Quant aux ensembles $E$ et $F$, on pourrait démontrer sans peine que pour tous les ensembles $Q$ et $R$ les inégalités

$$dQ < dE < dR$$

entraınent les inégalités

$$dQ < dF < dR$$

(quoiqu'on a $dE \neq dF$)[1].

[1]) Cf. le problème posé par M. Fréchet dans son livre cité, p. 31, note (1).

# On two-dimensional analysis situs with special reference to the Jordan curve-theorem.

By

D. W. Woodard (Philadelphia).

**Introduction.** R. L. Moore, in his paper, „*On the Foundations of Plane Analysis Situs*“, Transactions of the American Mathematical Society, Volume 17, 1916, proposed three systems of axioms, $\Sigma_1$, $\Sigma_2$, $\Sigma_3$, for the development of two-dimensional analysis situs. This paper will hereafter be referred to as „Moore“. To facilitate reference, Moore's notation will be followed as closely as practicable.

In Axiom 8, which belongs to all three systems, Moore assumes that every simple closed curve is the boundary of a region, that is, that every simple closed curve defines a bounded connected set of connected exterior having further properties implied by certain other axioms of the three systems.

The chief purpose of this investigation is to replace Moore's Axiom 8 by another axiom of such nature that no property of the simple closed curve is assumed. The Jordan curve-theorem in its most general form appears as the fundamental theorem of the set of theorems. Two systems of axioms, I and II, are presented. It is proved that (1) all of Moore's theorems follow as consequences of each of the systems of axioms, (2) every simple closed curve is the boundary of a set of points having all the properties of a region, (3) every space satisfying the set designated as Axioms I is homeomorphic with the Euclidean plane and (4) there exist spaces satisfying Axioms II that are „neither metrical, descriptive, nor separable“ [1]).

[1]) Axioms II are satisfied by the space thus described by Moore, page 164.

The proof of the Jordan curve-theorem for spaces of the type described in (4) constitutes, perhaps the most interesting result in the development that follows.

If Axiom 6 is omitted, it is still possible to prove Theorems 1—5 upon the basis of the remaining axioms, except that, in the case of Theorem 1, the last six words must be changed to read „$H_i'$ is a subset of $H_{i-1}'$“. The facts developed in the proofs of Theorems 1—5 constitute a proof of the Jordan curve-theorem. It is of interest to note that, under these circumstances, the proof of the Jordan curve-theorem is based upon a set of axioms which contains no assumption as to the character of the boundaries of regions. The foregoing statements apply to either of the two sets of axioms.

For an account of related investigations previously published, one is referred to the paper by J. R. Kline on *„Separation Theorems and their Relation to Recent Developments in Analysis Situs“* [1]).

I wish to express my deep obligation to Dr. J. R. Kline who suggested the problem and whose helpful criticism has been of inestimable value.

**Definitions.** Let $S$ denote the set of all elements (points) to be considered. Regions are certain subsets of $S$ having the properties implied in the axioms. For definitions of „limit point of a point-set“, „boundary of a point-set“, „connected set“, „Heine-Borel property“, „simple continuous arc“, „simple closed curve“, „bounded set“, see Moore, pages 132, 135, 136, 139. The set of points composed of the point-set $M$ and its boundary is denoted by $M'$. The expressions, „simple continuous arc $ABC$“, „arc $ABC$“ and „$ABC$“, denote the same object. Frequently an arc is designated by naming its two end-points, as, $AC$. The symbol $\underline{ABC}$ denotes the point-set (arc $ABC$) $- A - C$.

**Axioms I.** Axioms 1—5 are given precisely as they occur in Moore.

[1]) In this connection, attention is called to the Heidelberg thesis of Imgard Gawehn: *„Über unberandete 2-dimensionalen Mannigfaltigkeiten“*, Math. Ann. 98, 1927. In this thesis the validity of the Jordan curve-theorem im kleinen is postulated. In a reference to Moore's paper, the following incorrect statement is made: „ausserdem wird viel vorausgesetzt, z. B. die Existenz spezieller Systeme einander nicht schneidende Jordankurven“



**Axiom 1.** *There exists an infinite sequence of regions $K_1, K_2, K_3, \ldots$, such that (1) if $m$ is an integer and $P$ is a point, there exists an integer $n$, greater than $m$, such that $K_n$ contains $P$, (2) if $P$ and $\overline{P}$ are distinct points of a region $R$, then there exists an integer $\delta$ such that if $n > \delta$ and $K_n$ contains $P$, then $K_n'$ is a subset of $R - \overline{P}$.*

**Axiom 2.** *Every region is a connected set of points.*

**Axiom 3.** *If $R$ is a region, $S - R'$ is a connected set of points.*

**Axiom 4.** *If $R$ is a region, $R'$ possesses the Heine-Borel property.*

**Axiom 5.** *There exists an infinite set of points that has no limit point.*

**Axiom 6.** *If $P$ is a point on the boundary of a region $R$, then there exist two simple continuous arcs $AP$ and $BP$ such that $AP - P$ and $BP - P$ are subsets of $R$ and $S - R'$ respectively.*

**Axiom 7.** *If $AB$ is a simple continuous arc lying within a region $R$, then $R - AB$ is a connected set of points.*

**Axiom 8.** *If $\alpha_1$ and $\alpha_2$ are two finite connected sets of regions whose sum is a connected set, and $R_1$ and $R_2$ are two mutually exclusive regions without common boundary points such that $R_i$ $(i = 1, 2)$ contains at least one point common to $\alpha_1$ and $\alpha_2$, and all points common to $(\alpha_1)'$ and $(\alpha_2)'$ belong to $R_1' + R_2'$, then there exists at least one region $H$ containing $\alpha_1 + \alpha_2$ such that (1) the boundary of $H$ is a subset of the boundary of $\alpha_1 + \alpha_2$ and (2) $H$ contains a connected subset $M$ exterior to $R_1 + R_2$ and containing at least one point of $\alpha_1$ and one point of $\alpha_2$.*

**Independence examples.** The independence examples given by Moore, page 162, for Axioms 1—5, 8 of his $\Sigma_1$ system will serve for the above axioms bearing the same number. In the case of Axiom 6, Moore's $E_6$ or $E_7$ may be used to obtain a space with regions so defined that Axioms 1—5, 7, 8 hold true but for which one or the other of the arcs indicated in Axiom 6 does not exist. Certain types of regions, in addition to those indicated by Moore, must be defined in order that Axiom 8 may be satisfied in the independence examples for Axioms 4 and 6. The needed definitions are obvious.

