Finally, to see that $\mathscr{V}$ is locally finite, let $\omega \in (I, X)$. For some integers n and k, $F_n(\omega) > 1/k$. Define

$$R = \{\lambda \in (I, X) |\ F_n(\lambda) > 1/k\} .$$

The set R is open in (I, X) and contains ω. We claim that for $m > \max(k, n)$ and $s \in A^*$ with $\# s = m$, $R \cap V_s = \emptyset$. This will suffice since we know already that the collection $E\mathscr{U}_{\max(k,n)}$ is locally finite. For $\lambda \in R$,

$$F_{m-1}(\lambda) \geqslant F_n(\lambda) > 1/k .$$

Thus

$$mF_{m-1}(\lambda) > m(1/k) > 1 .$$

Since $f_s(\lambda) \leqslant 1$, it follows that $g_s(\lambda) = 0$. Thus λ is not in V_s. ∎

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Connections between different amoeba algebras

by

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Abstract. The "amoeba algebra" is the complete Boolean algebra which has the effect of making the union of all null Borel sets coded in the ground model have measure 0 in the corresponding Boolean extension. Six different versions of the amoeba algebra are studied, together with the localization algebra, and connections, in some cases isomorphism and in some cases forcing equivalence, are established between them.

§ 1. Introduction. A number of different versions of Martin and Solovay's original "amoeba" algebras have been considered. In their original application [5] the relevant set of conditions was taken to be the set of open subsets of the real line of measure less than a fixed ε, partially ordered by inclusion, approximating to an open set of measure ε. In [8] we took instead a "variable" ε. That is, a condition was a pair (p, ε) where p is an open subset of $\boldsymbol{R}$ of measure less than ε, giving the information about the generic open set X that $p \subseteq X$ and $\mu(X) < \varepsilon$. The main reason for this was to enable us to show that the amoeba set of conditions P satisfies $\mathrm{RO}(P) \cong \mathrm{RO}(P \times P)$ where $\mathrm{RO}(P)$ is the complete Boolean algebra associated with P (the "regular open" algebra). Whether this is true for Martin and Solovay's "fixed measure" case we still do not know. And then there are the amoeba algebras on compact intervals I (or equivalently on 2^{ω}) derived from the set of (relatively) open subsets of I of measure less than ε, which were used by Shelah in [7], and also by Miller and others in their investigations into the connections between measure and category on the real line.

What all these algebras $\boldsymbol{B}$ have in common is the following. In each case the Boolean value in $V^{\boldsymbol{B}}$ of the statement

$$\text{“}\mu\{x \in \boldsymbol{R}\colon x \text{ is not random over } V\} = 0\text{”}$$

is $\mathbf{1}$, where μ denotes Lebesgue measure. What ideally we would like to know is that this statement holds in an extension of V if and only if the extension contains a V-generic filter on $\boldsymbol{B}$. In the absence of this, however, the next best thing seems to be to show that as many as possible of the known versions of the amoeba algebra are isomorphic, or at any rate, are equivalent in the sense of forcing. This was in part achieved by Kutyłowski in [4], where he showed that of six versions, three "unbounded" ones are equivalent and three "bounded" ones are equivalent, even when the parameters are varied. We shall give our versions of these results and extend them as follows. In response to a question of Cichoń we establish the existence of isomorphisms between bounded amoeba algebras having different parameters, previously only known to be equivalent. In addition we include a proof, due to Fremlin, that the bounded and unbounded amoeba algebras are equivalent.

There are, of course, many other versions of the amoeba algebras one could concoct, but it seems pointless to treat them all individually; it would be better to prove a general result as indicated above. Other related algebras were discussed in [8], connected with category on the real line, and dominating functions. It is now known that these are strictly "smaller" than the amoeba algebra. Though related to Shelah's result [7] that "Solovay's inaccessible" is necessary for the construction of a model of ZF in which every set of reals is Lebesgue measurable, but not for the construction of a model in which every set has the property of Baire, this can be proved much more easily. Cichoń has pointed out that since the "amoeba algebra for category" is σ-centred, it actually adds no random reals at all, so certainly cannot contain the amoeba algebra as a complete subalgebra. The correct intermediate subalgebra to consider, which enabled Bartoszyński [1] to prove that the additivity of category is at least as great as the additivity of measure, is the "localization algebra". Though there are several versions of this algebra too, (which are not known to be equivalent) we shall just concentrate on one which we conjecture is equivalent to the amoeba algebras. We give evidence for this conjecture by producing an embedding in one direction and showing a strong connection (though not an embedding) in the other.

I would like to thank Jacek Cichoń and David Fremlin for valuable discussions, Mirosław Kutyłowski for sending me a copy of [4], David Fremlin for sending me his proof of Theorem 4.3, and the University of Wrocław for hospitality while some of this work was carried out.

§ 2. Definitions of the algebras. By an "algebra" I understand a complete Boolean algebra. In the context of forcing, however, it is generally much more convenient to work with a notion of forcing, i.e. a partially ordered set P regarded as a dense subset of the corresponding complete Boolean algebra $\boldsymbol{B} = \mathrm{RO}(P)$. To show, for example, that $\mathrm{RO}(P)$ and $\mathrm{RO}(Q)$ are isomorphic, it is sufficient to show that P and Q have isomorphic dense subsets. In terms of forcing, saying that $\mathrm{RO}(P)$ and $\mathrm{RO}(Q)$ are isomorphic amounts to saying that there is a V-definable function F such that whenever $\mathscr{F}$ is a V-generic subset of P then $F(\mathscr{F})$ is a V-generic subset of Q, and that F is 1-1 and onto the class of all V-generic subsets of Q. Intuitively, the passage from F to $F(\mathscr{F})$ neither loses any information, nor adds redundant information. In some cases we have to settle for "equivalence" of two notions of forcing, a weaker notion than isomorphism of their regular open algebras or even "embeddability" (i.e. in one direction only). We say that Q is *embeddable in P* if whenever $\mathscr{F}$ is a V-generic subset of P, $V[\mathscr{F}]$ contains a V-generic subset of Q. We say that P and Q are *equivalent* if each is embeddable in the other. It is perfectly possible to describe these notions in V, without reference to the Boolean extensions, but we shall not do so, since it is the relations between the extensions which is of primary interest to us. Notice also in this connection that we have been talking about generic extensions of "the universe" V, so that strictly speaking the discussion takes place in appropriate Boolean-valued universes $V^{\boldsymbol{B}}$. Alternatively, one may work with extensions of a suitable countable transitive model M. There are standard methods for passing between these two approaches, as described for example in [2].

There are seven types of algebra we consider here, six types of amoeba algebra, and the localization algebra. All are defined via the naturally associated notions of forcing. The amoeba algebras fall into two families, the "unbounded" ones, indicated by P, and the "bounded" ones, indicated by Q. In the Boolean algebras, the "stronger" condition will be nearer to $\boldsymbol{O}$, i.e. $p \leqslant q$ will mean that p is an extension of q. In the partial orderings the natural ordering very often goes the other way, (usually $\subseteq$). We shall try to avoid confusion by use of the word "extension" where possible. The partial orderings are as follows.

P^{I}_α is the set of open subsets of $\boldsymbol{R}$ of measure less than α, partially ordered by inclusion.

Q^{I}_α is the set of (relatively) open subsets of $[0, 1]$ of measure less than α, partially ordered by inclusion.

Here, and throughout α is a fixed real number. For P^{I}_α, α is any positive real number, and for Q^{I}_α, $\alpha \in (0, 1)$. The conditions approximate a "new" open subset X of $\boldsymbol{R}$ of measure α, and if $\mathscr{F}$ is a V-generic subset of P^{I}_α or Q^{I}_α then $X = \bigcup \mathscr{F}$. (More accurately, we should say that X is the union in $V[\mathscr{F}]$ of the open sets *coded* in the same way that members of $\mathscr{F}$ are, i.e. $X = \{x \in \boldsymbol{R}\colon (\exists a, b)\ (a < x < b\ \&\ (a, b) \in \mathscr{F})\}$. Under these circumstances we say that X *is* a V-generic subset of $\boldsymbol{R}$ of $[0, 1]$ *for* P^{I}_α or Q^{I}_α. We also say that X is *amoeba-generic of fixed measure*. The four other types of amoeba algebra are as follows.

$$\begin{aligned} P^{\mathrm{II}} &= \{(p, \varepsilon)\colon p \text{ an open subset of } \boldsymbol{R}\ \&\ \varepsilon \in \boldsymbol{R}\ \&\ \mu(p) < \varepsilon\},\\ Q^{\mathrm{II}} &= \{(p, \varepsilon)\colon p \text{ a relatively open subset of } [0, 1]\ \&\ \varepsilon \leqslant 1\ \&\ \mu(p) < \varepsilon\},\\ P^{\mathrm{III}}_\alpha &= \{(p, \varepsilon) \in P^{\mathrm{II}}\colon \varepsilon \leqslant \alpha\},\ (\alpha \in (0, \infty) \text{ fixed}),\\ Q^{\mathrm{III}}_\alpha &= \{(p, \varepsilon) \in Q^{\mathrm{II}}\colon \varepsilon \leqslant \alpha\},\ (\alpha \in (0, 1] \text{ fixed}). \end{aligned}$$

In these four cases, (q, δ) is an extension of (p, ε) if $p \subseteq q\ \&\ \delta \leqslant \varepsilon$. These conditions approximate $X = \bigcup \{p\colon (\exists \varepsilon)(p, \varepsilon) \in \mathscr{F}\}$ where $\mathscr{F}$ is a V-generic subset of the partial ordering, and (p, ε) gives the information that $p \subseteq X\ \&\ \mu(X) < \varepsilon$. We may call the X arising in this way an *amoeba-generic set of variable measure* (in that its measure is not fixed beforehand).

