver, I is a two-sided ideal in the quiver algebra KQ [3], [11] with $J^n \subset I \subset J^2$, $n \geqslant 2$, and J is the ideal in KQ generated by all arrows of Q. In case I is generated by a set of paths of length $\geqslant 2$, (Q, I) is called a *bound quiver with zero-relations* and $A = KQ/I$ an *algebra with zero-relations*. In the sequel we assume that $A = KQ/I$ for a bound quiver (Q, I) with zero-relations. We will consider both the algebra A and the K-linear category associated with (Q, I), and representations will be left modules or K-linear covariant functors from A to the category of K-vector spaces. Applying classical topological covering construction [15, Chap. 9] we get the *universal cover* $\tilde{Q} \xrightarrow{\pi} Q$ with the *fundamental group* G of Q. $\tilde{Q}$ is a locally finite tree, G is free, and π induces the canonical *Galois covering functor* $F\colon R \to A$ with group G [12], where $R = K\tilde{Q}/\tilde{I}$ is a *locally bounded* K-*category* [3] and $\tilde{I}$ is the ideal in $K\tilde{Q}$ generated by all paths ω in $\tilde{Q}$ such that $\pi(\omega) \in I$, which assigns to each object x of R its G-orbit Gx. An R-module is a K-linear covariant functor from R to the category of K-vector spaces. The *support* $\operatorname{supp} M$ of an R-module M is the full subcategory of R formed by all objects x of R such that $M(x) \neq 0$. The *dimension-vector* of an R-module M is the family $\dim M = (M(x)\colon K)_{x \in R}$ and $\dim M$ is finite if the dimension $\dim M = \sum_{x \in R} (M(x)\colon K)$ of M is finite. An R-module M is called *locally finite dimensional* if $(M(x)\colon K)$ is finite for all objects x in R, and M is finite dimensional if $\dim M$ is finite. We will denote by $\mathrm{MOD}\,R$ the category of all R-modules and by $\mathrm{Mod}\,R$ (resp. $\mathrm{mod}\,R$) the full subcategory of $\mathrm{MOD}\,R$ formed by all locally finite dimensional (resp. finite dimensional) R-modules. Further, we will denote by $\mathrm{Ind}\,R$ (resp. $\mathrm{ind}\,R$) the full subcategory of $\mathrm{Mod}\,R$ (resp. $\mathrm{mod}\,R$) formed by indecomposable modules. The group G acts on $\mathrm{MOD}\,R$ by the translations ${}^g(-)$, $g \in G$, which assign to each module M the R-module ${}^gM = M(g^{-1}(-))$. Finally denote by $F_\lambda\colon \mathrm{MOD}\,R \to \mathrm{MOD}\,A$ the *push-down functor* [3] which assigns to each R-module M the A-module $F_\lambda M$ such that $(F_\lambda M)(Gx) = \bigoplus_{g \in G} M(gx)$. It is well known [12, 3.5] that F_λ induces a bijection between the G-orbits of isoclasses of indecomposable objects in $\mathrm{mod}\,R$ and the isoclasses of indecomposable A-modules of form $F_\lambda M$, for $M \in \mathrm{ind}\,R$.

Following Drozd [8], [9] a locally bounded K-category Λ is called *tame* if for any finite dimension-vector d in Λ there exists a finite (parametrizing) family of functors $U_j \underset{K[T]}{\otimes} -\colon \mathrm{mod}\,K[T] \to \mathrm{mod}\,\Lambda$, $i = 1, \ldots, n_d$, where $K[T]$ is the polynomial algebra in one variable T, satisfying the conditions (a) and (b) below:

(a) For each i, $1 \leqslant i \leqslant n_d$, U_i is an Λ-$K[T]$-bimodule and a finitely generated free right $K[T]$-module.

(b) Every indecomposable Λ-module M with $\dim M = d$ is of the form $M \cong F_i(N)$ for some i and some simple $K[T]$-module N. For equivalent definitions of tame locally bounded categories we refer to [7]. One can prove (see [7]) that a locally bounded category Λ is tame if and only if every finite full subcategory of Λ is tame.

