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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Approximating homotopy equivalences of 3-manifolds by homeomorphisms

by

W. Jakobsche (Warszawa)

Abstract. We prove the 3-dimensional version of the α-approximation theorem of Chapman and Ferry.

1. Introduction. Let X and Y be topological spaces, let $f\colon X \to Y$ be proper map, i.e. a map such that inverse images of compacta are compact, and let α be an open cover of Y. We say that f is an *α-equivalence* provided that for some map $g\colon Y \to X$ there are homotopies θ_t from fg to the identity on Y and φ_t from gf to the identity on X, such that

(1) for each $x \in X$, there is a $U \in \alpha$ containing

$$\{f\varphi_t(x)\colon 0 \leqslant t \leqslant 1\};$$

(2) for $y \in Y$, there is a $U \in \alpha$ containing

$$\{\theta_t(y)\colon 0 \leqslant t \leqslant 1\}.$$

The aim of this note is to prove the following theorem:

THEOREM 1.1 (3-dimensional α-approximation theorem). *Let N^3 be a manifold of dimension 3. For every open cover α of N, there is an open cover β of N such that for any 3-manifold M^3 which contains no fake 3-cells, and for any β-equivalence $f\colon M \to N$ which is already a homeomorphism from ∂M to ∂N, f is α-close to a homeomorphism $h\colon M \to N$* (i.e. *for each $m \in M$, there is a $U \in \alpha$ containing $f(m)$ and $h(m)$*).

An α-approximation theorem was first proved for Q-manifolds by S. Ferry [F_1], and then for the manifolds of dimension $\geqslant 5$ by T. Chapman and S. Ferry. It fails in dimension 3 if the Poincaré conjecture is false; our Theorem 1.1 is a 3-dimensional version, with the additional assumption that the manifold M contains no fake 3-cells. A large part of our proof is identical to the proof of S. Ferry and T. Chapman [C–F], which is similar to Siebenmann's CE-approximation theorem. Our proof should be read together with the proof of Chapman and Ferry, because it uses the same notation and omits many arguments which are equal to the ones given in [C–F].

Actually, we have only to introduce some modifications in the proof of the "Handle Lemma" of [C–F] (p. 589). The proofs of the Main Theorem, and α-Approximation theorem of [C–F] work for dimension 3. We shall also use the "3-dimensional Splitting Theorem" proved in [J] (the n-dimensional, $n \geqslant 5$, version of this theorem was proved in [C–F] by different methods). As it is proved by S. Ferry [F_2], Theorem 1.1 implies the following theorem.

THEOREM 1.2. *If M is a 3-manifold and α is an open cover of M, then there is an open cover β of M such that if N is a 3-manifold containing no fake 3-cells and $g\colon (M, \partial M) \to (N, \partial N)$ is a proper β-map, then g is homotopic through α-maps to a homeomorphism.*

Theorem 1.2 gives another partial solution of the problems of [SB] (problem 97) and [S].

Note, that Theorem 1.2 is false if there exists a fake 3-cell and if we omit the assumption that N does not contain fake 3-cells. In the whole of this paper, we will use the notation of [C–F]. In particular, if $f\colon X \to Y$ is a proper map, $A \subset Y$, and α is an open cover of Y, then we say that f is an *α-equivalence over A* if there exist a map $g\colon A \to X$ and the homotopies $\vartheta_t\colon A \to Y$ from fg to id_A, and $\varphi_t\colon f^{-1}(A) \to X$ from $gf|f^{-1}(A)$ to $\mathrm{id}_{f^{-1}(A)}$ which satisfy conditions (1) and (2) with X replaced by $f^{-1}(A)$ and Y replaced by A. We say that f is an *ε-equivalence over A* if it is an α-equivalence over A for a cover α of A by the balls of radius $\leqslant \varepsilon$.

By R^m we will denote the Euclidean m-space, and by rB^m the ball in R^m of radius r. In particular, we write $B^m = 1B^m$. We will say that the map $f\colon X \to Y$ is at most 2 to 1 if, for every $y \in Y$, $f^{-1}(y)$ contains no more than two points.

2. The proof of the "Handle Lemma". In this section we prove Theorem 2.1 which is a 3-dimensional analogue of the "Handle Lemma" of [C–F].

For notation, let V^3 be a 3-dimensional manifold, $m+k = 3$, and let $f\colon V \to B^k \times R^m$ be a proper map such that $\partial V = f^{-1}(\partial B^k \times R^m)$ and f is a homeomorphism over $(B^k \setminus \frac{1}{2}\mathring{B}^k) \times R^m$.