Finally, the localization algebra is defined by the following set of conditions LOC.

LOC is the set of all $p \in \prod_{n\in\omega} [\omega]^{\leqslant 2^n}$ such that for some N, $(\forall n \geqslant N)|p(n)| \leqslant 2^N$. Here q is an extension of p if $(\forall n)$ $p_n \subseteq q_n$. The conditions approximate a function $\varphi \in \prod_{n\in\omega} [\omega]^{2^n}$. Thus for each n φ_n is a subset of ω of size 2^n, and the information about φ given by $p \in$ LOC is $(\forall n)$ $p_n \subseteq \varphi_n$. Any φ arising in this way is said to be *V-generic localizing.*

We have taken 2^n as being most convenient for our purposes, though by techniques described in [3], any other strictly increasing function of n could be used in its place.

The results we shall prove about these notions of forcing are as follows:

$P^{\rm I}_\alpha$, $P^{\rm II}_\alpha$, $P^{\rm III}_\alpha$, $Q^{\rm I}_\alpha$, $Q^{\rm II}_\alpha$, $Q^{\rm III}_\alpha$ are equivalent.

All members of $\{{\rm RO}(P^{\rm I}_\alpha)\colon \alpha \in (0, \infty)\}$ are isomorphic.

All members of $\{{\rm RO}(P^{\rm II})\} \cup \{{\rm RO}(P^{\rm III}_\alpha)\colon \alpha \in (0, \infty)\}$ are isomorphic.

All members of $\{{\rm RO}(Q^{\rm I}_\alpha)\colon \alpha \in (0, 1)\}$ are isomorphic.

All members of $\{{\rm RO}(Q^{\rm III}_\alpha)\colon \alpha \in (0, 1]\}$ are isomorphic (and, of course, $Q^{\rm II} = Q^{\rm III}_1$).

Any P or Q is embeddable in LOC.

The methods of proof will be as follows. In cases where the existence of an isomorphism is to be established, we shall construct an order-isomorphism between dense subsets of the Boolean algebras. In cases where an embedding is asserted to exist of T_1 into T_2 we shall show how to define a function F from V-generic subsets $\mathscr{F}$ of T_2 to V-generic subsets $F(\mathscr{F})$ of T_1. In this section we illustrate these methods in the simplest cases, and leave the more involved arguments to §§ 3 and 4.

THEOREM 2.1. *For any* $\alpha, \beta \in (0, \infty)$, $P^{\rm I}_\alpha \cong P^{\rm I}_\beta$ *(as partially ordered sets) and* $P^{\rm III}_\alpha \cong P^{\rm III}_\beta$, *so that* ${\rm RO}(P^{\rm I}_\alpha) \cong {\rm RO}(P^{\rm I}_\beta)$ *and* ${\rm RO}(P^{\rm III}_\alpha) \cong {\rm RO}(P^{\rm III}_\beta)$.

Proof. $\vartheta\colon P^{\rm I}_\alpha \to P^{\rm I}_\beta$ is given by $\vartheta(p) = \frac{\beta}{\alpha} p = \left\{\frac{\beta}{\alpha} x\colon x \in p\right\}$. $\vartheta\colon P^{\rm III}_\alpha \to P^{\rm II}_\beta$ is given by $\vartheta(p, \varepsilon) = \left(\frac{\beta}{\alpha} p, \frac{\beta}{\alpha} \varepsilon\right)$.

THEOREM 2.2 (Kutyłowski). ${\rm RO}(P^{\rm II}) \cong {\rm RO}(P^{\rm III}_1)$.

Proof. We give a rather more involved proof than Kutyłowski, since the ideas will be needed in § 3. The basic idea, however, is the same. We shall find isomorphic dense subsets of ${\rm RO}(P^{\rm II})$ and ${\rm RO}(P^{\rm III}_1)$. Firstly let $\{a_n\colon n \in \omega\}$ and $\{b_n\colon n \in \omega\}$ be infinite maximal antichains of $P^{\rm II}$ and $P^{\rm III}_1$ respectively. These are countable since each notion of forcing fulfils the c.c.c. The dense subsets of ${\rm RO}(P^{\rm II})$ and ${\rm RO}(P^{\rm III}_1)$ will then be $\{b \in {\rm RO}(P^{\rm II})\colon (\exists n)\ b \leqslant a_n\}$ and $\{b \in {\rm RO}(P^{\rm III}_1)\colon (\exists n)\ b \leqslant b_n\}$ respectively. To show that these are isomorphic, we show that for each n the set of extensions of a_n is itself isomorphic to ${\rm RO}(P^{\rm III}_1)$ (and similarly for b_n), and then all the individual isomorphisms can be fitted together. The point is that although $P^{\rm II}$ itself allows arbitrarily large values of ε, beyond any fixed element the situation is indistinguishable from that in $P^{\rm III}_1$.

Let $a_n = (p, \varepsilon)$. Firstly observe that, as in 2.1, the map ϑ given by $\vartheta(x) = \alpha x$, where $\alpha = 1/(\varepsilon - \mu(p))$, takes the set of extensions of (p, ε) in $P^{\rm II}$ isomorphically onto the set X of extensions of (q, δ) in $P^{\rm II}$, where $q = \alpha p$ and $\delta = \alpha\varepsilon$, and $\delta - \mu(q) = \alpha(\varepsilon - \mu(p)) = 1$. We show that ${\rm RO}(X) \cong {\rm RO}(P^{\rm III}_1)$. This is clear intuitively but there are some details to be checked.

Let q_1 be the set of points x of $\boldsymbol{R}$ at which q has Lebesgue density 1, i.e. such that $\lim_{\gamma\to 0} \frac{1}{2\gamma}\big(\mu((x-\gamma, x+\gamma) \cap q)\big) = 1$. By the Lebesgue density theorem [6, p. 17], q_1 differs from q by a set of measure 0. In addition, since q is open, $q \subseteq q_1$. Let f be the map which destroys all intervals of q_1, identifying their endpoints, and maps 0 to 0. This may be formally defined by

$$f(x) = \begin{cases} x - \mu((0, x) \cap q_1) & x \geqslant 0, \\ x + \mu((x, 0) \cap q_1) & x < 0. \end{cases}$$

f is then order-preserving, measure-preserving on $\boldsymbol{R} - q_1$, maps $\boldsymbol{R} - q_1$ onto $\boldsymbol{R}$, and is almost 1-1, in the sense that if $f(x) = f(y)$ for $x \neq y$ then x and y lie in the closure of some interval of q_1. Let D be the set of extensions (q', δ') of (q, δ) such that q' contains the closure of each interval of q_1. Since $q_1 \supseteq q$ and $\mu(q_1 - q) = 0$, D is dense in X (dense open actually). Let E be the set of $(q', \delta') \in P^{\rm III}_1$ such that q' contains all the images under f of endpoints of intervals of q_1. Similarly E is dense in $P^{\rm III}_1$. We may then map D to E by g where

$$g(q', \delta') = \big(f(q'), \delta' - \mu(q)\big),$$

and this establishes the isomorphism of ${\rm RO}(X)$ and ${\rm RO}(P^{\rm III}_1)$.

We now prove two "one-way" results. To formulate the first it is easiest to work with a modified version of $P^{\rm II}$, which we denote by $P^{\rm II}_+$. This is $\{(p, \varepsilon) \in P^{\rm II}\colon p \subseteq (0, \infty)\}$, with the same partial ordering as before. The proof of the following result is similar to that of Theorem 6.1 in [8].

THEOREM 2.3. (i) *If* A *is a* V*-generic subset of* $\boldsymbol{R}$ *for* $P^{\rm I}_1$, *then* $A \cap (0, \infty)$ *is a* V*-generic subset of* $(0, \infty)$ *for* $P^{\rm II}_+$.

(ii) ${\rm RO}(P^{\rm II}) \cong {\rm RO}(P^{\rm II}_+)$.

(iii) *If* $\mathscr{F}$ *is a* V*-generic subset of* $P^{\rm I}_1$ *then* $V[\mathscr{F}]$ *contains a* V*-generic subset of* $P^{\rm II}$.

(iv) *If* $\mathscr{F}$ *is a* V*-generic subset of* $Q^{\rm I}_\alpha$ *then* $V[\mathscr{F}]$ *contains a* V*-generic subset of* $Q^{\rm II}$.