Following Gabriel [13], for each finite poset S, an S-*space* (K-representation of S) is a system $\underline{V} = (V, V_x)_{x \in S}$, where V is a K-vector space, V_x are subspaces of V and $V_x \subset V_y$ provided $x \leqslant y$. A map $f\colon \underline{V} \to \underline{V}'$ between S-spaces is a K-linear map $f\colon V \to V'$ such that $f(V_x) \subset V'_x$ for all $x \in S$. The *coordinate-vector* of an S-space $\underline{V} = (V, V_x)_{x \in S}$ is the family $\mathrm{cdn}(\underline{V}) = (c, c_x;\ x \in S)$ where $c = (V\colon K)$ and $c_x = (V_x\colon K)$, $x \in S$, and $\mathrm{cdn}(\underline{V})$ is finite if c is finite. We will denote by S-SP the category of all S-spaces and by S-sp its full subcategory formed by all $\underline{V} \in S$-SP with $\mathrm{cdn}(\underline{V})$ finite. A finite poset S is called *tame* if for any finite coordinate-vector $\underline{c} = (c, c_x;\ x \in S)$ there is a finite (parametrizing) family of representations $(U^i, U^i_x)_{x \in S}$, $i = 1, \ldots, n$, of S in the category of finitely generated free right $K[T]$-modules such that all but a finite number of indecomposable S-spaces (over K) of coordinate-vector $\underline{c}$ are of the form $\big(U^i \underset{K[T]}{\otimes} N, \mathrm{Im}(U^i_x \underset{K[T]}{\otimes} N \xrightarrow{i_x \otimes 1} U^i \underset{K[T]}{\otimes} N)\big)_{x \in S}$ where $i_x\colon U^i_x \to U^i$ is the canonical injection and N is a simple $K[T]$-module. L. A. Nazarowa proved in [17] that a finite poset S is tame if and only if S does not contain full subposet whose Hasse-diagram has one of the forms



Indecomposable finite dimensional representations of tame posets are described in [1], [2], [17], [19].

Following Bongartz and Ringel [4] an R-module M has a *peak* $x \in R$, if for each arrow α leading to x, $M(\alpha)$ is an injection and for each arrow β going away from x, $M(\beta)$ is a surjection, where an arrow $\gamma\colon y \to z$ is said to be *lead to* x provided z and x belong to the same connected component of $\tilde{Q} \setminus \{\gamma\}$, otherwise γ is said to be *go away from* x. For each object x of R, we denote by $\mathscr{P}_x$ the full subcategory of $\mathrm{mod}\,R$ consisting of all representations having peak x. Moreover, for a connected convex subcategory Λ of R, we will denote by $\mathscr{P}_x(\Lambda)$ the full subcategory of $\mathscr{P}_x$ formed by all objects with support in Λ. Recall that a full subcategory $C = KT/KT \cap \tilde{I}$ of R is called *convex* if all vertices of any path in $\tilde{Q}$ connecting two points of T belong to T. It is shown in [4] that the set S_x (resp. $S_x(\Lambda)$) of all walks in R (resp. in Λ) with end point x, having no subsequence of the form $\alpha\alpha^{-1}$, $\alpha^{-1}\alpha$, w or w^{-1} with $w \in \tilde{I}$, admits a partial ordering and there are two canonical functors $F_x(\Lambda)\colon \mathscr{P}_x(\Lambda) \to S_x(\Lambda)$-sp, $G_x(\Lambda)\colon S_x(\Lambda)$-sp $\to \mathscr{P}_x(\Lambda)$ which yield an equivalence of $\mathscr{P}_x(\Lambda)$ and $S_x(\Lambda)$-sp. Moreover, every $S_x(\Lambda)$ is a full subposet of S_x and $S_x = \bigcup S_x(\Lambda)$ where the sum is taken over all finite convex subcategories Λ of R. Finally, observe that for $x \in R$, $g \in G$, $S_x = S_{gx}$, and hence we can associate with each vertex Gx of Q a partially ordered set $S_{Gx} = S_x$.

2. Main results. In this section we formulate the main results of the paper. Let us start with some definitions (see [6]). A full convex subcategory L of R is called a *line* provided L is isomorphic to the quiver algebra KQ_L where Q_L is a linear quiver of type A_n, A_∞ or A_∞^∞. A line L is called *G-periodic* if $G_L = \{y \in G;\ gL = L\}$ is nontrivial. Since G acts freely on the objects of R, for any G-periodic line L, G_L is an infinite cyclic subgroup of G. The group G acts on the set $\mathscr{L}$ of all G-periodic lines of R and denote by $\mathscr{L}_0$ some fixed set of representatives of G-orbits in $\mathscr{L}$. For each $L \in \mathscr{L}_0$ denote by B_L the indecomposable locally finite dimensional R-module with $\operatorname{supp} B_L = L$ defined by setting $B_L(x) = K$ for any object x of L and $B_L(\alpha) = \mathrm{id}_K$ for any arrow of Q_L. Then G_L equals to $\{g \in G;\ {}^gB_L \cong B_L\}$, the group algebra $K[G_L]$ is the algebra of Laurent polynomials $K[T, T^{-1}]$, $F_\lambda B_L$ has a structure of an A-$K[T, T^{-1}]$-bimodule and we have a functor