As an independence example for Axiom 7, consider the set of points $S_7$ contained in the ordinary straight line $\alpha$ and in $\beta$, the half-plane determined by $\alpha$ in a given Euclidean plane. A set of points is a region $R$ if and only if (1) $R$ is an ordinary Jordan region such that $R'$ is a subset of the half-plane $\beta$, or $R$ is the finite part of $S_7$ determined by an ordinary arc $ACB$ such that $A$ and $B$ belong to $\alpha$ and $\underline{ACB}$ is a subset of $\beta$. All axioms are satisfied except Axiom 7.

**Preliminary Theorems.** Theorems 1—22, Moore, which depend upon Axioms 1—6 are assumed. The following property of the simple continuous arc as stated by Moore, page 139, is of particular importance in our work: If an arc $AB$ has at least one point in common with a closed set $F$, then $AB$ has a first and a last point in common with $F$.

**Theorem 1.** *If $J$ is a closed curve, there is a sequence of pairs of region-sets $\{\alpha_1^{(i)}, \alpha_2^{(i)}\}$, $(i = 1, 2, 3, \ldots)$ such that (1) $J$ is the common part of $\alpha_1^{(1)} + \alpha_2^{(1)}$, $\alpha_1^{(2)} + \alpha_2^{(2)}, \ldots$ and (2) there exists a corresponding sequence of regions $\{H_i\}$ such that the boundary of $H_i$ is a subset of the boundary of $\alpha_1^{(i)} + \alpha_2^{(i)}$, $H_i$ contains $\alpha_1^{(i)} + \alpha_2^{(i)}$ and $H_i'$ is a subset of $H_{i-1}$.*

Proof. There exist points $A$ and $B$ separating on $J$ the points $C$ and $D$. All regions used are members of the fundamental sequence of regions postulated by Axiom 1. Let $R_A$ and $R_B$ be two regions about $A$ and $B$ respectively such that $R_A'$ and $R_B'$ have no common points, and $R_A'$ and $R_B'$ contain no points of the arcs $CBD$ and $CAD$ respectively. Cover each point of $ACB$ not in $R_A + R_B$ by a region having no point or boundary point in common with $ADB$. These regions, together with $R_A + R_B$, cover $ACB$. A finite subset of these regions, including $R_A$ and $R_B$, covers $ACB$ (Moore, Theorem 12). Let $\alpha_1^{(1)}$ denote this finite set of regions. Cover each point of $ADB$ not in $R_A + R_B$ by a region having no point or boundary point in common with $ACB$ or with any region of the set $\alpha_1^{(1)} - R_A - R_B$. As before, we obtain a finite set of regions, including $R_A$ and $R_B$, which cover $ADB$. Let $\alpha_2^{(1)}$ denote this finite set of regions. Evidently $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ satisfy the requirements of Axiom 8 and the corresponding region $H_1$ exists. $H_1$ is unique. This follows immediately from Theorem 21, Moore. In obtaining $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$, all regions used have subscripts greater than 2, and, in general, in obtaining $\alpha_1^{(i)}$, $\alpha_2^{(i)}$, all regions used have subscripts greater than $i$. There exist regions $R_1^{(2)}$ and $R_2^{(2)}$ about $A$ and $B$ respectively such that $R_1^{(2)}$ and $R_2^{(2)}$, together with their boundaries, are subsets of $R_A$ and $R_B$ respectively. As in the preceding, there exists a finite set of regions covering all points of $ACB$ not in $R_1^{(2)} + R_2^{(2)}$ such that every region of the set with its boundary is a subset of $\alpha_1^{(1)}$ and no region of the set has a point or boundary point in common with $ADB$. There is again a finite set of regions covering $ACB$ and including $R_1^{(2)}$ and $R_2^{(2)}$. We can obtain similarly a finite set of regions, including $R_1^{(2)}$ and $R_2^{(2)}$, covering $ADB$ such that each region of the set with its boundary is a subset of $\alpha_2^{(1)}$ and no region of the set other than $R_1^{(2)}$ and $R_2^{(2)}$ contains a point of $ACB$. Let $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ denote the first and second of these finite sets respectively. The corresponding region $H_2$ exists. Continuing in this manner, we can obtain the required sequence of pairs of region-sets and the corresponding sequence of regions $\{H_i\}$. By Moore, Theorem 21, $H_i'$ is a subset of $H_{i-1}$.

Let $Q$ be a point not belonging to $J$. Since $J$ is closed, there exists a region $R_Q$ about $Q$ such that $R_Q'$ contains no point of $J$. Let $Q_1$ be a point of $R_Q$ distinct from $Q$. By Axiom 1, there is a number $\delta$ such that for $n > \delta$, $K_n'$ is a subset of $R_Q - Q_1$, provided $K_n$ contains $Q$. In the process of obtaining the sequence of pairs of region-sets, every region of the fundamental sequence belonging to $\alpha_1^{(i)} + \alpha_2^{(i)}$ had a subscript greater than $i$. Then, no region of $\alpha_1^{(\delta+1)} + \alpha_2^{(\delta+1)}$ can contain $Q$. Hence $Q$ cannot belong to the common part of $\alpha_1^{(1)} + \alpha_2^{(1)}$, $\alpha_1^{(2)} + \alpha_2^{(2)}, \ldots$, and this common part is identical with $J$.