Proof. (i) Let D be a dense open subset of $P^{\rm II}_+$ lying in V, and let

$$E = \{p \in P^{\rm I}_1\colon \big((p \cap (0, \infty)), 1 - \mu(p \cap (-\infty, 0))\big) \in D\}.$$

We show that E is a dense open subset of P_1^{I}. Let $p\in P_1^{\mathrm{I}}$ be arbitrary. Then

$$\big((p\cap(0,\infty)),\,1-\mu(p\cap(-\infty,0))\big)\in P_+^{\mathrm{II}}$$

since

$$\mu(p\cap(0,\infty)) = \mu(p)-\mu(p\cap(-\infty,0))<1-\mu(p\cap(-\infty,0))\,.$$

Let (q,ε) be an extension of $\big((p\cap(0,\infty),\,1-\mu(p\cap(-\infty,0))\big)$ lying in D. Thus $p\cap(0,\infty)\subseteq q\ \&\ \varepsilon\leqslant 1-\mu(p\cap(-\infty,0))\big)$. Now

$$\mu(p\cup q) = \mu(p\cap(-\infty,0))+\mu(q)\leqslant 1-\varepsilon+\mu(q)\,,$$

so there is an open $p'\supseteq p\cup q$ of measure $1-\varepsilon+\mu(q)$ such that $p'\cap(0,\infty)=q$. Then $\mu(p')<1$ since $\mu(q)<\varepsilon$, and so $p'\in P_1^{\mathrm{I}}$. Also

$$\begin{aligned}\big(p'\cap(0,\infty),\,1-\mu(p'\cap(-\infty,0))\big) &= \big(q,\,1-(\mu(p')-\mu(p'\cap(0,\infty)))\big)\\ &= \big(q,\,1-\mu(p')+\mu(q)\big) = (q,\varepsilon)\in D\,,\end{aligned}$$

showing that p' is an extension of p lying in E. Hence E is dense open.

As A is V-generic on $\boldsymbol{R}$ for P_1^{I}, $p\in E$ for some $p\subseteq A$. Hence

$$(p\cap(0,\infty)),\,1-\mu(p\cap(-\infty,0))\in D\,.$$

Now

$$\mu(A\cap(0,\infty)) = 1-\mu(A\cap(-\infty,0))<1-\mu(p\cap(-\infty,0))\,.$$

Therefore $\{(q,\varepsilon)\in P_+^{\mathrm{II}}:\ q\subseteq A\cap(0,\infty)\ \&\ \mu(A\cap(0,\infty))<\varepsilon\}$ intersects D, as required for the V-genericity of $A\cap(0,\infty)$ for P_+^{II}.

(ii) The relevant isomorphism is induced between dense subsets of P_+^{II} and P^{II} by using the map which rearranges the semi-open intervals $(0,1]$, $(1,2]$, $(2,3]$, ... and which may be explicitly defined by

$$\vartheta(x) = \begin{cases} x-n & \text{if } x\in(2n,2n+1],\\ x-3n-2 & \text{if } x\in(2n+1,2n+2]\,.\end{cases}$$

Because of trouble with the integer points of division, the isomorphism should just be defined on the dense set $\{(p,\varepsilon)\in P_+^{\mathrm{II}}:\ (\forall n\geqslant 1)(n\in p)\}$.

(iii) follows immediately from (i) and (ii).

(iv) Let A be an amoeba V-generic subset of $[0,1]$ for Q_α^{I}. Then $B=[0,1]\cap 2A$ where $2A=\{2x:\ x\in A\}$ is seen to be amoeba V-generic for Q^{II} by the techniques of (i).

By similar methods we may establish the following.

THEOREM 2.4. (i) *If A is an amoeba V-generic subset of $\boldsymbol{R}$ for P_1^{I} then $A\cap[0,1]$ is amoeba V-generic for Q^{II}, and $V[A]$ also contains V-generic subsets of P_α^{III} and Q_α^{III}, all α.*

(ii) *If $\mathscr{F}$ is a V-generic subset of P^{II}, then $V[\mathscr{F}]$ contains V-generic subsets of Q^{II} and Q_α^{III}, all α.*

Proof. The only additional point to note is that the trick of intersecting may not work if the measure of A is too great. In this case, one applies a contraction (by $\beta\in(0,1)$ lying in V) initially so that its measure is less than the relevant α.

We have now shown how to pass from amoeba-generic sets of fixed measure to ones of variable measure (in most cases). Another method will show how to pass from amoeba-generic sets of variable measure to ones of fixed measure.

THEOREM 2.5. *Let A be an amoeba V-generic subset of $\boldsymbol{R}$ for P^{II} of measure <1 and let $x=\sup\{y:\ \mu(A\cup(0,y))\leqslant 1\}$. Then $A\cup(0,x)$ is amoeba V-generic for P_1^{I}.*

Proof. For any $(p,\varepsilon)\in P_1^{\mathrm{III}}$ let $f(p,\varepsilon)=\sup\{y:\ \mu(p\cup(0,y))\leqslant 1-\varepsilon+\mu(p)\}$. The idea of this definition is that $f(p,\varepsilon)$ should be the largest y such that (p,ε) tells us that $y\leqslant x$. Let $D\in V$ be a dense open subset of P_1^{I}, and let

$$E=\{(p,\varepsilon)\in P_1^{\mathrm{III}}:\ p\cup(0,f(p,\varepsilon))\in D\}\,.$$

We show that E is a dense open subset of P_1^{III}.

Let (p,ε) be an arbitrary member of P_1^{III}. Then by the definition of $x'=f(p,\varepsilon)$, $\mu(p\cup(0,x'))=1-\varepsilon+\mu(p)<1$, so that $p\cup(0,x')\in P_1^{\mathrm{I}}$. Let $p\cup(0,x')\subseteq q_1\in D$, and let $q=(q_1-[0,x'])\cup(p\cap[0,x'])$. Now $\mu(q_1)<1$ since $q_1\in P_1^{\mathrm{I}}$, so

$$\begin{aligned}\mu(q) &= \mu(q_1-[0,x'])+\mu(p\cap(0,x'))\\ &\leqslant \mu\big(q_1-(p\cup(0,x'))\big)+\mu\big((p\cup(0,x'))-[0,x']\big)+\mu(p\cap(0,x'))\\ &< 1-(1-\varepsilon+\mu(p))+\mu(p-[0,x'])+\mu(p\cap(0,x'))\\ &= \varepsilon-\mu(p)+\mu(p)=\varepsilon\,.\end{aligned}$$

Therefore $(q,\varepsilon)\in P_1^{\mathrm{III}}$.

Let $\mu(q)<\delta<\min\big(\varepsilon,\,\mu(q)+(1-\mu(q_1))\big)$. Then (q,δ) is an extension of (p,ε). Also, by choice of δ, $\mu(q\cup(0,x'))=\mu(q_1)<1-\delta+\mu(q)$. Therefore $f(q,\delta)\geqslant x'$ so that $q\cup(0,f(q,\delta))\supseteq q\cup(0,x')=q_1\in D$, showing that $(q,\delta)\in E$.

Since E is dense open in P_1^{III}, $E'=\{(p,\varepsilon)\in P^{\mathrm{II}}:\mu(p)\geqslant 1$ or $(p,\varepsilon)\in E\}$ is a dense open subset of P^{II}, and so $(p,\varepsilon)\in E'$ for some p,ε such that $p\subseteq A$ and $\mu(A)<\varepsilon$. Since $\mu(A)<1$ by hypothesis, $(p,\varepsilon)\in E$ and $p\cup(0,f(p,\varepsilon))\in D$. Clearly $f(p,\varepsilon)\leqslant x$, so the genericity of $A\cup(0,x)$ is established.

THEOREM 2.6. (i) *If $\mathscr{F}$ is V-generic on P^{II}, or P_α^{III} for some α, then $V[\mathscr{F}]$ contains a V-generic subset of P_β^{I}, any $\beta\in(0,\infty)$.*

(ii) *If $\mathscr{F}$ is V-generic on Q_α^{III} for some $\alpha<1$, then $V[\mathscr{F}]$ contains a V-generic subset of Q_β^{I}, any $\beta\in[\alpha,1)$.*

§ 3. Some isomorphisms. As remarked in Theorem 2.1, P_α^{I} and P_β^{I} are trivially isomorphic and so are P_α^{III} and P_β^{III}, for any $\alpha,\beta\in(0,\infty)$. We had to work slightly harder to establish the isomorphism of P^{II} and P_α^{III}, and that proof illustrates some of the ideas needed in the construction of isomorphisms between Q_α^{I} and Q_β^{I}, and between Q^{II}, Q_α^{III} and Q_β^{III} for $\alpha,\beta\in(0,1)$. Cichoń remarked that $\boldsymbol{B}=\mathrm{RO}(Q_\alpha^{\mathrm{I}})$

and $C = \mathrm{RO}(Q^{\mathrm{I}}_{\beta})$ are "locally isomorphic", in the sense that if $b \in \boldsymbol{B}$, $c \in \boldsymbol{C}$ are nonzero then there are nonzero $b' \leqslant b$ and $c' \leqslant c$ such that $\boldsymbol{B}$ and $\boldsymbol{C}$ are isomorphic below b' and c'. The problem therefore was to turn these local isomorphisms into a single isomorphism.