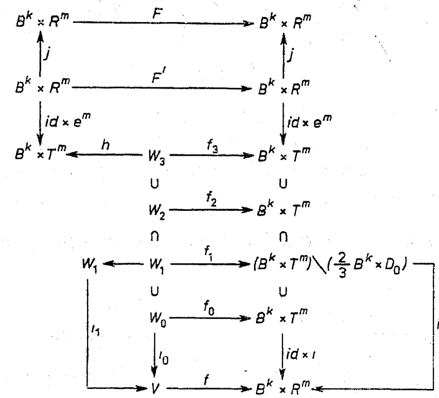
THEOREM 2.1 (Handle Lemma). *Suppose that V^3 contains no fake 3-cells. Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if f is a δ-equivalence over $B^k \times 3B^m$ then*

(1) *there exists an ε-equivalence $F\colon B^k \times R^m \to B^k \times R^m$ such that $F = \mathrm{id}$ over $(B^k \setminus \frac{5}{6}\mathring{B}^k) \times R^m \cup B^k \times (R^m \setminus 4B^m)$,*

(2) *there exists a homeomorphism $\varphi\colon f^{-1}(U) \to F^{-1}(U)$ such that $F\varphi = f|f^{-1}(U)$, where $U = (B^k \setminus \frac{5}{6}\mathring{B}^k) \times R^m \cup B^k \times 2B^m$.*

Proof. We will build the same diagram (*) of maps and spaces as in the proof of the "Handle Lemma" in [C–F]. In this diagram, $S^1 \subset R^2$ is a unit circle of complex numbers of absolute value 1, $e\colon R^1 \to S^1$ is a covering projection defined by $e(x) = e^{\pi i x/4}$.

$T^m = S^1 \times S^1 \times \ldots \times S^1$ is a torus, and $e^m = e \times e \times \ldots \times e$ is a covering map of R^m onto T^m. We define T_0^m to be $T^m \setminus D_0$, where D_0 is a disk contained in $T^m \setminus e^m(2B^m)$, (we put $T_0^m = T^m \setminus D_0$ rather than $T_0^m = T^m \setminus \{x_0\}$, $x_0 \in T_0^m$ as in [C–F], for a technical reason — it is easier to formulate Lemma 2.2; it is easy to see that this does not change the proof).



We start with the map f at the bottom of the diagram, and on the top we obtain the required map F.

The only difference between our proof and [C–F] is that to construct h in (*) we have to use the theorem of Waldhausen [W] (Lemma 2.6) rather than the surgery argument of [C–F]. This can be done if W_3 contains no fake 3-cells. To prove that W_3 contains no fake 3-cells we have only to prove that the same is true for W_1. Thus the problem reduces to showing that if i in (*) is appropriately chosen then W_1 contains no fake 3-cells. We use the facts that V contains no fake 3-cells and that we can construct an appropriate immersion of W_1 into V which extends i_0. It would be slightly easier to show that W_0 contains no fake 3-cells, but it does not help much in proving that W_1 contains no fake 3-cells. The construction of (*) contains some steps:

I. Construction of the immersions i' and i. We will construct an immersion

$$i'\colon (B^k \times T^m) \setminus (\tfrac{2}{3}B^k \times D_0) \to B^k \times 2.5\,\mathring{B}^m$$

such that the following conditions are satisfied:

(a) $i'|B^k \times T_0^m = \mathrm{id} \times i$ where $i\colon T_0^m \to 2.5\mathring{B}^m$ is an immersion such that $ie^m|2B^m\colon 2B^m \to 2B^m$ is the identity;

(b) $i'((B^k \setminus \frac{2}{3}B^k) \times T^m) \subset (B^k \setminus \frac{2}{3}B^k) \times 2.5\mathring{B}^m$;

(c) there exists a 3-manifold X and immersions $i_a\colon B^k \times T^m \setminus (\frac{2}{3}B^k \times D_0) \to X$ and $i_b\colon X \to 2.5\mathring{B}^m$ such that $i' = i_b \circ i_a$ and both i_a and i_b are at most 2 to 1.

We will apply the immersions i and i' to our diagram (∗). We need the following lemma:

LEMMA 2.2. (a) *For $m \leqslant 2$, there exists an immersion f of $[0,1]\times T^m$ into $[0,1]\times 2.5\mathring{B}^m$ such that $f|[0,1]\times T_0^m$ is a product map $f = \mathrm{id}_{[0,1]}\times\alpha$ for some immersion α and such that $\alpha e^m|2B^m\colon 2B^m \to 2B^m$ is the identity. Moreover, $f = f_a\circ f_b$ where $f_a\colon [0,1]\times T^m \to Y \subset [0,1]\times 2.5\mathring{B}^m$ and $f_b\colon Y\to [0,1]\times 2.5\mathring{B}^m$ are immersions which are at most 2 to 1, $f_a|\{0\}\times T_0^m = f|\{0\}\times T_0^m = \mathrm{id}_{\{0\}}\times\alpha$,*

$$f_b|\big(f_a(\{0\}\times T_0^m)\big) = \mathrm{id}$$

and Y is an $(m+1)$-submanifold of $[0,1]\times 2.5\mathring{B}^m$.