**Theorem 2.** *Corresponding to every simple closed curve $J$, there is a unique bounded set $I_J$ such that $I_J$ and $E_J$, the complement of $I_J + J$, are separated by $J$.*

Proof. The regions $H_1, H_2, H_3, \ldots$, obtained in the last theorem, have a common part $H_\omega$ which is closed (Moore, Theorem 14). Denote $H_\omega - J$ by $I_J$. Assume that $I_J$ is non-vacuous and contains a point $Q$. The point $Q$ cannot be a limit point of $E_J$. For, if $Q$ were a limit point of $E_J$, every region $R_Q$ containing $Q$ would contain a point $Q_1$ of $E_J$. Then, $Q_1$ belongs to $S - H_i'$ for some value of $i$. An arc in $R_Q$ joining $Q$ to $Q_1$ [1]) would contain a point of the boundary of $H_i$. Under the assumption, then, $Q$ is a limit point of the sum of the boundaries of the regions $H_i$. By the preceding theorem, $Q$ does not belong to the common part of $\alpha_1^{(1)} + \alpha_2^{(1)}$, $\alpha_1^{(2)} + \alpha_2^{(2)}, \ldots$. Then, there is a number $n$ such that $Q$ does not belong to $(\alpha_1^{(n)} + \alpha_2^{(n)})'$. There is a region $\overline{R}_Q$ containing $Q$ and no point of $(\alpha_1^{(n)} + \alpha_2^{(n)})'$. The boundary of $H_{n+j}$ $(j \geqq 1)$ is a subset of $\alpha_1^{(n)} + \alpha_2^{(n)}$. Then $\overline{R}_Q$ contains no point of the boundary of $H_{n+j}$.

[1]) Moore, Theorem 16.

Hence $Q$ is not a limit point of $E_J$. Further, in the same manner we may prove that no point of $E_J$ is a limit point of $I_J$.

The regions $R_1^{(i)}$ and $R_2^{(i)}$, containing all points common to $\alpha_1^{(i)}$ and $\alpha_2^{(i)}$, belong to $R_A$ and $R_B$ respectively. Then, by Axiom 8, there exists in $H_i$ a connected set $M_i$ exterior to $R_A+R_B$ such that $M_i$ contains a point $A_1^{(i)}$ of $\alpha_2^{(i)}$ and a point $A_2^{(i)}$ of $\alpha_2^{(i)}$. But $A_1^{(i)}$ and $A_2^{(i)}$ belong to $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ respectively. With due regard to the fact that $M_i$ is connected and that $\alpha_1^{(1)}-R_A-R_B$ and $\alpha_2^{(1)}-R_A-R_B$ have no common point or boundary point, it follows easily that $H_i$ contains a point $F_i$ exterior to $\alpha_1^{(1)}+\alpha_2^{(1)}$. We thus obtain a sequence of points $F_2, F_3, \ldots, F_n, \ldots$, such that $F_n$ belongs to $H_n$ and is exterior to $\alpha_1^{(1)}+\alpha_2^{(1)}$. Since each point belongs to $H_1$, the sequence is bounded and possesses a limit point $F$ (Moore, Theorem 13). The point $F$ obviously cannot belong to $\alpha_1^{(1)}+\alpha_2^{(1)}$ or to $E_J$ or to $J$. In any event, the point $F$ belongs to $I_J$. Then, $I_J$ is non-vacuous. It can easily be proved that $E_J$ is connected [1]).

Suppose that $I_J$ is not unique, that is, suppose that for two different processes of the kind described above, there were obtained two sets $H_\omega$ and $\overline{H}_\omega$ such that $\overline{H}_\omega$ contains a point $P$ not in $H_\omega$. The set $S-H_\omega$ is connected. Let $E$ be a point of $S-H_\omega$ and $S-\overline{H}_\omega$. Then, under the assumption made above, there exists an arc $PE$ in $S-H_\omega$ which contains a point $P_1$ on the boundary of $\overline{H}_\omega$. As proved above, no point of $\overline{H}_\omega - J$ is a boundary point of $\overline{H}_\omega$. Then, $P_1$ belongs to $J$. But $J$ belongs to $H_\omega$. This contradiction shows that $H_\omega$, and therefore $I_J$, is unique. The set $I_J$, being a subset of $H_1$, is bounded.

**Theorem 3.** *Every point of a simple closed curve $J$ is a limit point of $I_J$.*

Proof. Assume that $P$, any point of $J$, is not a limit point of $I_J$. An arc joining a point of $I_J$ to a point of $E_J$ [2]) contains a first point on $J$. Then $J$ contains a non-vacuous subset $N$ consisting of points which are limit points of $I_J$. The set $N$ is closed. Then $P$ is not a limit point of $N$. There exists a region $R_P$ about $P$ such that $R_P'$ contains no point of $N$. There is in $R_P$ an arc $APB$ of $J$. The arc $APB$ contains no point of $N$. Let $C$ be a point of $J$ such

[1]) In fact, it can be shown that every two points of $E_J$ are the extremities of an arc lying wholly in $E_J$.

[2]) Moore, Theorem 15, 22.



that $P$ and $C$ separate $A$ and $B$ on $J$. Let $D_1$ and $D_2$ respectively be the first points that $PAC$ and $PBC$ have in common with the closed set $N$. The set $D_1CD_2+I_J$ is closed and connected.