LEMMA 3.1. *Suppose that S is a countable dense subset of (α_0, α), where $1-\sqrt{1-\alpha} \leqslant \alpha_0 < \alpha < 1$. Then there is a maximal antichain A of Q^{I}_{α} such that $A = \{A_s\colon s \in S\}$ where each A_s is countably infinite, and for each s, if $p \in A_s$ then $\mu(p) = s$.*

Proof. Firstly let $\{p_n\colon n \in \omega\}$ be infinitely many independent open subsets of $[0, 1]$ of measure α_0. The point of the choice of α_0 is that the p_n are then pairwise incompatible in Q^{I}_{α}. For if $m \neq n$, $\mu(p_m \cup p_n) = 1-(1-\alpha_0)^2 \geqslant \alpha$. Hence $A' = \{p_n\colon n \in \omega\}$ is an antichain in Q^{I}_{α}. Now S is countable, so we may let $A' = \bigcup\{A'_s\colon s \in S\}$ where the A'_s are infinite and pairwise disjoint. If $p \in A'_s$, let p' be an extension of p of measure s, and let $A'' = \{p'\colon p \in A\}$. Then A'' is also an antichain. Finally, let A be an antichain of Q^{I}_{α} containing A'', maximal subject to the measure of each of its members lying in S. Since such elements form a dense subset of Q^{I}_{α}, A is a maximal antichain of Q^{I}_{α}.

LEMMA 3.2. *Let $\alpha, \beta \in (0, 1)$. Then there are α', β' such that*

$$1-\sqrt{1-\alpha} \leqslant \alpha' < \alpha, \quad 1-\sqrt{1-\beta} \leqslant \beta' < \beta \quad \textit{and} \quad \frac{\alpha-\alpha'}{1-\alpha'} = \frac{\beta-\beta'}{1-\beta'}.$$

Proof. Assume without loss of generality that $\alpha \leqslant \beta$. Let

$$\alpha' = 1-\sqrt{1-\alpha} \quad \text{and} \quad \beta' = 1-\frac{1-\beta}{\sqrt{1-\alpha}}.$$

Since $0 < \alpha < 1$, $1-\alpha < \sqrt{1-\alpha}$. Thus $1-\sqrt{1-\alpha} < \alpha$. Also, since $0 < \alpha \leqslant \beta < 1$, $\dfrac{1}{\sqrt{1-\alpha}} \leqslant \dfrac{1}{\sqrt{1-\beta}}$, and so $1-\beta' = \dfrac{1-\beta}{\sqrt{1-\alpha}} \leqslant \sqrt{1-\beta}$, giving $1-\sqrt{1-\beta} \leqslant \beta'$. Now $1 < \dfrac{1}{\sqrt{1-\alpha}}$, and thus

$$\beta' = 1-\frac{1-\beta}{\sqrt{1-\alpha}} < 1-(1-\beta) = \beta.$$

Finally,

$$\frac{1-\alpha}{1-\alpha'} = \frac{1-\alpha}{\sqrt{1-\alpha}} = \sqrt{1-\alpha} = \frac{1-\beta}{(1-\beta)/\sqrt{1-\alpha}} = \frac{1-\beta}{1-\beta'},$$

from which it follows that

$$\frac{\alpha-\alpha'}{1-\alpha'} = \frac{\beta-\beta'}{1-\beta'}.$$

THEOREM 3.3. *If $\alpha, \beta \in (0, 1)$ then $\mathrm{RO}(Q^{\mathrm{I}}_{\alpha}) \cong \mathrm{RO}(Q^{\mathrm{I}}_{\beta})$.*

Proof. The key to the proof is Cichoń's remark that if

$$\frac{\alpha-\mu(p)}{1-\mu(p)} = \frac{\beta-\mu(q)}{1-\mu(q)} = \gamma$$

where $p \in Q^{\mathrm{I}}_{\alpha}$ and $q \in Q^{\mathrm{I}}_{\beta}$, then the extensions of p in $\mathrm{RO}(Q^{\mathrm{I}}_{\alpha})$ are isomorphic to the extensions of q in $\mathrm{RO}(Q^{\mathrm{I}}_{\beta})$. This is clear intuitively, and is proved as in Theorem 2.2 by showing that each set of extensions is isomorphic to $\mathrm{RO}(Q^{\mathrm{I}}_{\gamma})$. The measure of the set "left over" by p is $1-\mu(p)$ and $\alpha-\mu(p)$ is the remaining measure which may be covered.

Now let us choose α', β' by Lemma 3.2 such that

$$1-\sqrt{1-\alpha} \leqslant \alpha' < \alpha, \quad 1-\sqrt{1-\beta} \leqslant \beta' < \beta, \quad \text{and} \quad \frac{\alpha-\alpha'}{1-\alpha'} = \frac{\beta-\beta'}{1-\beta'}.$$

Let S be any countable dense subset of (α', α), and let ϑ be the map given by $\vartheta(x) = cx+d$, which takes α' to β' and α to β $\left(\text{so that } c = \dfrac{\beta-\beta'}{\alpha-\alpha'} \text{ and } d = \dfrac{\alpha\beta'-\alpha'\beta}{\alpha-\alpha'}\right)$.

Let $T = \vartheta(S)$. Then T is a countable dense subset of (β', β). Also since $\dfrac{\alpha-\alpha'}{1-\alpha'} = \dfrac{\beta-\beta'}{1-\beta'}$, $\alpha-\alpha'-\alpha\beta'+\alpha'\beta = \beta-\beta'$, so that $1-d = c$. Thus, for each $s \in S$,

$$\frac{\beta-\vartheta(s)}{1-\vartheta(s)} = \frac{\vartheta(\alpha)-\vartheta(s)}{1-\vartheta(s)} = \frac{c(\alpha-s)}{1-cs-d} = \frac{\alpha-s}{1-s}.$$

By Lemma 3.1 there are maximal antichains $A = \bigcup\limits_{s\in S} A_s$ and $B = \bigcup\limits_{t\in T} B_t$ of Q^{I}_{α} and Q^{I}_{β} respectively such that $|A_s| = |B_t| = \aleph_0$, and members of A_s, B_t have measure s, t respectively. Let φ map A_s 1-1 onto $B_{\vartheta s}$ for each s. Since $\dfrac{\beta-\vartheta(s)}{1-\vartheta(s)} = \dfrac{\alpha-s}{1-s}$, φ extends to an isomorphism between dense subsets of $\mathrm{RO}(Q^{\mathrm{I}}_{\alpha})$ and $\mathrm{RO}(Q^{\mathrm{I}}_{\beta})$ by Cichoń's remark.

THEOREM 3.4. *$\mathrm{RO}(Q^{\mathrm{II}})$ and $\mathrm{RO}(Q^{\mathrm{III}}_{\alpha})$ for $\alpha \in (0, 1)$ are all isomorphic.*

Proof. As in the proof of Lemma 3.1 it is clear that $Q^{\mathrm{III}}_{\alpha}$ has an infinite maximal antichain. Moreover, $D = \{(p, \varepsilon) \in Q^{\mathrm{II}}\colon \varepsilon < 1\}$ is dense open in Q^{II}, so if A is an infinite maximal antichain of D, A will also be a maximal antichain of Q^{II}. By the methods of 3.3 it is sufficient to show that any bijection between these antichains can be extended to an isomorphism between the algebras. Now the algebra below (p, ε) is isomorphic to Q^{III}_{β} where $\beta = \dfrac{\varepsilon-\mu(p)}{1-\mu(p)}$, so it is sufficient to show that for any $\alpha, \beta \in (0, 1)$, $\mathrm{RO}(Q^{\mathrm{III}}_{\alpha}) \cong \mathrm{RO}(Q^{\mathrm{III}}_{\beta})$.

LEMMA 3.5. *Let $\alpha, \beta \in (0, 1)$. Then there are countably infinite families $\mathscr{U} = \{U_n\colon n \in \omega\}$ and $\mathscr{V} = \{V_n\colon n \in \omega\}$ of open intervals such that*

(i) $\bigcup_{n\in\omega} U_n$ *is dense in* $(0,\alpha)$ *and* $\bigcup_{n\in\omega} V_n$ *is dense in* $(0,\beta)$,

(ii) $m \neq n \to U_m\cap U_n = V_m\cap V_n = \emptyset$,

(iii) *if* $(\gamma,\delta)\in\mathscr{U}\cup\mathscr{V}$ *then* $1-\sqrt{1-\delta}\leqslant\gamma<\delta$,

(iv) *if* $(\gamma,\delta)\in\mathscr{U}$ *then there is* $(\gamma',\delta')\in\mathscr{V}$ *such that* $\dfrac{\delta-\gamma}{1-\gamma}=\dfrac{\delta'-\gamma'}{1-\gamma'}$, *and similarly if* $(\gamma',\delta')\in\mathscr{V}$ *is given.*

Proof. Suppose $\alpha\leqslant\beta$. Let $\alpha_0=\alpha$ and $\alpha_{n+1}=1-\sqrt{1-\alpha_n}$. Then $1-\alpha_n = (1-\alpha)^{1/2^n}$, so (α_n) is monotonic decreasing tending to 0. We let

$$\mathscr{U}=\{(\alpha_{n+1},\alpha_n):\ n\in\omega\}.$$

Then (i), (ii), (iii) hold for $\mathscr{U}$.