$$\Phi^L = F_\lambda B_L \underset{K[T,T^{-1}]}{\otimes} -: \operatorname{mod} K[T, T^{-1}] \to \operatorname{mod} A$$

where $\operatorname{mod} K[T, T^{-1}]$ denotes the category of all finite dimensional (over K) left $K[T, T^{-1}]$-modules.

Moreover, denote by $\mathscr{D}$ the family of all finite convex connected subcategories of R and by $\mathscr{D}_0$ a fixed set of representatives of the G-orbits in $\mathscr{D}$. For any $\Lambda \in \mathscr{D}_0$, $x \in R$, consider the composed functor

$$\psi_x(\Lambda)\colon S_x(\Lambda)\text{-sp} \xrightarrow{G_x(\Lambda)} \mathscr{P}_x(\Lambda) \hookrightarrow \operatorname{mod} R \xrightarrow{F_\lambda} \operatorname{mod} A.$$

We can now formulate our main result.

THEOREM 1. *Assume that each indecomposable finite dimensional R-module has a peak. Then in our notation the following statements hold*:

(i) *Every finite dimensional indecomposable A-module is isomorphic either to $\psi_x(\Lambda)(V)$ for some $x \in R$, $\Lambda \in \mathscr{D}_0$ containing x and an indecomposable finite dimensional $S_x(\Lambda)$-space V, or to $\Phi^L(W)$ for some $L \in \mathscr{L}_0$ and an indecomposable finite dimensional $K[T, T^{-1}]$-module W.*

(ii) *A is tame if and only if every poset S_a, associated with each vertex a of Q, contains no full subposet whose Hasse-diagram has one of the forms* $(1, 1, 1, 1, 1)$, $(1, 1, 1, 2)$, $(2, 2, 3)$, $(1, 3, 4)$, $(1, 2, 6)$ *or* $(N, 5)$.

For each $x \in R$, let R_x be the full subcategory of R formed by all objects of $\operatorname{supp} M$, for all $M \in \operatorname{ind} R$ such that $M(x) \neq 0$. Following [5] R is called *locally support-finite* if R_x is finite (has a finite number of objects) for each object x of R. Recall also that G acts freely on $\operatorname{Ind} R$ if ${}^gM \ncong M$ for any $M \in \operatorname{Ind} R$ and each $1 \neq g \in G$. From a general fact [5] it is known that if R is locally support-finite then every $X \in \operatorname{ind} A$ is of the form $F_\lambda M$ for some $M \in \operatorname{ind} R$. The following theorem shows that in our case the converse is also true.

THEOREM 2. *Assume that each indecomposable finite dimensional R-module has a peak. Then the following statements are equivalent*:

(i) *Every finite dimensional indecomposable A-module is isomorphic to $\psi_x(\Lambda)(V)$ for some $x \in R$, $\Lambda \in \mathscr{D}_0$ containing x, and an indecomposable finite dimensional $S_x(\Lambda)$-space V.*

(ii) *R is locally support-finite.*

(iii) *For each vertex a of Q, S_a is finite.*

(iv) *G acts freely on* $\operatorname{Ind} R$.

(v) $\operatorname{Ind} R = \operatorname{ind} R$.

3. Tamness of a tree category. The main aim of this section is to prove the following proposition which we shall use in the proof of Theorem 1.

PROPOSITION 1. *Let Λ be a finite locally bounded tree category such that each indecomposable finite dimensional Λ-module has a peak. Then Λ is tame if and only if all posets $S_x(\Lambda)$, $x \in \Lambda$, are tame.*

For the proof of this proposition we need the following lemma.