Form $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ in accordance with Axiom 8 as follows: $R_1^{(1)}$ and $R_2^{(1)}$ are regions about $D_1$ and $D_2$ respectively such that $R_1^{(1)}$ and $R_2^{(1)}$ have no point or boundary point in common. The set $\alpha_1^{(1)}$ consists of $R_1^{(1)}+R_2^{(1)}$ and a finite set of regions covering all points of $D_1PD_2$ not in $R_1^{(1)}+R_2^{(1)}$ such that no region of $\alpha_1^{(1)}-R_1^{(1)}-R_2^{(1)}$ has a point or boundary point belonging to $D_1CD_2+I_J$. The set $\alpha_2^{(1)}$ consists of $R_1^{(1)}+R_2^{(1)}$ and a finite set of regions covering all points of $D_1CD_2+I_J$ not in $R_1^{(1)}+R_2^{(1)}$ such that no region of $\alpha_2^{(1)}-R_1^{(1)}-R_2^{(1)}$ has a point or boundary point in common with a region of $\alpha_1^{(1)}-R_1^{(1)}-R_2^{(1)}$. Evidently regions of the kind described exist so that $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ can be obtained so as to satisfy the requirements of Axiom 8. Let $R_1^{(2)}$ and $R_2^{(2)}$ be regions which, with their boundaries, lie in $R_1^{(1)}$ and $R_2^{(1)}$ respectively and contain the points $D_1$ and $D_2$ respectively. The set $\alpha_1^{(2)}-R_1^{(2)}-R_2^{(2)}$ is a finite set of regions which cover all points of $D_1PD_2$ not in $R_1^{(2)}+R_2^{(2)}$ and which, with their boundaries, are subsets of $\alpha_1^{(1)}$. The set $\alpha_2^{(2)}-R_1^{(2)}-R_2^{(2)}$ is a finite set of regions which cover all points of $D_1CD_2$ not in $R_1^{(2)}+R_2^{(2)}$ and which, with their boundaries are subsets of $\alpha_2^{(1)}$. The sets $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ evidently exist in satisfaction of the requirements of Axiom 8. From this point proceed precisely as in Theorem 1 and obtain the corresponding sequence of regions, $H_1, H_2, H_3, \ldots$, whose common part is $I_J+J$. By the preceding theorem, $I_J$ contains a point $F$ not in $\alpha_1^{(1)}+\alpha_2^{(1)}$. But $\alpha_1^{(1)}+\alpha_2^{(1)}$ contains $I_J$. Thus, we are led to a contradiction by the assumption that any point $P$ of $J$ is not a limit point of $I_J$.

**Corollary.** *No region is a subset of an arc.*

This follows easily by the proof given by Moore of Theorem 23. The interior of a simple closed curve is to be interpreted to be an $I_J$-set of points.

**Theorem 4.** *Every point of a simple closed curve $J$ is a limit point of $E_J$.*

Proof. Let $P$ be any point of $J$ and $R_P$ any region containing $P$. Then $R_P$ contains an arc $APB$ of $J$. The curve $J$ contains at least one point $Q$ which is a limit point of $E_J$. By Therem 1, there is a region $H$ containing $J$. There is a region $R_Q$ containing $Q$ such that $R_Q$ lies in $H$ and contains a point $E$ of $E_J$. By Axiom 7, there exists in $H$ an arc $EP$ containing no point of arc $(J-\overline{APB})$ such that $P_1$ is the first point that $EP$ has on $\overline{APB}$. $P_1$ is a limit point of $E_J$ and lies in $R_P$. Then any region containing $P$ contains a point of $J$ which is a limit point of $E_J$. Hence $P$ is a limit point of the subset of $J$ consisting of limit points of $E_J$. Then, $P$ is a limit point of $E_J$.

**Theorem 5.** *The set $I_J$ defined by any simple closed curve $J$ is connected.*

Proof. Let $Q$ be a point of $I_J$ and $D_Q$, the maximal connected subset of $I_J$ containing $Q$. There is an arc $QE$ joining $Q$ to a point $E$ of $E_J$ having $D_1$ as the first point after $Q$ on $J$. Let $R_{D_1}$ be a region containing $D_1$ such that $Q+E$ does not belong to $R'_{D_1}$. There exists an arc $QE$ in $S-R'_{D_1}$ having $D_2$ as the first point after $Q$ on $J$. Then $D_1$ and $D_2$ are limit points of $D_Q$. Let $N$ be the subset of $J$ containing all points of $J$ which are limit points of $D_Q$. Suppose that a point $P$ of $J$ is not a limit point of $D_Q$, and let $C$ be a point on $J$ such that $P$ and $C$ separate $D_1$ and $D_2$ on $J$. As in Theorem 3, there exist two points $A$ and $B$ on $J$ such that $A$ and $B$ are limit points of $D_Q$ and the arc $ACB$ contains $N$. There is an arc $XYZ$ such that $\underline{XYZ}$ belongs to $D_Q$ and $X$ and $Z$ are points on $J$ in $R_A$ and $R_B$, regions about $A$ and $B$ respectively, such that $R'_A+R'_B$ does not contain $P$, and $R'_A$ and $R'_B$ have no common points. Under the assumption that $P$ is not a limit point of $D_Q$, no point of $APB$ not in $R_A+R_B$ is a limit point of $D_Q$. Let $T_1$ and $V_1$ be points of $R_A$, and $T_2$ and $V_2$ points of $R_B$ such that these points lie on $J_1$ (the simple closed curve $PXYZP$) in the order $PT_1XV_1YV_2ZT_2$, and $T_1XV_1$ and $T_2ZV_2$ are subsets of $R_A$ and $R_B$ respectively. $\underline{XCZ}$ belongs to $E_{J_1}$ (Theorem 4). Let $S_1$ denote the set of points of $I_{J_1}$ not in $R_A+R_B$. Then each point of $S_1$ belongs to one and only one of the following classes.

(1) A point $P_1$ of $S_1$ belongs to $C_1$ if there exists an arc lying in $I_{J_1}$ except for one end-point and joining $P_1$ to a point of $T_1PT_2$ and a similar arc joining $P_1$ to a point of $V_1YV_2$.

(2) A point $P_2$ of $S_1$ belongs to $C_2$ if there exists an arc lying in $I_{J_1}$ except for one end-point and joining $P_2$ to a point of $T_1PT_2$, but there is no similar arc joining $P_2$ to a point of $V_1YV_2$.

(3) A point $P_3$ of $S_1$ belongs to $C_3$ if there exists an arc lying in $I_{J_1}$ except for one end-point and joining $P_3$ to a point of $V_1YV_2$, but there is no similar arc joining $P_3$ to a point of $T_1PT_2$.