(b) *There exists an immersion α of T_0^3 in $2.5\mathring{B}^m$ such that $\alpha e^3|2B^3\colon 2B^3\to 2B^3$ is the identity and such that $\alpha = \alpha_b\circ\alpha_a$ where $\alpha_a\colon T_0^3\to Y_1$ and $\alpha_b\colon Y_1\to 2.5\mathring{B}^3$ are immersions which are at most 2 to 1 and $\mathring{Y}_1$ is a 3-manifold.*

Proof. (a) For $m = 1$, immersion f can be described by Fig. 1. Here $f_a = f$ and $f_b = \mathrm{id}_Y$, $Y = \mathrm{Im} f$.

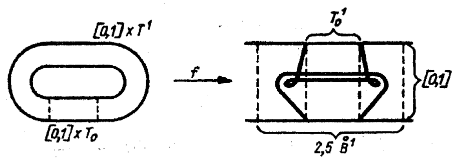


Fig. 1

Now let $m = 2$. We first find an immersion $\alpha\colon T_0^2\to 2.5\mathring{B}^2$ as in Fig. 2. Then we find the corresponding immersion

$$f_a|[0,1]\times T_0^2 \to [0,\tfrac{1}{2}]\times 2.5\mathring{B}^2,\quad f_a = l\times\alpha$$

where $l\colon [0,1]\to[0,\frac{1}{2}]$ is given by $l(x) = x/2$ (see Fig. 3).

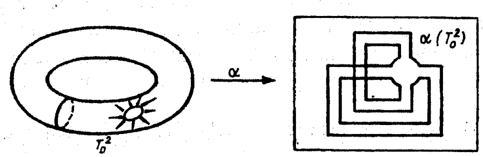


Fig. 2

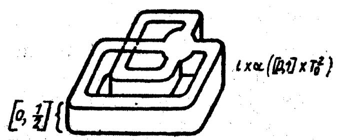


Fig. 3

We have to extend f_a to all $[0,1]\times T^2$. We have $T_0^2 = T\setminus\mathring{D}^2$ where D^2 is a 2-ball in T^2. So we have to extend f_a to $[0,1]\times D^2$. To do that, let us add to $f_a([0,1]\times T_0^2)$ an immersed disk $\bar{D}$ such that $\bar{D} = i(D^2)$. Here i is an immersion which is 2 to 1, and $\partial\bar{D}$ is equal to a bold immersed closed curve in Fig. 3, and $\bar{D}\cap f_a([0,1]\times T_0^2) = \partial\bar{D} = i(\partial D^2)$, and $\bar{D}\subset[\frac{1}{2},1]\times 2.5B^2$. To picture $\bar{D}$, let us first divide D^2 into three disks B_1, B_2 and B_3 as in Fig. 4. Then Fig. 5 shows that $i|B_1$, $i|B_2$ and $i|B_3$,

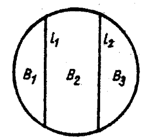


Fig. 4

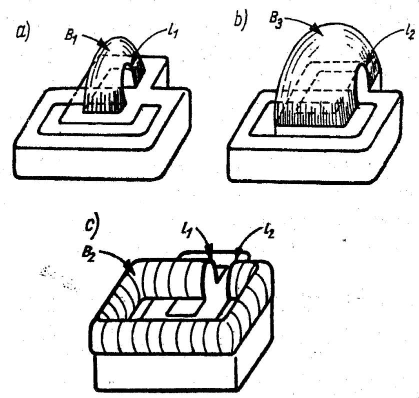


Fig. 5

respectively. It is easy to see that $i|B_k$, $k = 1,2,3$ are embeddings, so $i = i|B_1\cup\cup i|B_2\cup i|B_3$ is at most 2 to 1, and we can put $\bar{D} = i(D^2)$. Thickening $i(D^2)$, we can easily get the required immersion f_a. We put $Y = \mathrm{im}(f_a)$. Then it is easy to find a 2 to 1 immersion $f_b\colon Y\to[0,1]\times 2.5\mathring{B}^m$, which is $m\times\mathrm{id}_{\alpha(T_0^2)}$ on $f_a([0,1]\times T_0^2) = l([0,1])\times\alpha(T_0^2)$, where $m\colon[0,\frac{1}{2}]\to[0,1]$ is given by $m(x) = 2x$.

We can require that f_b does not contain more points of Y outside of a small neighbourhood of $f_a([0,1]\times T_0^2)$, so that the resulting immersion is 2 to 1. Of course,

$$f_a|\{0\}\times T_0^2 = f|\{0\}\times T_0^m = \mathrm{id}\times\alpha \quad\text{and}\quad f_b|\big(f_a(\{0\}\times T_0^m)\big) = \mathrm{id}.$$

The condition that $\alpha e^m|2B^m$ is the identity can easily be achieved by the Schoenflies theorem.