LEMMA 1. (a) *Let Γ be an integral domain and $\varphi\colon \Gamma^n \to \Gamma^m$ be a monomorphism of left free Γ-modules where $m, n \in N$. Then there exists an element $h \in \Gamma$ such that $1 \underset{\Gamma}{\otimes} \varphi\colon (\Gamma_h \underset{\Gamma}{\otimes} \Gamma^n) \to (\Gamma_h \underset{\Gamma}{\otimes} \Gamma^m)$ is a splittable monomorphism of left free Γ-modules.*

(b) *Let $\Gamma = K[T]_h$, for some $h \in K[T]$ and $\varphi\colon \Gamma^n \to \Gamma^m$ be a homomorphism of free left Γ-modules given by $(m \times n)$-matrix $[f_{ij}]_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}}$ with coefficients in Γ. If a homomorphism of K-vector spaces $\varphi(t)\colon K^n \to K^m$, defined by the scalar matrix $[f_{ij}(t)]_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}}$ is a monomorphism (resp. epimorphism) for some $t \in K$ which is not a root of h, then so is φ (resp. $1 \underset{\Gamma}{\otimes} \Gamma_{h_1}$, for some $h_1 \in \Lambda$).*

Proof. We start with the following remark. Let Γ be a commutative ring and $\varphi\colon \Gamma^n \to \Gamma^m$ be a homomorphism given by $m \times n$-matrix $f = [f_{ij}]_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}}$ having a nonzero minor h of order p, where $p = \min(m, n)$; then $1 \underset{\Gamma}{\otimes} \varphi$ is a splittable monomorphism (resp. epimorphism) if $m \geqslant n$ (resp. $m \leqslant n$). Indeed, if $m \geqslant n$ and, for example, the matrix $\bar{f} = [f_{ij}]_{1 \leqslant i, j \leqslant n}$ has nonzero determinant h, then the matrix $\bar{f}$ considered as a matrix with coefficients in Γ_h is invertible and a homomorphism $\psi\colon (\Gamma_h)^m \to (\Gamma_h)^n$ defined by matrix $[(\bar{f})^{-1}, 0]$ furnishes a retraction for $1 \otimes \varphi$. Dual case can be proved analogously.

Now we will prove (a). Let Γ and φ satisfy the assumption of (a). Observe that f produces a monomorphism of $\Gamma_{(0)}$ vector spaces $(\Gamma_{(0)})^n \to (\Gamma_{(0)})^m$. Then there exists a nonzero minor h of order n and (a) is a consequence of the remark.

(b) Let $f = [f_{ij}]_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}}$ be a matrix with coefficients in $\Gamma = K[T]_h$ for some $h \in K[T]$, and let $\varphi(t)\colon K^n \to K^m$ be a monomorphism (resp. epimorphism) for some t which is not a root of h. Then $f(t)$ admits a nonzero minor of order $p = \min(m, n)$. Without loss of generality, we can assume that the determinant of the matrix $f(t) = [f_{ij}(t)]_{1\leqslant i,j\leqslant p}$ is nonzero. Consequently the determinant of f is nonzero. Now, for $n \leqslant m$, the columns of f are lineary independent over A and so φ is monomorphism. Case $m \leqslant n$ follows from the remark.

COROLLARY 1. *Let $\Gamma = K[T]_h$, for some $h \in K[T]$, let Λ be a locally bounded tree K-category, and U an Λ-Γ-bimodule, finitely generated free as a right Γ-module and $x \in \Lambda$. Assume that the set W of all isoclasses of indecomposable Λ-modules of the form $U \underset{\Gamma}{\otimes} N$, for some simple Γ-module N, belonging to $\mathscr{P}_x(\Lambda)$ is nonempty. Then there exists an element $h' \in \Gamma$ such that the set of isoclasses of all indecomposable Λ-modules of the form $U' \underset{\Gamma'}{\otimes} N'$, for some simple Γ'-module N', where $\Gamma' = \Gamma_{h'}$ and $U' = U \underset{\Gamma}{\otimes} \Gamma'$, consists of all but a finite number of isoclasses from W, ${}_\Lambda U'$ has a peak in x, and $U'(\alpha)$ is splittable monomorphism or a splittable epimorphism for all arrows α in the ordinary quiver Q_Λ of Λ.*