(4) All other points of $S_1$ belong to $C_4$.

No point of $C_i$ is a limit point of $C_j$ $(i\neq j)$. It is to be noted that the boundary points in $I_{J_1}$ of $R_A$ and of $R_B$ belong to one of these four classes.

*Suppose that $C_1$ is vacuous.* Form $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ of Axiom 8 as follows: Let $R_A$ and $R_B$ be $R_1^{(1)}$ and $R_2^{(1)}$ respectively. (In this case, and in several instances which follow, $R_1^{(i)}$ and $R_2^{(i)}$ are common to $\alpha_1^{(i)}$ and $\alpha_2^{(i)}$). Since every point of $I_{J_1}$ can be joined to some point of $J$ by an arc lying in $I_{J_1}$ except for one end-point, every point of $C_4$ can be joined to some point on the boundary of $R_A$ or $R_B$ by an arc lying in $I_{J_1}$. $C_4$ may be vacuous. All regions used in the following are regions of the fundamental seqence of regions of Axiom 1. Cover each point of $T_1PT_2$ not in $R_A+R_B$ with a region having no point or boundary point in common with $C_3+XYZ$. Cover each point of $C_2+C_4$ with a region that lies in $I_{J_1}$ and has no point or boundary point in common with $C_3$. Then, these regions, together with $R_A$ and $R_B$, cover the closed set $XPZ+C_2+C_4$. A finite subset, including $R_A+R_B$ will cover the same set of points. Denote this finite set of regions by $\alpha_1^{(1)}$. The set $\alpha_1^{(1)}$ is connected. Cover each point of $XYZ$ not in $R_A+R_B$ and each point of $C_3$ with a region having no point or boundary point in common with $XPZ$ or with any region of $\alpha_1^{(1)}-R_A-R_B$. A finite subset of these regions, together with $R_A+R_B$, will cover the closed set $C_3+XYZ$. Denote this finite set, which includes $R_A+R_B$, by $\alpha_2^{(1)}$. Keeping in mind that $R_A$ and $R_B$ take the place of $R_1^{(1)}$ and $R_2^{(1)}$ respectively, it is evident that $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ satisfy the requirements of Axiom 8. Form $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ as follows, using only regions of subscript greater than 2. The regions $R_1^{(2)}$ and $R_2^{(2)}$ are regions about $X$ and $Z$ respectively such that $R_1^{(2)}$ and $R_2^{(2)}$, with their boundaries, are subsets of $R_A$ and $R_B$ respectively. $\alpha_1^{(2)}-R_1^{(2)}-R_2^{(2)}$ consists of a finite number of suitably chosen regions covering points of $XPZ$ not in $R_1^{(2)}+R_2^{(2)}$ and such that these regions with their boundaries lie in $\alpha_1^{(1)}$. The set $\alpha_2^{(2)}-R_1^{(2)}-R_2^{(2)}$ consists of a finite set of regions covering points of $XYZ$ not in $R_1^{(2)}+R_2^{(2)}$. These regions evidently exist in such wise that $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ satisfy the requirements of Axiom 8. From this point proceed as in Theorem 1 and obtain the sequence of regions, $H_1, H_2, H_3, \dots$ whose common part is $I_{J_1}+J_1$. But, as proved previously, $I_{J_1}$ contains a point $F$ not in $\alpha_1^{(1)}+\alpha_2^{(1)}$. But $\alpha_1^{(1)}+\alpha_2^{(1)}$ contained $I_{J_1}$. This contradiction shows that $C_1$ is not vacuous. Then, there is an arc [1]) in $I_{J_1}$ joining a point of $T_1PT_2$ to a point of $V_1YV_2$. ($I_{J_1}$ is obviously a subset of $I_J$). This contradicts the fact that $APB$ contains no point which is a limit point of $D_Q$. Hence $P$ is a limit point of $D_Q$. Then, every point of $J$ is a limit point of $D_Q$.

Let $P$ and $Q$ be any two points of $I_J$. There exist $D_P$ and $D_Q$ as defined above. Suppose that there is no arc in $I_J$ joining a point of $D_P$ to a point of $D_Q$. Since every point of $J$ is a limit point of $D_P$ and of $D_Q$, there are two arcs, $P_1P_2P_3$ and $Q_1Q_2Q_3$, such that $P_1, P_3, Q_1, Q_3$ are distinct and lie on $J$, $\underline{P_1P_2P_3}$ and $\underline{Q_1Q_2Q_3}$ belong to $D_P$ and $D_Q$ respectively and, under the assumption, have no point in common. The points $P_1, P_3, Q_1, Q_3$, together with two other points $A$ and $B$, may obviously be so chosen that they lie on $J$ in the order $AP_1Q_1BQ_3P_3$. Let $J_1$ denote the closed curve $P_1P_2P_3Q_3Q_2Q_1P_1$. The set $P_1AP_3 + Q_1BQ_3$ is exterior to $I_{J_1} + J_1$, since $I_{J_1} + J_1$ is a subset of $I_J + J$ and $A$ and $B$ are limit points of $E_J$. The set $I_{J_1}$ contains a point $L$. There exists $D_L$ of $I_{J_1}$ and every point of $J_1$ is a limit point of $D_L$. Then, there is an arc $L_1L_2L_3$ such that $L_1$ and $L_3$ lie on $\underline{P_1P_2P_3}$ and $\underline{Q_1Q_2Q_3}$ respectively, and $\underline{L_1L_2L_3}$ is a subset of $I_{J_1}$ and, consequently, of $I_J$. Hence we have an arc in $I_J$ joining a point of $D_P$ to a point of $D_Q$ contrary to the assumption made above. But $P$ and $Q$ were any points of $I_J$. Then $I_J$ is connected, in fact, connected in the strong sense.