(b) First, let us take the immersions

$$f_a\colon [0,1]\times T^2 \to Y = \operatorname{im} f_a \subset [0,1]\times 2.5\mathring{B}^2,$$
$$\alpha\colon T_0^2 \to R^2$$

obtained in the proof of (a). Then let us take the sum

$$Y_1 = Y\cup_p([0,1]\times\alpha(T_0^2))$$

with the identification homeomorphism $p\colon A_1\to A_2$,

$$A_1 = i(\{0,1\}\times T_0^2) = \{0,\tfrac{1}{2}\}\times\alpha(T_0^2)\subset Y,$$
$$A_2 = \{0,1\}\times\alpha(T_0^2)\subset[0,1]\times\alpha(T_0^2).$$

In this way we get an immersion

$$\alpha_a\colon T_0^3 = ([0,1]\times T^2)\cup([0,1]'\times T_0^2)\to Y_1$$

where $([0,1]\times T^2)\cup([0,1]'\times T_0^2)$ is a sum with identifications such that the intervals $[0,1]$ and $[0,1]'$ add up so as to give an S^1 factor in $T^3\supset T_0^3$. Then we can decompose $Y_1 = Z_1\cup Z_2$, so that $Z_1 = \bar{\alpha}_a(S_1\times T_0^2)$ and $Z_2 = \bar{f}_a([0,1]\times D^2)$ (here $S^1 = [0,1]\cup[0,1]'\supset[0,1]$ and $D^2 = T^2\setminus\mathring{T}_0^2$). It is easy to find $\alpha_b\colon Y_1\to 2.5\mathring{B}^3$ such that $\alpha_b|Z_1$ and $\alpha_b|Z_2$ are embeddings, so that α_b is 2 to 1. This ends the proof of (2.2).

Now, if $m=3$, then $\mathring{B}\times T^3\setminus(\tfrac{2}{3}\mathring{B}\times D_0) = T^3\setminus D_0 = T_0^3$, and we can put $i'=\alpha$, where α is an immersion guaranted by Lemma 2.2.

To describe i' for $m<3$, we will use the following notation: We identify $B^k\setminus\tfrac{2}{3}\mathring{B}^k$ with $S^{k-1}\times[0,1]$, so that $S^{k-1}\times\{0\}$ is identified with $\partial(\tfrac{2}{3}B^k)$ and we define an immersion

$$j\colon S^{k-1}\times[0,1]\times T^m\to S^{k-1}\times[0,1]\times 2.5\mathring{B}^m$$

as $\mathrm{id}_{S^{k-1}}\times f$, where f is an immersion given by Lemma 2.2. Then $j = j_b\circ j_a$, where

$$j_a = \mathrm{id}_{S^{k-1}}\times f_a\colon S^{k-1}\times[0,1]\times T^m\to S^{k-1}\times Y$$

and

$$j_b\colon \mathrm{id}_{S^{k-1}}\times f_b\colon S^{k-1}\times Y\to S^{k-1}\times[0,1]\times 2.5\mathring{B}^m$$

are immersions which are at most 2 to 1. By our identification, j maps $(B^k\setminus\tfrac{2}{3}\mathring{B}^k)\times T^m$ into $(B^k\setminus\tfrac{2}{3}\mathring{B}^k)\times 2.5\mathring{B}^m$. Now, we define i' by

$$i'|(B^k\setminus\tfrac{2}{3}B^k)\times T^m = j|(B^k\setminus\tfrac{2}{3}B^k)\times T^m$$

and

$$i'|\tfrac{2}{3}B^k\times T_0^m = \mathrm{id}_{\frac{2}{3}B^k}\times\alpha\,;$$

it is easy to see that if f and α satisfy the requirements of 2.2, then our immersion i' is well defined and satisfies conditions (a)–(c).

We only show how to construct i_a, i_b and X. If $m\leqslant 2$ we put

$$i_a(x) = \begin{cases} j_a(x) & \text{for } x\in(B^k\setminus\tfrac{2}{3}B^k)\times T^m,\\ (\mathrm{id}_{\frac{2}{3}B^k}\times\alpha)(x) & \text{for } x\in\tfrac{2}{3}B^k\times T_0^m,\end{cases}$$

and $X = (S^{k-1}\times Y)\cup\tfrac{2}{3}B^k\times f_a(\{0\}\times T_0^m)$ (note that $S^{k-1}\times Y\subset S^{k-1}\times[0,1]\times 2.5\mathring{B}^m$ and $S^{k-1}\times\{0\} = \partial(\tfrac{2}{3}B^k)$). Then we put

$$i_b(x) = \begin{cases} j_b(x) & \text{for } x\in j_a(B^k\setminus\tfrac{2}{3}B^k)\times T^m,\\ x & \text{for } x\in\tfrac{2}{3}B^k\times(f_a(\{0\}\times T_0^m)).\end{cases}$$

If $m=3$ we just put $i_a=\alpha_a$, $i_b=\alpha_b$ and $X=Y$.

II. Construction of i_1', i_0, W_0 and W_1 and W_1'. We construct the spaces W_0 and W_1' by taking the pullbacks:

$$W_0 = \{(x,y)\in V\times(B^k\times T_0^m)\colon f(x) = (\mathrm{id}\times i)(y)\},$$
$$W_1' = \{(x,y)\in V\times(B^k\times T^m)\setminus(\tfrac{2}{3}B^k\times D_0)\colon f(x) = i'(y)\}\,;$$

by i_0 and i_1' we denote the restrictions of the projections of W_0 and W_1', respectively, onto the first coordinate, and by f_0 and f_1' the restrictions of the projections onto the second coordinate. It is easy to see that i_0 and i_1' are immersions of the manifolds W_0 and W_1' into V, and that W_1', W_0, and i_0 is the restriction of i.