We define β_ξ by transfinite induction on ξ as follows. Firstly $\beta_0=\beta$. If β_ξ is defined for all $\xi<\lambda$ where λ is a limit ordinal, we let $\beta_\lambda=\inf_{\xi<\lambda}\beta_\xi$. Otherwise suppose β_ξ has been defined, and $\beta_\xi>0$. Then for some n, $\beta_\xi\geqslant\alpha_n$. Let n be the least such, and let $\beta_{\xi+1}=1-\dfrac{1-\beta_\xi}{\sqrt{1-\alpha_n}}$. Since $0<1-\alpha_n<1$, $\beta_{\xi+1}<1-(1-\beta_\xi)=\beta_\xi$, so the sequence (β_ξ) is monotonic decreasing. Also, as $\alpha_{n+1}<\alpha_n\leqslant\beta_\xi$, $1-\beta_\xi<1-\alpha_{n+1}=\sqrt{1-\alpha_n}$, so $0<1-\dfrac{1-\beta_\xi}{\sqrt{1-\alpha_n}}=\beta_{\xi+1}$. Thus (β_ξ) is a monotonic decreasing sequence of non-negative numbers, and so $\beta_\xi=0$ for some countable ordinal ξ. We let $\mathscr{V}=\{(\beta_{\xi+1},\beta_\xi):\ \xi\in 0_n\}$, so that $\mathscr{V}$ is a countable family. Clearly (i) and (ii) hold for $\mathscr{V}$, and (iii) holds, since from $\alpha_n\leqslant\beta_\xi$ and $0<1-\alpha_n<1$ it follows that

$$1-\sqrt{1-\beta_\xi}=1-\frac{1-\beta_\xi}{\sqrt{1-\beta_\xi}}\leqslant 1-\frac{1-\beta_\xi}{\sqrt{1-\alpha_n}}=\beta_{\xi+1}<\beta_\xi .$$

Finally we have to establish (iv). If $(\gamma,\delta)\in\mathscr{V}$ then $(\gamma,\delta)=(\beta_{\xi+1},\beta_\xi)$ for some ξ, and $\beta_{\xi+1}=1-\dfrac{1-\beta_\xi}{\sqrt{1-\alpha_n}}$ for some n. Thus

$$\frac{1-\beta_\xi}{1-\beta_{\xi+1}}=\sqrt{1-\alpha_n}=\frac{1-\alpha_n}{\sqrt{1-\alpha_n}}=\frac{1-\alpha_n}{1-\alpha_{n+1}},\quad\text{and}\quad\frac{\beta_\xi-\beta_{\xi+1}}{1-\beta_{\xi+1}}=\frac{\alpha_n-\alpha_{n+1}}{1-\alpha_{n+1}}$$

follows. Conversely we have to show that for any n there is some ξ such that n is the least satisfying $\alpha_n\leqslant\beta_\xi$. Since $\inf_{\xi\in 0_n}\beta_\xi=0$, there is a least ξ' such that $\beta_{\xi'}<\alpha_n$. As $\alpha\leqslant\beta$, ξ' must be a successor, so we let $\xi'=\xi+1$ and $\beta_{\xi+1}<\alpha_n\leqslant\beta_\xi$. Let m be the least such that $\alpha_m\leqslant\beta_\xi$. Then

$$\alpha_{m+1}=1-\sqrt{1-\alpha_m}=1-\frac{1-\alpha_m}{\sqrt{1-\alpha_m}}\leqslant 1-\frac{1-\beta_\xi}{\sqrt{1-\alpha_m}}=\beta_{\xi+1},$$

so $n<m+1$ giving $m=n$. In other words, n *is* the least satisfying $\alpha_n\leqslant\beta_\xi$, as desired.

We may now conclude the proof of Theorem 3.4. Let $\mathscr{U}=\{U_n:\ n\in\omega\}$ and $\mathscr{V}=\{V_n:\ n\in\omega\}$ be countable families as provided by Lemma 3.5. For each n, if $U_n=(\alpha_n',\alpha_n)$, let A_n be an infinite family of open subsets of $[0,1]$ whose members are independent of measure α_n'. Then as $1-\sqrt{1-\alpha_n}\leqslant\alpha_n'<\alpha_n$, $A_n'=\{(p,\alpha_n):\ p\in A_n\}$ is an antichain of $Q_\alpha^{\rm III}$. Moreover if $m\neq n$, members of A_m' and A_n' are incompatible, since (α_m',α_m) and (α_n',α_n) are disjoint. Hence $A'=\bigcup\{A_n':\ n\in\omega\}$ is an antichain of $Q_\alpha^{\rm III}$. In a similar way an antichain $B'=\bigcup\{B_n':\ n\in\omega\}$ is chosen.

Let $\Gamma=\left\{\dfrac{\alpha_n-\alpha_n'}{1-\alpha_n'}:\ n\in\omega\right\}$. By the choice of $\mathscr{U}$ and $\mathscr{V}$, if $\mathscr{V}=\{(\beta_n',\beta_n):\ n\in\omega\}$, we also have $\Gamma=\left\{\dfrac{\beta_n-\beta_n'}{1-\beta_n'}:\ n\in\omega\right\}$. For each $\gamma\in\Gamma$, let $n=n(\gamma)\in\omega$ be such that $\gamma=\dfrac{\alpha_n-\alpha_n'}{1-\alpha_n'}$ and let S_γ be a countable dense subset of (α_n',α_n). For arbitrary m there will be a unique n such that $n=n(\gamma)$ where $\gamma=\dfrac{\alpha_m-\alpha_m'}{1-\alpha_m'}$. Let ϑ_{nm} be the linear mapping taking α_n' to α_m' and α_n to α_m and let $S_m=\vartheta_{nm}S_\gamma$. (In particular this implies that $S_n=S_\gamma$). Similarly for arbitrary m there is a unique n such that $n=n(\delta)$ where $\delta=\dfrac{\beta_m-\beta_m'}{1-\beta_m'}$, and we let $T_m=\varphi_{nm}S_\delta$ where φ_{nm} is the linear mapping taking α_n' to β_m' and α_n to β_m.

As $|A_n'|=\aleph_0$ we may write A_n' in the form $\bigcup\{A_{rs}':\ r,s\in S_n,\ r<s\}$ where the A_{rs}' are infinite and disjoint. For each $(p,\varepsilon)\in A_{rs}'$ let (p',ε') be an extension of (p,ε) such that $\mu(p')=r\ \&\ \varepsilon'=s$. Let A'' be an antichain of $Q_\alpha^{\rm III}$ containing every such (p',ε') maximal subject to all its members being of the form (q,δ) where $(\exists n)\ (\mu(q),\delta\in S_n)$. Now since $\bigcup_{n\in\omega}U_n$ is dense in $(0,\alpha)$ the set of all such (q,δ) is dense in $Q_\alpha^{\rm III}$, so A'' is a maximal antichain of $Q_\alpha^{\rm III}$. Also for each n, and $r,s\in S_n$ with $r<s$, $A_{rs}''=\{(q,\delta)\in A'':\ \mu(q)=r\ \&\ \delta=s\}$ is countably infinite. Let us fix enumerations of A_{rs}'' for each such r, s.

In a similar way we choose a maximal antichain

$$B''=\bigcup\{B_{rs}'':\ r<s,\ r,s\in T_n,\ n\in\omega\}$$

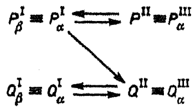
of $Q_\beta^{\rm III}$ such that whenever $(q,\delta)\in B_{rs}''$, $\mu(q)=r\ \&\ \delta=s$ and $|B_{rs}''|=\aleph_0$, and fix an enumeration of each such B_{rs}''.

For $\gamma\in\Gamma$ let $M_\gamma=\left\{n:\ \dfrac{\alpha_n-\alpha_n'}{1-\alpha_n'}=\gamma\right\}$ and $N_\gamma=\left\{n:\ \dfrac{\beta_n-\beta_n'}{1-\beta_n'}=\gamma\right\}$. Then $1\leqslant|M_\gamma|,\ |N_\gamma|\leqslant\aleph_0$. Let F_γ be a 1-1 map from $M_\gamma\times\omega$ onto $N_\gamma\times\omega$. The 1-1 map ϑ from A'' onto B'' is now given as follows. For $(p,\varepsilon)\in A''$ let (p,ε) be the nth member of A_{rs}'' where $r,s\in S_m$ and let $F_\gamma(m,n)=(m',n')$, where $m\in M_\gamma$. Then $\vartheta(p,\varepsilon)=$ the nth member of $B_{r's'}''$, where $r'=\varphi_{mm'}(r)$ and $s'=\varphi_{mm'}(s)$.