Proof of Proposition 1. Let Λ be a tame finite tree category. Take any $x \in \Lambda$ and a finite coordinate-vector $\underline{c} = (c, (c_x)_{x\in S_x(\Lambda)})$ such that there exists infinitely many isoclasses of indecomposable $\underline{V} \in S_x(\Lambda)$-sp with $\operatorname{cdn}(\underline{V}) = \underline{c}$. Then by our assumption there exists a finite family of Λ-$K[T]$-bimodules U_i, $i = 1, \ldots, n$, finitely generated free $K[T]$-module such that for any indecomposable $M \in \operatorname{mod}\Lambda$ with $\dim M = d$, where $d = \dim G_x(\Lambda)(\underline{V})$, for some $\underline{V} \in S_x(\Lambda)$-sp with $\operatorname{cdn}(\underline{V}) = \underline{c}$, M has the form $U_i \otimes N$, for some i and a simple $K[T]$-module N. By Corollary 1 we can assume that there exists a family of bimodules ${}_\Lambda U'_{\Gamma_i}$, $i = 1, \ldots, n'$, $n' \leqslant n$ such that $\Gamma_i = K[T]_{h_i}$ for some $h_i \in K[T]$, all U'_i are free finitely generated right Γ_i-module, all Λ-modules U'_i have peak in x and, for any arrow α in Q_Λ, $U'_i(\alpha)$ is a splittable monomorphism or a splittable epimorphism, and moreover for all but a finite number of isoclasses of indecomposable modules $M \in P_x(\Lambda)$ with $\dim M = d$, M has the form $M = U'_i \underset{\Gamma_i}{\otimes} N$ for some i and simple Γ_i-module N. Thus by definition of the functor $F_x(\Lambda)$ we get that $F_x(\Lambda)(U'_i \underset{\Gamma_i}{\otimes} N) = F_x(\Lambda)(U'_i) \underset{\Gamma}{\otimes} N$ for any i and N. Hence by [7, Lemma 3.] $S_x(\Lambda)$ is tame.

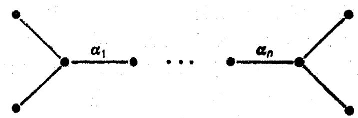
Let now $S_x(\Lambda)$ be tame for all $x \in \Lambda$. Take a dimension-vector d in Λ such that there exists $M \in \operatorname{ind}\Lambda$ with $\dim M = d$. By our general assumption M has some peak x. Since $S_x(\Lambda)$ is tame, there exists a finite number U_i^x of representations of $S_x(\Lambda)$ in finitely generated free right $K[T]$-modules, $i = 1, \ldots, n_x$, such that for all but a finite number of isoclasses of indecomposable objects $\underline{V} \in S_x(\Lambda)$-sp, with $\operatorname{cdn}(\underline{V}) = c$, where $c = c(d, x) = \operatorname{cdn}(F_x(\Lambda)(M))$ for some M with $\dim M = d$, $\underline{V}$ has the form $\underline{V} \cong U_i \underset{K[T]}{\otimes} N$ for some i and a simple $K[T]$-module N. Since $S_x(\Lambda)$

are finite, by Lemma 1 we can assume that all inclusions in U_i^x are splittable monomorphism. Hence, by the definition of $G_x(\Lambda)$, $G_x(\Lambda)(U_i^x \otimes N) = G_x(\Lambda)(U_i^x) \otimes N$ for any i and N. Thus the family of bimodules $G_x(\Lambda)(U_i^x)$, where $x \in \Lambda$ and $i = 1, \ldots, n_x$, parametrizes all but a finite set of isoclasses of indecomposable Λ-modules of dimension-vector d. Consequently by [7, Proposition 1] Λ is tame.

4. Proof of Theorem 1. Assume that each indecomposable finite dimensional R-module has a peak. Let $\Lambda = KT/I_T$, where T is a connected subtree of $\tilde{Q}$ and let $I_T = KT \cap \tilde{I}$, be a convex subcategory of R. Denote by T_0 the set of vertices of T and by T_1 the set of all its arrows. The degree $\deg_T(x)$ of a vertex $x \in T_0$ is the number of times n is used as an endpoint or a startpoint of the arrows in T. A vertex $x \in T_0$ with $\deg_T(x) = 1$ is called a *tip* of T. Denote by $\Delta(T)$ the maximum of $\deg_T(x)$, for $x \in T_0$, and by $\Omega(T)$ the number of tips of T if it is finite and ∞ otherwise. Recall that $M \in \operatorname{ind}\Lambda$ is *sincere* if $M(x) \neq 0$ for all $x \in T_0$. For each sincere $M \in \operatorname{ind}\Lambda$ we will denote by $\Lambda_M = KT_M/I_M$ the *shrinked algebra* of Λ obtained by shrinking (in the sense of [4, § 3]) of all arrows $x \xrightarrow{\beta} y$ for which $M(\beta)$ are isomorphism and x, y are not tips of T, and by $\overline{M}$ the Λ_M-module corresponding to M. Then $\operatorname{mod}\Lambda_M$ can be interpreted as the full subcategory of $\operatorname{mod}\Lambda$ containing only modules for which all these shrinked arrows β are represented by isomorphisms.