We write $T_0^m = Y_0\cup(S^{m-1}\times[0,\infty))$, and let $Y_t = Y_0\cup(S^{m-1}\times[0,t])$. Then $T^m\setminus\mathring{Y}_t$ is an m-ball containing D_0. As in [C–F] we can prove that f_0 is a δ_0-equivalence over $B^k\times Y_3$ if δ_0 was chosen sufficiently small.

III. Construction of W_1, f_1 and i_1. By adding a copy of $(B^k\setminus\tfrac{2}{3}B^k)\times D_0$ to W_0 we can form manifold W_1 containing W_0 such that f_0 extends to a proper map $f_1\colon W_1\to(B^k\times T^m)\setminus(\tfrac{2}{3}B^k\times\{x_0\})$ which is a homeomorphism over $(B^k\setminus\tfrac{2}{3}B^k)\times T^m$. As in [C–F], it is a δ_1-equivalence over $B^k\times T^m\setminus[\tfrac{3}{4}\mathring{B}^k\times(T^m\setminus Y_2)]$ if δ_0 is sufficiently small.

Now, let us notice that W_1 is homeomorphic to W_1'. Actually,

$$f|f^{-1}(B^k\setminus\tfrac{2}{3}B^k)\times R^m\to(B^k\setminus\tfrac{2}{3}B^k)\times R^m$$

is a homeomorphism, and we have the following comutative diagram:

$$\begin{array}{ccc} (i_1')^{-1}\big(f^{-1}(B^k\setminus\tfrac{2}{3}B^k)\times R^m\big) & \xrightarrow{\;f_0\;} & (i')^{-1}\big((B^k\setminus\tfrac{2}{3}B^k)\times R^m\big)\\ \downarrow{\scriptstyle i_1'|} & & \downarrow{\scriptstyle i'|}\\ f^{-1}\big((B^k\setminus\tfrac{2}{3}B^k)\times R^m\big) & \xrightarrow{\;f\;} & (B^k\setminus\tfrac{2}{3}B^k)\times R^m \end{array}$$

This implies that

$$f_0\colon (i_1')^{-1}\big(f^{-1}(B^k\setminus\tfrac{2}{3}B^k)\times R^m\big)\to(i')^{-1}\big((B^k\setminus\tfrac{2}{3}B^k)\big)\times R^m$$

is a homeomorphism. This, by the construction of W_1, implies that there is a homeomorphism $\beta\colon W_1'\to W_1$ such that $\beta(W_0) = W_0$ (note that $W_0\subset W_1$ and $W_0\subset W_1'$). Now, we put $i_1 = i_1'\beta$. We want to prove that W_1' (and consequently W_1) contains no fake 3-cells.

First we prove some lemmas.

LEMMA 2.3. *i_1' can be expressed as a composite function $i_1' = i_b' \circ i_a'$, where $i_a'\colon W_1' \to X'$ and $i_b'\colon X' \to V$ are immersions which are at most 2 to 1, and X' is a 3-manifold.*

Proof. Let $X' = \{(x, y) \in V \times X\colon f(x) = i_b(y)\}$ where X and i_a are given by the definition of $i'(c)$. W_1' can be expressed as

$$W_1' = \{(x, y) \in X' \times (B^k \times T^m) \setminus (\tfrac{2}{3} B^k \times D_0)\colon \hat{f}(x) = i_a(y)\}$$

where i_b', i_a' and $\hat{f}$ are restrictions of projections of X' on V, W_1' on X' and X' on X, respectively. Of course, i_a' and i_b' are immersions. Moreover, they are at most 2 to 1. We prove it for i_a' (for i_b' the proof is identical).

Suppose that there are three points, $a_1, a_2, a_3 \in W_1'$, such that $a_k \neq a_l$ for $k \neq l$, and $i_a'(a_1) = i_a'(a_2) = i_a'(a_3)$. Let $a_k = (x_k, y_k)$, for $k = 1, 2, 3$. Then $x_1 = x_2 = x_3 \in X'$ so $y_k \neq y_l$ for $k \neq l$. But $\hat{f}(x_1) = \hat{f}(x_2) = \hat{f}(x_3)$, and so by the definition of W_1' $i_a(y_1) = i_a(y_2) = i_a(y_3)$. This contradicts the fact that i_a is at most 2 to 1.