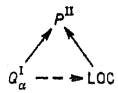
By definition of $\varphi_{mm'}$, $r',s'\in T_{m'}$ and $r'<s'$. The main point is to check that

$\frac{s'-r'}{1-r'} = \frac{s-r}{1-r}$ which means that the algebras in $\mathrm{RO}(Q_\alpha^{\mathrm{III}})$ below (p, ε) and in $\mathrm{RO}(Q_\beta^{\mathrm{III}})$ below $\vartheta(p, \varepsilon)$ are isomorphic. This is proved as in Theorem 3.3.

§ 4. The localization algebra and the bounded to unbounded case. Using $\equiv$ to indicate the existence of an isomorphism between the regular open algebras of two notions of forcing, and $P \to Q$ to indicate that Q is embeddable in P, the following is what we have so far established.



We conclude by fitting LOC into the picture as far as possible, and by showing how to pass from Q_α^{I} to P^{II}:



The tentative arrow -------> indicates that we can establish some connection in that direction, though not embeddability. Thus if $B = \mathrm{RO}(Q_\alpha^{\mathrm{I}})$, V^B contains a "poor man's V-generic set" on LOC. Let us say that a function $\varphi \in \prod_{n\in\omega} [\omega]^{\leqslant 2^n}$ is *V-localizing* if for any $f \in \omega^\omega \cap V$, $\{n: f(n) \notin \varphi_n\}$ is finite. The key property of a V-generic localizing function is that it is V-localizing, but it is easy to construct V-localizing functions which are not V-generic localizing. Indeed it is not clear that $V[\varphi]$ has to contain a V-generic localizing function whenever φ is V-localizing. Hence the dotted arrow.

THEOREM 4.1. *Let X be amoeba V-generic on $Q_{\frac12}^{\mathrm{I}}$. Then if $\{J_{ni}: n, i \in \omega, n \geqslant 2\}$ are independent open subsets of* $[0, 1]$ *such that $\mu(J_{ni}) = \frac{1}{2^{n-1}}$, the function φ given by $\varphi_n = \{i \in \omega: J_{ni} \subseteq X\}$ is V-localizing and lies in $\prod_{n\in\omega} [\omega]^{\leqslant 2^n}$.*

Proof. Firstly we show that $|\varphi_n| \leqslant 2^n$ for each n. For $n < 2$, $\varphi_n = \emptyset$ so this is clear. For $n \geqslant 2$ let K be a finite subset of φ_n. Then since $\left(1-\frac{1}{2^{n-1}}\right)^{-|K|} \geqslant \frac{|K|}{2^{n-1}}$,

$$\frac{2^{n-1}}{|K|} \geqslant \left(1-\frac{1}{2^{n-1}}\right)^{|K|} = 1-\mu\big(\bigcup_{i\in K} J_{ni}\big) \geqslant 1-\mu(X) = \tfrac12,$$

and hence $|K| \leqslant 2^n$, showing that $|\varphi_n| \leqslant 2^n$.

Now let $f \in \omega^\omega \cap V$, and let $D = \{p \in Q_{\frac12}^{\mathrm{I}}: \{n: J_{nf(n)} \not\subseteq p\}$ is finite$\}$. It is sufficient to show that D is dense open. Let $p \in Q_{\frac12}^{\mathrm{I}}$. Then $\mu(p) < \frac12$ so for some n, $\sum_{i\geqslant n} \frac{1}{2^{i-1}} < \frac12 - \mu(p)$ and $q \in D$ where $q = p \cup \bigcup \{J_{if(i)}: i \geqslant n\}$.

This argument is a modification of that in [8, p. 610]. Fremlin suggested the use of independent sets to deduce the existence of a V-localizing function from a V-generic set on $Q_{\frac12}^{\mathrm{I}}$ rather than P_1^{I}.

In the other direction, genericity can be established.

THEOREM 4.2. *Suppose that φ is V-generic localizing. Let $\mathscr{V}_n$ be the set of finite unions of open intervals with rational endpoints of measure $\leqslant \frac{1}{4^n}$, and let $\{I_{ni}: i \in \omega\}$ enumerate $\mathscr{V}_n$. Then $X = \bigcup \{I_{ni}: i \in \varphi_n, n \in \omega\}$ is amoeba V-generic on P^{II}.*

Proof. Firstly we observe that $\mu(X) < \infty$ since

$$\mu(X) \leqslant \sum \{\mu(I_{ni}): i \in \varphi_n, n \in \omega\} \leqslant \sum \frac{|\varphi_n|}{4^n} = 2.$$

Now for any $p \in \mathrm{LOC}$, let $X_p = \bigcup \{I_{ni}: i \in p_n, n \in \omega\}$, and let

$$\varepsilon_p = \mu(X_p) + \sum_{n\in\omega} \left(\frac{2^n - |p_n|}{4^n}\right).$$

The same argument shows that $\mu(X_p)$ and ε_p are finite, and clearly $\mu(X_p) < \varepsilon_p$. The idea is that the information given us by p about X is precisely that $X_p \subseteq X$ & $\mu(X) < \varepsilon_p$.

Let $D \in V$ be a dense open subset of P^{II}, and let $E = \{p \in \mathrm{LOC}: (X_p, \varepsilon_p) \in D\}$. We show that E is dense open in LOC. Let $p \in \mathrm{LOC}$. Now $(X_p, \varepsilon_p) \in P^{\mathrm{II}}$, so it has an extension (Y, ε) in D. This means that $X_p \subseteq Y$ & $\mu(Y) < \varepsilon \leqslant \varepsilon_p$. Since $\mu(Y) < \varepsilon_p$ there is some N such that $\mu(Y) < \mu(X_p) + \sum_{i<N} \left(\frac{2^i - |p_i|}{4^i}\right)$. Hence we may express Y in the form $X_p \cup \bigcup \{J_i: i < N\}$ where each J_i is open and of measure $< \frac{2^i - |p_i|}{4^i}$. The expression now has to be modified. Each J_i may be written in the form $\bigcup \{I_{ij}: j \in U_i\} \cup K_i$ where $|U_i| = 2^i - |p_i|$ and K_i has arbitrarily small measure. We choose K_i so that $\mu(K_i) < \frac{4}{3N\cdot 4^N}$ for each i. Then $\mu(\bigcup_{i<N} K_i) < \frac{4}{3\cdot 4^N} = \sum_{N\leqslant i} \frac{1}{4^i}$ so that we may write $\bigcup_{i<N} K_i = \bigcup \{I_{in(i)}: N \leqslant i\}$. We let

$$q_i = \begin{cases} p_i \cup U_i & i < N, \\ p_i \cup \{n(i)\} & N \leqslant i. \end{cases}$$

Then $X_q = Y$. Since q is an extension of p, $\varepsilon_q \leqslant \varepsilon_p$. Finally we extend q to q' if necessary (i.e. if $\varepsilon < \varepsilon_q$) by adding in finitely many new points so as to bring $\varepsilon_{q'}$ below ε while keeping $\mu(X_{q'} - X_q)$ small. (This is the reason we included sets of measure $< \frac{1}{4^n}$ in $\mathscr{V}_n$ and not just those of measure equal to $\frac{1}{4^n}$.) Thus q' is an extension of p lying in E.

Since E is dense open, and φ is V-generic localizing, $p \in E$ for some p such that $(\forall n)\ p_n \subseteq \varphi_n$. Thus $(X_p, \varepsilon_p) \in D$ and as $X_p \subseteq X$ and $\mu(X) < \varepsilon_p$ this establishes the amoeba V-genericity of X for P^{II}.

THEOREM 4.3 (Fremlin). *Suppose that X is amoeba V-generic for $Q^{\mathrm{I}}_{\frac{1}{2}}$, and let $\mathscr{V}_n$, I_{ni} be as in Theorem 4.2, where in addition we assume that each member of $\mathscr{V}_n$ occurs infinitely often in the enumeration $\{I_{ni}\}_{i \in \omega}$. If $\{J_{ni}\}_{n, i \in \omega}$ are independent relatively open subsets of* $[0, 1]$ *such that* $\mu(J_{ni}) = \dfrac{1}{2^{n+1}}$, *then*

$$Y = \bigcup \{I_{ni}\colon n, i \in \omega, \mu(X - J_{ni}) = 0\}$$

is amoeba V-generic for P^{II}.

Proof. For $p \in Q^{\mathrm{I}}_{\frac{1}{2}}$ let $Y_p = \bigcup \{I_{ni}\colon \mu(J_{ni} - p) = 0\}$ and

$$\varepsilon_p = \sup\{\mu(Y_q)\colon p \subseteq q \in Q^{\mathrm{I}}_{\frac{1}{2}}\}.$$

The idea is that, as in the proof of Theorem 4.2, (Y_p, ε_p) lies in P^{II} and is precisely the information about Y given us by p. The proof is carried out by a series of lemmas.