**Theorem 6.** *If $R$ is a region and $ABC$ an arc of a closed curve $J$ such that $A$ and $C$ are on the boundary of $R$ and $\underline{ABC}$ is a subset of $R$, and if $M$ is the set of all points that can be joined to a point of $\underline{ABC}$ by an arc whose every point, except an end-point on $ABC$, is common to $I_J$ and $R$, then $M$ is a connected set of points.*

Proof. Let $P$ and $Q$ be any two points of $M$ and $PP_1$ and $QQ_1$ two arcs common to $I_J$ and $R$ except for end-points $P_1$ and $Q_1$ which belong to $\underline{ABC}$. If $\underline{PP_1}$ and $\underline{QQ_1}$ have a common point, $P$ and $Q$ being any two points of $M$, the theorem is proved. Suppose that $\underline{PP_1}$ and $\underline{QQ_1}$ have no common points and that $P_1$ is distinct from $Q_1$. If $P_1 \equiv Q_1$, the proof follows, with suitable modifications, from that given below. Since $I_J$ is connected, there is an arc $PXQ$ such that $PXQ$ is a subset of $I_J$. A subset of $PP_1 + PXQ + QQ_1$ is an arc $P_1YQ_1$ such that $\underline{P_1YQ_1}$ belongs to $I_J$. Suppose that $\underline{P_1YQ_1}$ contains at least one point of $S - R'$, and let $T_1$ and $T_2$ be the first and last points respectively that $P_1YQ_1$ has in common with the boundary of $R$. (If $P_1YQ_1$ contains no point of $S - R'$, but does contain points of the boundary of $R$, the theorem follows immediately, provided such an arc can be obtained in connection with any two points $P$ and $Q$ of $M$). Let $B_1$ be a point of $ABC$ between $P_1$ and $Q_1$. Let $J_1$ denote the closed

[1]) This arc lies in $I_{J_1}$ except for end-points.



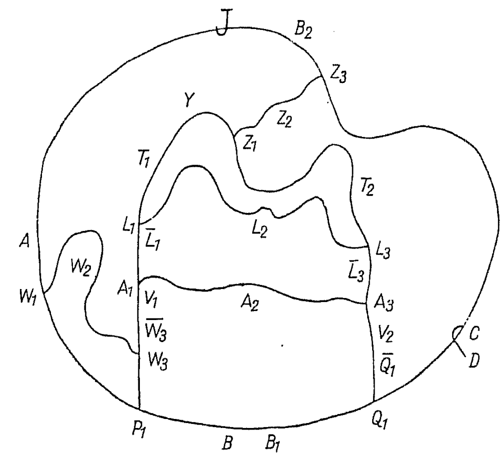


Fig. 1.

curve $P_1B_1Q_1YP_1$. The set $I_{J_1}$ is a subset of $I_J$. There exists an arc $L_1L_2L_3$ such that $L_1$ and $L_3$ lie on $J_1$ in the order $P_1L_1T_1YT_2L_3Q_1$ and $\underline{L_1L_2L_3}$ is a subset of $I_{J_1}$. Suppose that $L_1L_2L_3$ contains a point of $S - R'$ (see remark above). Let $J_2$ denote the closed curve $P_1B_1Q_1L_3L_2L_1P_1$. Then, $I_{J_2}$ is a subset of $I_{J_1}$ and, consequently, of $I_J$. Let $V_1$ and $V_2$ be two points on $J_2$ in the order $P_1V_1L_1L_2L_3V_2Q_1$. Let $J_3$ denote the simple closed curve $P_1T_1YT_2Q_1B_2AP_1$, where $B_2$ is a point on $J$ which with $B_1$ separates $A$ and $C$ on $J$. Then, $I_J = I_{J_1} + I_{J_3} + \underline{P_1T_1YT_2Q_1}$ (Moore, Theorem 25. The proof of this theorem follows without modification if the interior of a simple closed curve is now interpreted to be an $I_J$-set of points). There exists an arc $W_1W_2W_3$ such that $\underline{W_1W_2W_3}$ is a subset of $I_{J_3}$ and $W_1$ and $W_3$ lie on $J_3$ in the order

$AW_1P_1W_3V_1$. Let $J_4$ denote the closed curve $W_1W_2W_3 + W_3P_1$ (on $P_1V_1T_1$) $+ P_1W_1$ (on $ABC$). There exists a region $R_C$ about $C$ such that $R'_C$ contains no point of $J_1 + J_4$. There is in $R_C$ an arc $CD$ joining $C$ to a point $D$ of $E_J$. There is a number $\delta$ such that $D$ lies in $S - H'_\delta$, where $H_\delta$ is one of the regions of the sequence whose common part is $I_J + J$. There exists an arc $Z_1Z_2Z_3$ such that $Z_1$ lies on $T_1YT_2$, $Z_3$ lies on $Q_1CB_2$ between $B_2$ and $C$ and $Z_1Z_2Z_3$ lies in $I_{J_3}$ and contains no point of $W_1W_2W_3$.