In the next lemma we will use the following notation: Let $f\colon X \to Y$ be a map. For $i \geqslant 2$ we write

$$S_k(f) = \{x \in X\colon \|f^{-1}(f(x))\| = k\}, \quad S(f) = \bigcup_{k \geqslant 2} S_k(f),$$

$$S_1(f) = S(f) \setminus \bigcup_{k \geqslant 2} S_k(f) \quad \text{and} \quad \Sigma_i(f) = f(S_i(f)).$$

LEMMA 2.4. *Suppose that F is a fake 3-cell, and that V is a 3-manifold containing no fake 3-cells. Then there in no PL-immersion $j\colon F \to V$ such that j is at most 2 to 1.*

Proof. Let F be any fake 3-cell, and let $D_1, D_2, \ldots, D_n$ be disjoint discs in $F \setminus \partial F$. Then the manifold $F \setminus \mathring{D}_1 \setminus \mathring{D}_2 \setminus \ldots \setminus \mathring{D}_n$ will be called a *fake 3-cell with holes*. Let us observe that if F is a fake 3-cell with n-holes, then ∂F consists of $(n+1)$ 2-spheres, and for every simple closed curve α in ∂F the manifold F' obtained from F by attaching to F an index 2 handle H^2, so that α is an attaching sphere of H^2, is a fake 3-cell with $n+1$ holes. We will prove 2.4 not only for fake 3-cells but more generally for fake 3-cells with holes.

Suppose $j\colon F \to V$ is an immersion of a fake 3-cell with holes F which is at most 2 to 1. Changing j slightly in the collar of ∂F in F, if necessary, we can assume that $j| = j|\partial F\colon \partial F \to V$ is in general position with respect to ∂F (see [H], p. 10 for definition). This together with the fact that j is at most 2 to 1 means that $S_k(j|\ ) = \emptyset$ for $k > 2$. j is an immersion, whence $S_1(j|\ ) = \emptyset$. $S_2(j|\ )$ is a sum of a collection $C(j)$ of simple closed curves in ∂F which are pairwise glued by j, and $j|$ is transverse at every point of $\Sigma_2(j|\ )$ (see [H], Def. 1.11 (vi)). Thus we will consider only immersions $j\colon F \to V$ of fake 3-cells with holes into V with the above properties.

By Lemma 2.5 there are no such immersions j if the cardinality $\|C(j)\|$ is zero, i.e. $C(j) = 0$. Let us assume inductively that there are no such immersions j with $\|C(j)\| < n$, and suppose that there is an immersion $j\colon F \to V$ with $\|C(j)\| = n$ for some fake 3-cell with holes F.

First suppose that there exists in $C(j)$ a curve c which satisfies the following condition:

(a) c bounds in ∂F a disk D such that D contains no other element of $C(j)$ and that $j(D \setminus \partial D) \subset j(F \setminus \partial F)$.

Condition (a) implies that $j^{-1}(j(D))$ consists of two disks: one of them is D and the other is a disk D' such that $D' \setminus \partial D' \subset F \setminus \partial F$ and $\partial D' = c'$, where c' is a curve in ∂F which is identified with c by j (see schematic Fig. 6). Let D'' be a disk bounded by c' in ∂F such that $\operatorname{Int} D'' \cap \operatorname{Int} D' = \emptyset$. Then $D'' \cup D'$ is a sphere which bounds some homotopy 3-cell with holes F_1 in F (we define a homotopy 3-cell with holes as $F \setminus \bigcup \mathring{D}_i$, where F is a cell or a fake 3-cell). Of course, $F_2 = \overline{F \setminus F_1}$ is also a homotopy 3-cell with holes. Let B a small regular neighbourhood of F_1 in F, and let $F_2' = \overline{F \setminus B}$. Of course, F_2' is homeomorphic to F_2. Now, let $j' = j|F'$ where F' is that one of the manifolds F_1 and F_2' which is a fake 3-cell with holes (i.e., it is not a normal 3-cell with holes). Then $j'\colon F \to V$ is an immersion with $\|C(j')\| < n$

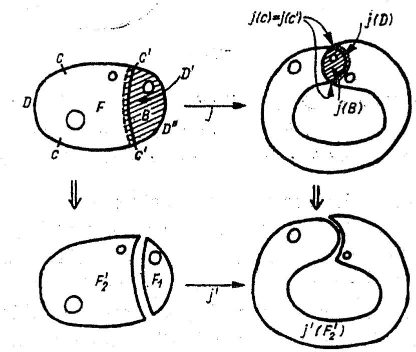


Fig. 6

(we have eliminated c, c' and possibly some other elements of $C(j)$). This contradicts the inductive assumption. If there is no curve $c \in C(j)$ satisfying condition (a), then we can always find a curve $c \in C(j)$ which satisfies the following condition:

(b) c bounds in ∂F a disk D such that D contains no other element of $C(j)$ and that $j(D \setminus \partial D) \cap j(F \setminus \partial F) = \emptyset$.

Let c' be a curve in $C(j)$, $c' \neq c$, which is identified with c by j. Then, let F' be a fake 3-cell with holes obtained from F by attaching an index 2 handle H to F (see Fig. 7), so that c' is an attaching sphere of H and that $H \cap F = \partial H \cap \partial F$ is a regular neighbourhood of c' in ∂F which intersects no other element of $C(j)$. Let $j_1\colon H \to V$ be a homeomorphic imbedding of H in V such that $j_1(H)$ is a regular neighbourhood of $j(D)$ in the manifold $\overline{V \setminus j(F)}$ and that $j_1|\partial F \cap \partial H = j|\partial F \cap \partial H$ (we can define such a j_1 because, by (b), $j(D) \cap j(F \setminus D) = j(\partial D)$).