LEMMA 4.4. *If $p \in Q^{\mathrm{I}}_{\frac{1}{2}}$ then $\mu(\bigcup \{I_{ni}\colon n \geqslant m, \mu(J_{ni} - p) = 0\}) \leqslant 4/2^m$ for each m.*

Proof. Let $U_n = \{i\colon \mu(J_{ni} - p) = 0\}$. Then

$$\frac{1}{2} < 1 - \mu(p) \leqslant 1 - \mu(\bigcup \{J_{ni}\colon i \in U_n\}) = \left(1 - \frac{1}{2^{n+1}}\right)^{|U_n|}$$

so that

$$|U_n| = 2^{n+1}\left[\left(1 + \frac{|U_n|}{2^{n+1}}\right) - 1\right] \leqslant 2^{n+1}\left[\left(1 - \frac{1}{2^{n+1}}\right)^{-|U_n|} - 1\right] < 2^{n+1}[2 - 1] = 2^{n+1}.$$

Therefore

$$\mu(\bigcup \{I_{ni}\colon n \geqslant m, \mu(J_{ni} - p) = 0\}) = \mu(\bigcup \{I_{ni}\colon i \in U_n, n \geqslant m\}) \leqslant \sum_{n=m}^{\infty} \frac{2^{n+1}}{4^n} = \frac{4}{2^m}.$$

We now let $K_m(p) = \{(n, i)\colon n \leqslant m, i \in \omega, \mu(J_{ni} - p) = 0\}$ where $p \in Q^{\mathrm{I}}_{\frac{1}{2}}$ and $m \in \omega$.

LEMMA 4.5. *If $p \in Q^{\mathrm{I}}_{\frac{1}{2}}$ and $m, n \in \omega$ there is $k \in \omega$ such that*

$$K_m(p \cup J_{nj}) = K_m(p) \cup \{(n, j)\} \quad \textit{for every } j \geqslant k.$$

Proof. Suppose on the contrary that $\{j\colon K_m(p \cup J_{nj}) \neq K_m(p) \cup \{(n, j)\}\}$ is infinite. For each j in this set let

$$(n_j, i_j) \in K_m(p \cup J_{nj}) - (K_m(p) \cup \{(n, j)\}).$$

Case 1. For some (n', i), $A = \{j\colon (n', i) = (n_j, i_j)\}$ is infinite. Thus if $j \in A$, $\mu((J_{n'i}) - (p \cup J_{nj})) = 0$ & $\mu(J_{n'i} - p) \neq 0$ & $(n', i) \neq (n, j)$. We have

$$\mu((J_{n'i} - J_{nj}) - p) = 0$$

for each $j \in A$ so that

$$\mu(\bigcup_{j \in A} (J_{n'i} - J_{nj}) - p) = \mu((J_{n'i} - \bigcap_{j \in A} J_{nj}) - p) = 0.$$

But as A is infinite and the J_{nj} are independent, $\mu(\bigcap_{j \in A} J_{nj}) = 0$, giving $\mu(J_{n'i} - p) = 0$, a contradiction.

Case 2. For each (n', i), $\{j\colon (n', i) = (n_j, i_j)\}$ is finite. Then there is an infinite set A such that if $j, j' \in A$ with $j \neq j'$, $(n_{j'}, i_{j'}) \neq (n_j, i_j)$ and $(n, j') \neq (n_j, i_j)$, $(n_{j'}, i_{j'})$. Thus $\{J_{n_j i_j}\colon j \in A\} \cup \{J_{nj}\colon j \in A\}$ is independent. For $j \in A$, $\mu((J_{n_j i_j} - J_{nj}) - p) = 0$ as in Case 1, so as $\{J_{n_j i_j} - J_{nj}\colon j \in A\}$ is an infinite independent family, $\mu(\bigcup_{j \in A} (J_{n_j i_j} - J_{nj})) = 1$ giving $\mu(p) = 1$, a contradiction.

LEMMA 4.6. *Suppose that p, V, n, δ, $\mathscr{W}$ are such that $p \in Q^{\mathrm{I}}_{\frac{1}{2}}$, $V \in \mathscr{V}_n$, $n \in \omega$, $\delta > 0$, $\mathscr{W}$ is a finite family of measurable subsets of* $[0, 1]$, *and* $\mu(p) + \dfrac{1}{2^{n+1}} < \frac{1}{2}$. *Then there is $p' \in Q^{\mathrm{I}}_{\frac{1}{2}}$ extending p such that $V \subseteq Y_{p'}$, $\mu(p') \leqslant \mu(p) + \dfrac{1}{2^{n+1}}$, $\mu(Y_{p'} - (Y_p \cup V)) \leqslant \delta$, and for every $W \in \mathscr{W}$, $\mu(W - p') \geqslant \left(1 - \dfrac{1}{2^n}\right)\mu(W - p)$.*

Proof. Let m be such that $\dfrac{2}{2^m} \leqslant \delta$. By Lemma 4.5 there is k such that $K_m(p \cup J_{nj}) = K_m(p) \cup \{(n, j)\}$ for every $j \geqslant k$. By independence of the J_{nj} there is k' such that for every $W \in \mathscr{W}$ and $j \geqslant k'$, $\mu((W - p) \cap J_{nj}) \leqslant \dfrac{1}{2^n}\mu(W - p)$. Let $j \geqslant \max(k, k')$ be such that $V = I_{nj}$ and let $p' = p \cup J_{nj}$.

Then $\mu(p') \leqslant \mu(p) + \dfrac{1}{2^{n+1}} < \frac{1}{2}$ and for every $W \in \mathscr{W}$,

$$\mu(W - p') \geqslant \left(1 - \frac{1}{2^n}\right)\mu(W - p).$$

Also, since $K_m(p') = K_m(p) \cup \{(n, j)\}$,

$$Y_{p'} \subseteq Y_p \cup I_{nj} \cup \bigcup \{I_{ri}\colon r > m, i \in \omega, \mu(J_{ri} - p') = 0\}$$

from which by Lemma 4.4 we find that $\mu(Y_{p'} - (Y_p \cup V)) \leqslant 4/2^{m+1} \leqslant \delta$.

LEMMA 4.7. *Suppose that p, V and δ are such that $p \in Q^{\mathrm{I}}_{\frac{1}{2}}$, V is an open subset of $\mathbf{R}$, and $\delta > 0$, and suppose that $\mu(p) + \frac{1}{2^n} < \frac{1}{2}$ and $\mu(V - Y_p) \leqslant \frac{1}{4^n}$. Then there is $p' \in Q^{\mathrm{I}}_{\frac{1}{2}}$ extending p such that $V \subseteq Y_{p'}$ and $\mu(Y_{p'} - V) \leqslant \delta + \mu(Y_p - V)$.*

Proof. Since $\mu(V - Y_p) \leqslant \frac{1}{4^n}$ there is a sequence (V_k) such that $V_k \in \mathscr{V}_{n+k}$ and $V - Y_p \subseteq \bigcup_{k \in \omega} V_k$. By Lemma 4.6 we may choose inductively a sequence (p_k) of members of $Q^{\mathrm{I}}_{\frac{1}{2}}$ such that

$$p = p_0 \subseteq p_1 \subseteq p_2 \subseteq \ldots, \qquad V_k \subseteq Y_{p_{k+1}},$$

$$\mu(p_{k+1}) \leqslant \mu(p_k) + \frac{1}{2^{n+k+1}} \leqslant \mu(p) + \frac{1}{2^n} - \frac{1}{2^{n+k+1}},$$

$$\mu(Y_{p_{k+1}} - (Y_{p_k} \cup V_k)) \leqslant \frac{\delta}{2^{k+1}},$$

$$\mu(J_{mi} - p_{k+1}) \geqslant \left(1 - \frac{1}{2^{n+k}}\right) \mu(J_{mi} - p_k), \quad \text{all } m, i \leqslant k.$$

We let $p' = \bigcup_{k \in \omega} p_k$. Then $\mu(p') \leqslant \mu(p) + \frac{1}{2^n} < \frac{1}{2}$, so $p' \in Q^{\mathrm{I}}_{\frac{1}{2}}$. Since $V_k \subseteq Y_{p_{k+1}}$, $V \subseteq Y_{p'}$. Now we show that $Y_{p'} = \bigcup_{k \in \omega} Y_{p_k}$. For let m and i be such that $\mu(J_{mi} - p') = 0$, and let $k = \max(m, i)$. Then if $l \geqslant k$,

$$\mu(J_{mi} - p_{l+1}) \geqslant \left(1 - \frac{1}{2^{n+l}}\right) \mu(J_{mi} - p_l)$$

so that $0 = \mu(J_{mi} - p') = \lim_{l \to \infty} \mu(J_{mi} - p_l) \geqslant \prod_{l \geqslant k} \left(1 - \frac{1}{2^{n+l}}\right) \mu(J_{mi} - p_k)$. Now $\sum_{l \geqslant k} \frac{1}{2^{n+l}} < \infty$, so $\prod_{l \geqslant k} \left(1 - \frac{1}{2^{n+l}}\right) > 0$. Hence $\mu(J_{mi} - p_k) = 0$, and $Y_{p'} = \bigcup_{k \in \omega} Y_{p_k}$ follows.