Consider $J_2$. Every boundary point of $R$ lying in $I_{J_2}$ is a limit point of the set of points of $I_{J_2}$ in $S - R'$. Every point of $I_{J_2}$ in $S - R'$ can be joined to some point of $L_1L_2L_3$ by an arc lying in $I_{J_2}$ and $S - R'$ (except for an end-point on $L_1L_2L_3$). Then $L_1L_2L_3$ together with the boundary points of $R$ lying in $I_{J_2}$ and the points of $I_{J_2}$ belonging to $S - R'$ form a closed connected set. Now form $\alpha_1$ and $\alpha_2$ of Axiom 8, using only regions that lie in $I_J$ and contain no point or boundary point in common with $J_4 + CD + P_1W_3V_1 + P_1B_1Q_1V_2$. The regions $R_1$ and $R_2$ are to be suitably chosen regions containing $L_1$ and $L_2$ respectively. The set $\alpha_1 - R_1 - R_2$ is a finite set of regions covering points of $L_1L_2L_3$ not in $R_1 + R_2$ and all points of the boundary of $R$ that belong to $I_{J_2}$ and all points of $I_{J_2}$ lying in $S - R'$ such that no region of the set has a point or boundary point on $L_1T_1YT_2L_3$. The set $\alpha_2 - R_1 - R_2$ is a finite set of regions covering points of $L_1T_1YT_2L_3$ not in $R_1 + R_2$ such that no region of the set has a point or boundary point in common with a region of $\alpha_1 - R_1 - R_2$. Such sets of regions evidently exist so that $\alpha_1$ and $\alpha_2$ satisfy Axiom 8. By this axiom, there is a region $H$ containing $\alpha_1 + \alpha_2$ with a boundary which is a subset of the boundary of $\alpha_1 + \alpha_2$. The simple closed curve $J_4$ evidently lies in $S - H'$, since for each point of $J_4$ there is an arc joining this point to $D$ and containing no point on the boundary of $H$. In a similar manner there may be obtained a region $H_1$ containing $J_4 + P_1B_1Q_1$ such that $H_1$ lies in $H_\delta$, $Z_1Z_2Z_3 + V_1L_1T_1 + V_2L_3T_2 + CD + BZ_3C + H'$ lies in $S - H'_1$, and $H'_1$ lies in $S - H'$. By the reasoning employed in Theorem 5, it can be shown that there exists an arc $A_1A_2A_3$ such that $A_1$ and $A_3$ lie on $\underline{W_3V_1L_1}$ and $\underline{Q_1V_2L_3}$ respectively and $\underline{A_1A_2A_3}$ is a subset of the common part of $I_{J_2}$ and $R$. (In this case, however, the points of the classes $C_1$, $C_2$, $C_3$, $C_4$ are subsets of the set of points of $I_{J_2}$ not $H + H_1$. The arcs $\overline{L_1}V_1\overline{W_3}$ and $\overline{L_3}V_2\overline{Q_1}$ respectively correspond to the arcs $T_1PT_2$ and $V_1YV_2$ mentioned in Theorem 5, where $\overline{L_1}$ and $\overline{W_3}$ are points of $\underline{L_1V_1W_3}$, $\overline{L_3}$ and $\overline{Q_1}$ are points of $\underline{Q_1V_2L_3}$, and the arcs $\overline{L_1}L_1L_2L_3\overline{L_3}$ and $\overline{W_3}W_3P_1B_1Q_1\overline{Q_1}$ are subsets of $H$ and $H_1$ respectively. The arcs used in defining $C_i$ ($i = 1, 2, 3, 4$) are to lie in the common part of $I_{J_2}$ and $R$ with the exception of a single end-point in each instance). But $A_1$ can be joined to $P$ and $A_2$ to $Q$ by arcs lying in $M$. Then, $P$ can be joined to $Q$ by an arc lying in $M$. Therefore, $M$ is connected.



**Theorem 7.** *The boundary of every region is a simple closed curve.*

Proof. Let $C$ and $D$ be any two points of the boundary of a region $R$. By Axioms 2, 3, 6, there exists a simple closed curve $CY_1DY_2C$ such that the arcs $CY_1D$ and $CY_2D$ belong to $R$ and $S - R'$ except for the points $C$ and $D$. Let $\overline{J}$ denote the simple closed curve $CY_1DY_2C$. Then, $I_{\overline{J}}$ contains a non-vacuous subset $M$ of the boundary of $R$. Suppose that $M$ is not connected. Then, $M = M_1 + M_2$, where $M_1$ and $M_2$ are two sets such that neither set contains a limit point of the other set. Let $P_1$ be a point of $M_1$. There exists an arc $Z_1P_1W_1$ such that $Z_1$ and $W_1$ belong to $\underline{CY_1D}$ and $\underline{CY_2D}$ respectively, $\underline{Z_1P_1W_1}$ is a subset of $I_{\overline{J}}$ and $P_1$ is the only point of $M$ on $Z_1P_1W_1$. Let $\overline{J_1}$ and $\overline{J_2}$ denote the simple closed curves $Z_1P_1W_1CZ_1$ and $Z_1P_1W_1DZ_1$ respectively. Then, $I_{\overline{J}} = I_{\overline{J}_1} + I_{\overline{J}_2} + \underline{Z_1P_1W_1}$ (Moore, Theorem 25). There exists a point $P_2$ of $M_2$ in either $I_{\overline{J}_1}$ or $I_{\overline{J}_2}$. Suppose that $P_2$ belongs to $I_{\overline{J}_2}$. There is a region $\overline{R}$ about $P_2$ such that $\overline{R}'$ lies in $I_{\overline{J}_2}$. By Axiom 6 there is an arc $A_2P_2B_2$ in $\overline{R}$ such that $A_2$ and $B_2$ belong to $R$ and $S - R'$ respectively and $P_2$ is the only point that this arc has on the boundary of $R$. There exist arcs $Y_1A_2$ in $R$ and $Y_2B_2$ in $S - R'$ such that $Z_2$ and $W_2$ respectively are the first points that these arcs have on $\overline{J}_2$. A subset of $Y_1A_2 + A_2P_2B_2 + Y_2B_2$ is an arc $Z_2P_2W_2$ such that $\underline{Z_2P_2W_2}$ is a subset of $I_{\overline{J}_2}$. Suppose that $Z_2$ and $W_2$ lie on $\overline{J}$ in the order $Z_1Z_2DW_2W_1$. (If one or both of the points $Z_2$ and $W_2$ lie on $Z_1P_1W_1$, the following argument still holds). Let $A$ and $B$ be two points of $\overline{J}$ in the order $Z_1AZ_2DW_2BW_1$. Let $J$ denote the simple closed curve $Z_1AZ_2P_2W_2BW_1P_1Z_1$. The set $I_J$ contains a subset of $M$. Suppose that $\overline{M_1}$ and $\overline{M_2}$ respectively are non-vacuous subsets of $M_1$ and $M_2$ in $I_J$ ($I_J$ is a subset of $I_{\overline{J}}$). If one of these sets is vacuous, it will be obvious that the desired result still obtains. In the following proof the sets $\overline{Z_1 C W_1}$ and $\overline{Z_2 D W_2}$ belonging to $\overline{J}$ are to be disregarded in naming arcs. Thus, $\overline{AP_2B}$ means the arc $AP_2B$ of $J$. There exists a region $R_B$ containing $B$ and no point of $R'$. In $R_B$ there is an arc $BE$ joining $B$ to a point $E$ of $E_J$ and having $B_1$ as the last point on $J$. There is a number $\delta$ such that $E$ lies in $S - H'_\delta$ where $H_\delta$ is a region of the sequence of regions whose common part is $I_J + J$. There is a region $R_{P_1}$