Now we define $F' = H \cup F$ and we define j' by $j'|F = j$ and $j'|H = j_1$. One can easily see that $||C(j')|| < n$, because we have eliminated the curves c and c'.
To complete the proof of 2.4 we now only have to prove the following lemma.

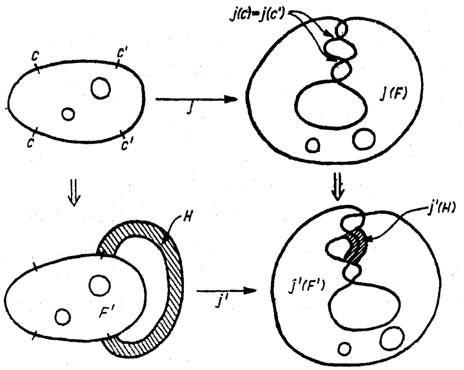


Fig. 7

LEMMA 2.5. *Let F be a fake 3-cell with holes, let V be a 3-manifold, and let j: $M \to V$ be an immersion such that $j|\partial M$ is a homeomorphic embedding. Then there exists a fake 3-cell with holes F' and an embedding i: $F' \to V$.*

Proof. We consider the closure of the components of $F \setminus j^{-1}(j(\partial F))$. At least one of them is a fake 3-cell with holes. Let us denote it by F'. $j|F'$ is an embedding. Now we can prove that W_1' contains no fake 3-cells. The immersion i_1': $W_1' \to V$ can be expressed, by Lemma 2.3, as $i_1' = i_b' \circ i_a'$ where i_b': $X' \to V$ and i_a': $W_1' \to X'$ are at most 2 to 1. Let us consider the PL structures induced by i_b' on X' and by i_a' on W_1'. By Lemma 2.4, X', and consequently W_1', contain no fake 3-cells, because V contains no fake 3-cells. This implies that W_1 contains no 3-cells.

IV. Construction of W_2. We consider the open set

$$G = [\tfrac{4}{5}\mathring{B}^k \times (T^m \setminus Y_1)] \setminus (\tfrac{3}{4}B^k \setminus \mathring{Y}_2)$$

(see II for the definition of Y_t). We identify G with $S^2 \times R$. If δ_1 is sufficiently small, then we can find a 2-sphere $S \subset f_1^{-1}(S^2 \times (-1, 1))$ which is bicollared, which separates $f_1^{-1}(S^2 \times \{1\})$ from $f_1^{-1}(S^2 \times \{-1\})$ and for which $f_1|S$: $S \to G$ is a homotopy equivalence. We find S by using the 3-dimensional "splitting theorem" of [J]. We define a 3-ball $D^3 = \frac{3}{4}B^k \times (T^m \setminus \mathring{Y}_2)$, and we let W_2 be the closure of the component of $W_1 \setminus S$ containing $f_1^{-1}(Y_0)$. Our map f_2: $W_2 \to B^k \times T^m \setminus D^3$ is defined by $f_2 = f_1|W_2$. It is well defined for δ_1 sufficiently small. Of course, W_2 contains no fake 3-cells, because W_1 contains no fake 3-cells.

V. Construction of W_3. W_3 is constructed from W_2 by attaching to W_2 the cone over S. W_3 is a compact 3-manifold which is homotopy equivalent to $B^k \times T^m$ and contains no fake 3-cells. We show, as in [C–F], that for δ_1 sufficiently small there is a δ_3-equivalence f_3: $W_3 \to B^k \times T^m$ which agrees with f_1 over $(B^k \times \frac{5}{6}B^k) \times T^m \cup B^k \times Y_0$.

VI. Construction of h. We want h to be a homeomorphism which agrees with f_3 over $(B^k \setminus \frac{5}{6}B^k) \times T^m$, and which is homotopic to f_3. Such an h can be obtained by using the following lemma:

LEMMA 2.6 (Waldhausen [W]). *Let M and N be connected, compact, orientable, irreducible* PL-3-*manifolds such that N is sufficiently large, and let f: $M \to N$ be a* PL-*homotopy equivalence such that $f^{-1}(\partial N) = \partial M$, and that f is a homeomorphism on the boundary. Then f is a homotopic relative boundary to a* PL-*homeomorphism.*

Lemma 2.6 is not stated in [W] as a theorem but it is proved as part of the proof of Theorem 6.1 of [W]. It was first used in the torus argument by Hamilton [H].

VII. Construction of F'. F': $B^k \times R^m \to B^k \times R^m$ is the covering of $f_3 h^{-1}$ which is an identity on $(B^k \setminus \frac{5}{6}\mathring{B}^k) \times T^m$. F' is bounded, and it is an ε-equivalence if δ_3 is small (see [C–F]).

VIII. Construction of j. Let J: $R^3 \to 4\mathring{B}^k \times 4\mathring{B}^m$ be a radial homeomorphism which is fixed on $2B^k \times 2B^m$. Then an open embedding j: $B^k \times R^m \to B^k \times R^m$ is defined by restricting J.