We deduce that $\mu(Y_{p'} - V) = \lim_{k \to \infty} \mu(Y_{p_k} - V) \leqslant \mu(Y_p - V) + \sum \frac{\delta}{2^{k+1}} = \mu(Y_p - V) + \delta$.

LEMMA 4.8. *If $p \in Q^{\mathrm{I}}_{\frac{1}{2}}$ and $\mu(Y_p) < \varepsilon$ there is an extension p' of p in $Q^{\mathrm{I}}_{\frac{1}{2}}$ such that $\varepsilon_{p'} \leqslant \varepsilon$.*

Proof. Let m be such that $\mu(Y_p) + \frac{2}{2^m} \leqslant \varepsilon$. We shall choose $p' \supseteq p$ in $Q^{\mathrm{I}}_{\frac{1}{2}}$ in such a way that if p'' is any extension of p' in $Q^{\mathrm{I}}_{\frac{1}{2}}$ then $K_m(p'') = K_m(p)$. For such a p'' it follows from Lemma 4.4 that $\mu(Y_{p''}) \leqslant \mu(Y_p) + \frac{4}{2^{m+1}}$, so that $\varepsilon_{p'} \leqslant \varepsilon$ as required.

The idea in the choice of p' is to look at those J_{ni} which would be in danger of becoming subsets of an extension of p, and hence increasing K_m, and to ensure that each is incompatible in $Q^{\mathrm{I}}_{\frac{1}{2}}$ with p'. This is done by adding to p in order to form p' enough sets of the form J_{rj} for large r. These J_{rj} will be sufficiently independent with respect to p from the J_{ni} it is desired to avoid, and since we shall have $r > m$, their adjunction will not increase the measure of p beyond ε.

Let $L = \{(n, i)\colon n \leqslant m,\ i \in \omega,\ \mu(J_{ni} - p) > 0\}$. Then for $n \leqslant m$,

$$\lim_{i \to \infty} \mu(J_{ni} - p) = \frac{1}{2^{n+1}}(1 - \mu(p))$$

by the independence of the J_{ni}, from which it follows that

$$\delta = \inf\{\mu(J_{ni} - p)\colon (n, i) \in L\} > 0.$$

Let $\gamma > 0$ be chosen so that $\gamma < \min\left(\frac{\delta}{6}, \frac{1}{2^{m+3}}(\frac{1}{2} - \mu(p))\right)$ and let $r > m$ be such that $\frac{1}{2^{r+2}} \leqslant \gamma$. Then $(\frac{1}{2} - \gamma) \div \frac{1}{2} \leqslant 1 - \frac{1}{2^{r+1}}$, so for some l, $\frac{1}{2} - \gamma \leqslant \left(1 - \frac{1}{2^{r+1}}\right)^l \leqslant \frac{1}{2}$. We let

$$E_j = \bigcup \{J_{ri}\colon lj \leqslant i < l(j+1)\}$$

for each $j \in \omega$. Thus the E_j are independent and of measure $1 - \left(1 - \frac{1}{2^{r+1}}\right)^l \in [\frac{1}{2}, \frac{1}{2} + \gamma]$.

Since $\lim_{i \to \infty} \mu(J_{ni} - p) = \frac{1}{2^{n+1}}(1 - \mu(p))$ there is $k \in \omega$ such that if $i \geqslant k$ and $n \leqslant m$, $\mu(J_{ni} - p) \geqslant \frac{1}{2^{n+1}}(1 - \mu(p)) - \gamma$. Now if $(n, i) \in L$ then $\lim_{j \to \infty} \mu(E_j \cap J_{ni} - p) \leqslant (\frac{1}{2} + \gamma)\mu(J_{ni} - p)$ by independence of the E_j so that there is $j \in \omega$ such that $\mu(E_j \cap J_{ni} - p) \leqslant (\frac{1}{2} + 2\gamma)\mu(J_{ni} - p)$ whenever $(n, i) \in L$ and $i < k$. Let $p'_1 = p \cup E_j$.

We now show that if $(n, i) \in L$ then $\mu(J_{ni} - p'_1) > \gamma$.

Case 1. If $i < k$ then

$$\begin{aligned} \mu(J_{ni} - p'_1) &= \mu(J_{ni} - p) - \mu(E_j \cap J_{ni} - p) \\ &\geqslant \mu(J_{ni} - p) - (\tfrac{1}{2} + 2\gamma)\mu(J_{ni} - p) \\ &\geqslant \delta(\tfrac{1}{2} - 2\gamma) > 3\gamma - 2\gamma\delta \quad \text{(by choice of } \gamma) \\ &\geqslant \gamma, \quad \text{(as } \delta \leqslant 1). \end{aligned}$$

Case 2. If $i \geqslant k$ then

$$\begin{aligned} \mu(J_{ni} - p'_1) &\geqslant \mu(J_{ni} - p) - \mu(E_j \cap J_{ni}) \\ &\geqslant \frac{1}{2^{n+1}}(1 - \mu(p)) - \gamma - (\tfrac{1}{2} + \gamma)\frac{1}{2^{n+1}} \end{aligned}$$

$$= \frac{1}{2^{n+1}}(\tfrac{1}{2}-\mu(p)-\gamma)-\gamma$$

$$> \frac{1}{2^{n+1}}(2^{m+3}\cdot\gamma-\gamma)-\gamma \quad \text{(by choice of } \gamma)$$

$$\geqslant \gamma, \quad \text{(as } m \geqslant n).$$

Now $\mu(p_1') \geqslant \mu(E_j) \geqslant \frac{1}{2}$ so there is $p' \in Q^{\mathrm{I}}_{\frac{1}{2}}$ with $p \subseteq p' \subseteq p_1'$ having measure greater than $\frac{1}{2}-\gamma$. This is the desired p', and we have to show that if $p' \subseteq p'' \in Q^{\mathrm{I}}_{\frac{1}{2}}$ then $K_m(p'') = K_m(p)$. In other words we have to show that if $(n, i) \in L$, $\mu(J_{ni}-p'') > 0$. But this follows since

$$\mu(J_{ni}-p'') \geqslant \mu(J_{ni}-p_1') - \mu(p''-p') > 0 .$$

We are now able to complete the proof of Theorem 4.3. Let $D \in V$ be a dense open subset of P^{II}, and let $E = \{p \in Q^{\mathrm{I}}_{\frac{1}{2}}\colon (Y_p, \varepsilon_p) \in D\}$. We have to show that E is dense open in $Q^{\mathrm{I}}_{\frac{1}{2}}$. Let $p \in Q^{\mathrm{I}}_{\frac{1}{2}}$ be arbitrary. Then from Lemma 4.7 we deduce that $(Y_p, \varepsilon_p) \in P^{\mathrm{II}}$ (i.e. there is an extension p' of p such that $\mu(Y_{p'}) > \mu(Y_p)$, giving $\mu(Y_p) < \varepsilon_p$). Let n be such that $\mu(p) + \dfrac{1}{2^n} < \frac{1}{2}$, and let $\delta_1 = \min\left(\varepsilon_p, \mu(Y_p) + \dfrac{1}{4^n}\right)$. Then (Y_p, δ_1) is an extension of (Y_p, ε_p) in P^{II}. As D is dense open in P^{II} there is an extension (Y, δ) of (Y_p, δ_1) lying in D. By Lemma 4.7, since $\mu(Y-Y_p) \leqslant \dfrac{1}{4^n}$ there is an extension q of p in $Q^{\mathrm{I}}_{\frac{1}{2}}$ such that $Y \subseteq Y_q$ and $\mu(Y_q - Y) < \varepsilon - \mu(Y) + \mu(Y_p - Y) = \varepsilon - \mu(Y)$. Thus $\mu(Y_q) < \varepsilon$. Finally by Lemma 4.8 there is an extension q' of q in $Q^{\mathrm{I}}_{\frac{1}{2}}$ such that $\varepsilon_{q'} \leqslant \varepsilon$. Putting these together, q' is an extension of p in $Q^{\mathrm{I}}_{\frac{1}{2}}$ such that $(Y_{q'}, \varepsilon_{q'})$ is an extension of (Y, δ) in P^{II}, so that $(Y_{q'}, \varepsilon_{q'}) \in D$ showing that $q' \in E$.

Since E is a dense open subset of $Q^{\mathrm{I}}_{\frac{1}{2}}$ lying in V, $p \in E$ for some $p \subseteq X$. Thus $(Y_p, \varepsilon_p) \in D$ and since it is clear that $Y_p \subseteq Y \,\&\, \mu(Y) < \varepsilon_p$, this establishes the V-genericity of Y for P^{II}.

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