LEMMA 2. *Assume T is finite and $\Delta(T) \geqslant 2$. Then, for any sincere indecomposable finite dimensional Λ-module M, $\Delta(T_M) \leqslant 2\Delta(T) - 2$.*

Proof. Obviously, for $\Delta(T) = 2$, $\Delta(T_M) = \Delta(T) = 2\Delta(T) - 2$. Assume $\Delta(T) \geqslant 3$. Since any quiver algebra of a Dynkin type $\tilde{D}_n$, $n \geqslant 5$, has an indecomposable module without peak and M is sincere, T has no subtree of the form



where $n \geqslant 1$ and $M(\alpha_1), \ldots, M(\alpha_n)$ are isomorphisms. Then a simple analysis yields the required inequality.

LEMMA 3. *Assume T is finite and $2 \leqslant \Delta(T) \leqslant e$. Then for any $M \in \operatorname{ind}\Lambda$ with $(M(x)\colon K) \leqslant d$ for all $x \in T_0$, holds $\Omega(\operatorname{supp} M) \leqslant (2e-2)^d$.*

Proof. First observe that $\Omega(\operatorname{supp} M) = \Omega(\operatorname{supp}\overline{M})$ and for each arrow $y \xrightarrow{\alpha} z$ in T_M such that y and z are not tips, $M(\alpha)$ is a proper monomorphism or a proper epimorphism. Let x be a peak of M in Λ. Then x is a peak of $\overline{M}$ in Λ_M and from the above remark, the lengths of the walks

$$x - x_1 - \ldots - x_r$$

in T_M are bounded by d. Since by Lemma 2, $\Delta(T_M) \leqslant 2e-2$, we get $\Omega(\operatorname{supp} M) \leqslant (2e-2)^d$.

Proof of Theorem 1. In order to prove (i), it is enough by [6, Theorem 3.6] to show that the support of any $Y \in \operatorname{Ind} R$ with $G_Y = \{g \in G;\ {}^gY \cong Y\}$ nontrivial and $\operatorname{supp} Y/G_Y$ finite is a line. Let Y be such an R-module. We will apply *fundamental sequences* introduced in [6, § 4]. For a full subcategory C of R we will denote by $\hat{C}$ the full subcategory of R formed by all objects x such that $R(x, y) \neq 0$ or $R(y, x) \neq 0$ for some $y \in C$. For X and $Y \in \operatorname{MOD} C$, $X \in\mid Y$ denotes that X is isomorphic to a direct summand of Y. Moreover, if D is a full subcategory of C and $X \in \operatorname{MOD} C$, then $X|_D \in \operatorname{MOD} D$ is the restriction of X to D. Denote by C_n, $n \in N$, a fixed family of finite full subcategories of R defined by setting C_0 is a subcategory given by a fixed object x of R and $C_{n+1} = \hat{C}_n$ for $n \in N$. Observe that R is the union of C_n, $n \in N$. Then Y produces a sequence $(Y_n, u_n)_{n\in N}$ where $Y_n \in \operatorname{mod} C_n$ and $u_n\colon Y_n \to Y_{n+1}|_{C_n}$ is a C_n-homomorphism for all $n \in N$, satisfying the following conditions:

(a) For each $n \in N$, $Y_n \in \operatorname{ind} C_n$ or $Y_n = 0$.
(b) $Y_n \neq 0$ for some $n \in N$.
(c) For each $n \in N$, u_n is a splittable monomorphism.
(d) For each $n \in N$, $Y_n \in\mid Y|_{C_n}$.

Observe that, since $\operatorname{supp} Y/G_Y$ is finite and Y is locally finite dimensional, there is a common bound d for $(Y_n(y)\colon K)$, $y \in R$, $n \in N$. Moreover, we know by [6, Corollary 4.4] that $Y = \varinjlim (Y_n, u_n)$. Let $D_n = \operatorname{supp} Y_n$, $n \in N$, and $D = \operatorname{supp} Y$. Obviously, $D = \bigcup_{n\in N} D_n$. Moreover $D = KT_D/I_D$, $D_n = KT_{D_n}/I_{D_n}$, $n \in N$, for connected subtrees T_D and T_{D_n} of $\tilde{Q}$, and $I_D = KT_D \cap \tilde{I}$, $I_{D_n} = KT_{D_n} \cap \tilde{I}$. D is a line if and only if $\Delta(T_D) \leqslant 2$. Suppose $\Delta(T_D) \geqslant 3$ and denote by U the set of all vertices x of T_D with $\deg_{T_D}(x) \geqslant 3$. We claim that U is finite. Indeed, $e = \Delta(\tilde{Q}) = \Delta(Q)$ is finite and hence $2 \leqslant \Delta(T_{D_n}) \leqslant e$ for D_n containing at least three objects, say for $n \geqslant n_0$. Thus there is $m \geqslant n_0$ such that $\Delta(T_{D_{n+1}}) = \Delta(T_{D_n})$ for $n \geqslant m$. Hence U is contained in T_{D_m} and consequently is finite. On the other hand, $G_D = \{g \in G;\ gD = D\}$ contains $G_Y = \{g \in G;\ {}^gY \cong Y\}$ so there is $1 \neq g \in G$ such that $gU = U$. But this is impossible since G is free and U is finite. Therefore, $\Delta(T_D) \leqslant 2$ and D is a line. This finishes the proof of (i).