IX. Construction of F. We define F: $B^k \times R^m \to B^k \times R^m$ as follows:

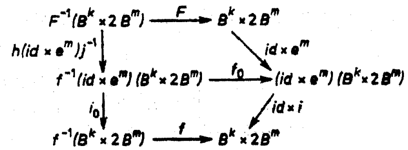
$$F(x) = \begin{cases} jF'j^{-1}(x) & \text{for } x \in j(B^k \times R^m), \\ x & \text{for } x \notin j(B^k \times R^m). \end{cases}$$

$$F = \text{id} \quad \text{on} \quad [(B^k \setminus \tfrac{5}{6}\mathring{B}^k) \times R^m] \cup [B^k \times (R^m \setminus 4\mathring{B}^m)],$$

$$F \neq F'j^{-1} \quad \text{over} \quad B^k \times 2B^m,$$

and F is still an ε-equivalence.

X. Construction of φ. We have a commutative diagram



The vertical arrows are homeomorphisms, and by composing the inverses of the two on the left we get a homeomorphism

$$\psi: f^{-1}(B^k \times 2B^m) \to F^{-1}(B^k \times 2B^m)$$

which satisfies $F\psi = f|f^{-1}(B^k \times 2B^m)$. ψ extends to a homeomorphism $\varphi: f^{-1}(U) \to F^{-1}(U)$ defined by $\varphi = f$ on $f^{-1}((B^k \setminus \frac{5}{6}\mathring{B}^k) \times R^m)$. This ends the proof of Theorem 2.1.

3. Concluding remarks. To prove Theorem 1.1 we now repeat the argument of [C–F]. First we prove a theorem corresponding to the "Main theorem" of [C–F].

For notation, let V^3 be a 3-manifold, $3 = m+k$, and let $F: V \to B^k \times R^m$ be a proper map such that $\partial V = f^{-1}(\partial B^k \times R^m)$ and f is a homeomorphism over $(B^k \times \frac{1}{2}\mathring{B}^k) \times R^m$.

THEOREM 3.1 (main theorem). *Suppose that V contains no fake 3-cells. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if f is a δ-equivalence over $B^k \times 3B^m$ then there exists a proper map $\tilde{f}: V \to B^k \times R^m$ such that:*

(1) *$\tilde{f}$ is an ε-equivalence over $B^k \times 2.5B^m$,*

(2) *$\tilde{f} = f$ over $[(B^k \setminus \frac{2}{3}\mathring{B}^k) \times R^m] \cup [B^k \times (R^m \setminus 2\mathring{B}^m)]$,*

(3) *$\tilde{f}$ is a homeomorphism over $B^k \times B^m$.*

The proof of 3.1 is precisely as in [C–F]. We have only to use the fact that V and subsets of $B^k \times S^m$ contain no fake 3-cells, and the 3-dimensional "Splitting theorem" of [J]. Having proved Theorem 3.1, we prove 1.1 as in [C–F].

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On indecomposable representations of quivers with zero-relations

by

Mirosław Dembiński, Piotr Dowbor and Andrzej Skowroński (Toruń)

Abstract. Let A be a bound quiver algebra KQ/I with zero-relations and $R = KQ/\tilde{I}$ its universal Galois covering. Applying new covering techniques [5], [6] we give a simple description of indecomposable finite dimensional representations of A in case each indecomposable finite dimensional representation of R has a peak [4].

0. Introduction. It is well known that in many cases [4], [10], [12], [14], [16], [18] the representation theory of finite dimensional algebras over an algebraically closed field can be reduced to that for partially ordered sets, shortly posets. In particular, if A is a tree algebra KQ/I of a finite tree Q with zero-relations I, then by [4] A is representation-finite, that is admits only finitely many nonisomorphic finite dimensional indecomposable representations, if and only if the partially ordered sets associated to all vertices of Q are representation-finite, and in this case each indecomposable representation of A has a peak. Similarly, by coverings techniques, the classification problem of indecomposables of a representation-finite quiver algebra with zero-relations can be reduced [12], [14] to that for representation-finite tree algebras (with zero-relations), and consequently to posets.

The purpose of this paper is to give a rather simple description of indecomposable finite dimensional representations of an arbitrary quiver algebra with zero-relations for which every indecomposable finite dimensional representation of its universal Galois covering, being a locally bounded tree category with zero-relations, has a peak. Applying the covering techniques developed recently for representation-infinite algebras by the second and third author [5], [6], we reduce the classification problem of indecomposable to that for the corresponding posets and to the classification of indecomposable finite dimensional representations over the algebra $K[T, T^{-1}]$ of Laurent polynomials. In particular, we will show that any such algebra is tame if and only if the corresponding posets are tame.

1. Notation and conventions. Throughout this paper, we denote by K an algebraically closed field. By an observation of Gabriel [3], [11] a basic connected finite dimensional K-algebra A can be written as $A = KQ/I$, where Q is a finite connected