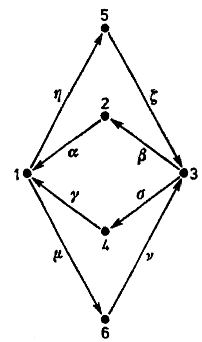
In order to prove (ii) observe that for each $x \in R$ the posets $S_x(\Lambda)$, where $\Lambda \in \mathcal{D}$ with $x \in \Lambda$, form a family of full subposet of S_x such that any finite subposet of S_x is contained in some $S_x(\Lambda)$. Then (ii) is an immediate consequence of Proposition 1 and [7, Theorem].

5. Proof of Theorem 2. (i) $\to$ (iii). Assume that every indecomposable finite dimensional A-module M is of the form $F_\lambda N$ for some $N \in \operatorname{ind} R$. We will show that S_a is a finite poset for any vertex a of Q. Suppose that S_a is infinite for some $a \in Q$. Then we claim that R contains a G-periodic line. Indeed, $\tilde{Q}$ is a locally finite tree and $\tilde{I}$ contains the ideal $\tilde{J}^n$ of $K\tilde{Q}$, generated by all paths of length n, for some $n \in N$, so there exist finite lines in R with arbitrary large number of changes of orientation. Thus, since Q is finite, there exists a finite line C in R whose tips b and c are both ending points and belong to the same G-orbit in R. Hence the full subcategory L of R formed by the union of subcategories $g^z C$, $z \in Z$, where $gb = c$, is a G-periodic line. Then, since $G_{B_L} = \{g \in G;\ {}^gB_{B_L} \cong B_L\}$ is nontrivial, we get a contradiction with [6, Proposition 2.4].

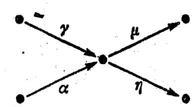
(iii) $\to$ (ii). First observe that if an indecomposable R-module M has a peak x and $M(y) \neq 0$ then there exists a (unique) finite line in R with tips x and y. Now assume that all posets S_a, $a \in Q_0$, are finite. We will show that the category R is locally support-finite. For any object $x \in R$, denote by U_x the set of all $y \in R$ such that there exists a finite line in R with tips x and y. The sets U_x, $x \in R$, are finite because the posets S_a, $a \in Q_0$, are finite. Take any objects $x \in R$ and $X \in \operatorname{ind} R$, with $X(x) \neq 0$. Since by our general assumption X has a peak, from the above observation $\operatorname{supp} X$ is contained in the full subcategory W_x of R formed by the union of sets U_y, $y \in U_x$. Consequently R is locally support-finite because, for each $x \in R$, R_x is contained in the finite category W_x.

(ii) $\to$ (v). Follows from [6, Proposition 2.5].
(v) $\to$ (iv). Follows directly from the fact that G is free.
(iv) $\to$ (i). This is a consequence of [6, Corollary 2.3].

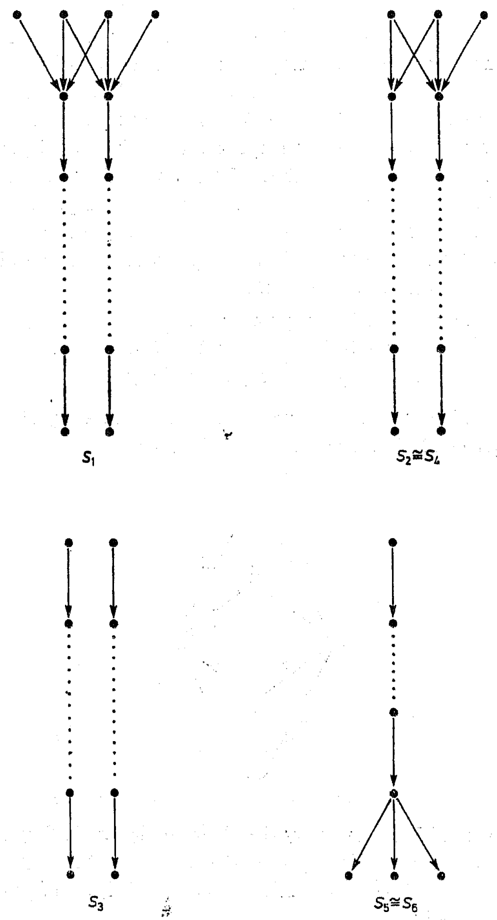
6. Examples. We end the paper with examples illustrating Theorem 1. Let A be the bound quiver algebra KQ/I where Q is the quiver



and I is the ideal in KQ generated by the elements $\beta\xi$, $\sigma\nu$, $\eta\alpha\beta$, $\eta\gamma\sigma$, $\mu\alpha\beta$, $\mu\gamma\sigma$, $\nu\mu$ and $\xi\eta$. Similarly as in [6, 5.3], one proves that the support of any indecomposable R-module, where $R = K\tilde{Q}/\tilde{I}$ is the universal cover of A, is either a finite line or a subquiver of the quiver

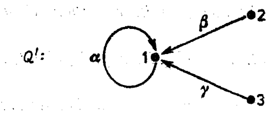


Thus every indecomposable finite dimensional R-module has a peak, by well-known classification of indecomposable representations of the above quiver of extended Dynkin type $\tilde{D}_4$. Moreover, the partially ordered sets associated with the vertices of Q have the following form:



Hence by Theorem 1(ii) A is tame.

Finally, we shall present an example showing that the assumption in Theorem 1 is for its form essential. Let $A' = KQ'/I'$ be the zero-relation algebra given by the quiver



and the ideal I' in KQ' generated by the element α^2, and R' the universal cover of A'. One can show that the classification problem of indecomposable finite dimensional R'-modules contains difficult classification problem for the Ringel's pattern $\left(\tilde{D}_n, \begin{matrix} n-2 \\ n-2 \end{matrix}\right)$ [18]. Hence there are indecomposable finite dimensional R'-modules without peak and thus they are not determined by indecomposable representations of the associated posets. Moreover, there are modules $Y \in \operatorname{Ind} R'$ with nontrivial stabilizers and nonlinear supports which create 1-parameter series of indecomposable finite dimensional A'-modules which are not in the image of the push-down functor F'_λ: $\operatorname{mod} R' \to \operatorname{mod} A'$ (modules of the second kind in the sense of [6]).

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Quadratic form schemes and quaternionic schemes

by

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Abstract. Quaternionic schemes, quaternionic structures and abstract Witt rings are known to be equivalent abstract versions of the algebraic theory of quadratic forms. This paper establishes a relationship between quadratic form schemes and the three other axiomatic approaches to quadratic forms. It is shown that cancelative quadratic form schemes coincide with quaternionic schemes.

Introduction. The algebraic theory of quadratic forms focuses on quadratic forms over fields. However, it has become clear that some parts of the theory are best treated by using an appropriate abstract language. Several authors have had ideas of this kind and as a result we are confronted with at least four distinct abstract approaches to quadratic form theory. These are:

(i) Quadratic form schemes ([3], [4], [5], [6], [10], [11], [12]).
(ii) Quaternionic structures ([1], [8], [9]).
(iii) Abstract Witt rings ([1], [8], [9] and earlier papers cited there).
(iv) Quaternionic schemes ([1]).

The relationships among (ii), (iii), and (iv) are fully known. Marshall's book [8] shows that (ii) and (iii) are equivalent and Carson and Marshall [1] prove that (ii) and (iv) are equivalent. It is the aim of this paper to clarify the role of (i) among the abstract theories of quadratic forms.

In Section 1 we exhibit several equivalent sets of axioms for (i) and in Section 2 we do the same for (iv). In both cases we have found that the generally accepted sets of axioms for (i) and (iv) are dependent and we reduce the number of axioms in each case to a pair of independent axioms and even to a single axiom in each case.

Section 3 explains the relationship between two concepts of isometry of forms used in abstract theories (chain isometry and inductively defined isometry following [8] and [9]). The main result, Theorem 3.5, establishes the actual equivalence of quadratic form schemes with cancellation property and quaternionic schemes. A corollary to this result asserts that the classical Witt cancellation theorem for forms of any dimension is a consequence of the cancellation property for 2-dimensional forms.

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