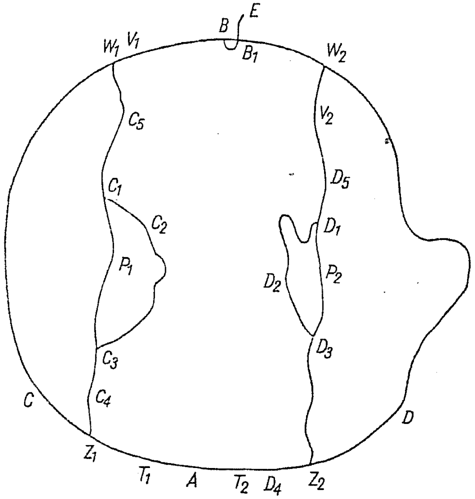


Fig. 2.

about $P_1$ such that $R'_{P_1}$ lies in $H_\delta$ and contains no point of $B_1E + \overline{M_2} + AP_2B$. As an almost immediate consequence of Theorem 6, it can be shown that there is an arc $C_1C_2C_3$ such that $\overline{C_1C_2C_3}$ is common to $R_{P_1}$ and $I_J$, and $C_1$ and $C_3$ lie on $J$ in the order $C_1P_1C_3A$. Let $J_1$ denote the simple closed curve $C_1C_2C_3P_1C_1$. No point of $\overline{M_2}$ lies in $I_{J_1}$ since $I_{J_1}$ is a subset of $R_{P_1}$. Cover each point and boundary point of the subset of $M_1$ not in $I_{J_1}$ ($P_1$ is not a limit point of this set) by a region which, with its boundary, lies in $I_J$ and has no point or boundary point in common with $M_2$. A finite subset of these regions, $[K]$, will cover the given closed set. Consider any region $K_1$ of $[K]$. Such a region has a point $Q_1$ in $R$.



There is in $R$ an arc joining $Q_1$ to $C_3$ having $\overline{Q_1}$ as a first point on the set composed of the arcs $C_1C_2C_3$ and $P_1AP_2$. There exist arcs of this type for each region of $[K]$. Since $[K]$ is a finite set, all of the end-points $\overline{Q_i}$ lie on $C_1C_2C_3$ or $P_1C_3D$, where $D$ is a point on $P_1AP_2$ between $C_3$ and $P_2$. Form $\alpha_1$ and $\alpha_2$ of Axiom 8 as follows: $R_1$ and $R_2$ are regions about $C_1$ and $P_1$ respectively. The set of regions $\alpha_1 - R_1 - R_2$ is a finite set covering all points of arc $P_1C_1$ (of $P_1C_1B_1$) not in $R_1 + R_2$. No region of $\alpha_1 - R_1 - R_2$ is to have a point or boundary point on $P_1AP_2B_1C_1 + B_1E + C_1C_2C_3$. The set $\alpha_2 - R_1 - R_2$ consists of $[K]$ and a finite set of regions covering all points of $C_1C_2C_3 + P_1C_3D +$ all arcs $Q_i\overline{Q_i}$ not in $[K] + R_1 + R_2$. All regions used in obtaining $\alpha_1 + \alpha_2$ are to lie in $H_\delta$ and to have no point or boundary point in common with $B_1E + \overline{M_2} + P_2B_1$ (of $P_2B_1P_1$). Obviously, there exist regions as indicated which form $\alpha_1$ and $\alpha_2$ in satisfaction of the requirements of Axiom 8. According to the same axiom, there is a region containing $\alpha_1 + \alpha_2$ whose boundary is a subset of the boundary of $\alpha_1 + \alpha_2$. Denote this region by $R_{P_1}$. There exists a region $R_{P_2}$ lying in $H_\delta$ and containing $P_2$ such that $R'_{P_2}$ contains no point of $P_1C_1B_1E + \overline{R'_{P_1}}$. As before, there is an arc $D_1D_2D_3$ such that $\overline{D_1D_2D_3}$ is common to $I_J$ and $R_{P_2}$, and $D_1$ and $D_3$ lie on $J$ in the order $D_3P_2D_1B_1$. Cover each point of $\overline{M_2}$ with a region which, together with its boundary, lies in $I_J$ and has no point or boundary point in common with any region of $\alpha_1 + \alpha_2$. Cover $P_2$ with a region belonging to $H_\delta$ and containing no point or boundary point in common with $\overline{R'_{P_1}} + P_1C_1B_1E + D_1D_2D_3$. A finite subset, $G$, of all these regions, including the region about $P_2$, will cover the closed set $\overline{M_2} + P_2$. If any region of $G$ contains a point of $\overline{R_{P_1}}$, then this region lies wholly in $\overline{R_{P_1}}$. Let $G_1$ be the set of all regions of $G$ that lie in the exterior of $\overline{R_{P_1}}$. The set $G_1$ is not vacuous, as the region used to cover $P_2$ belongs to $G_1$. Let $\overline{G_1}$ be the set of all regions of $G_1$ which, together with their boundaries, form with $D_1D_2D_3$ a connected set. If there is any region of $\overline{G_1}$ that has only a set of boundary points in common with other regions of $\overline{G_1}$ and no point or boundary point on $D_1D_2D_3$, add to $\overline{G_1}$ a finite set of regions covering these boundary points such that each region of this additional set lies in $I_J$ and has no point or boundary point in common with $\overline{R}'_{P_1}+J$ or with any region of $G_1$ not in $\overline{G_1}$. These additional regions evidently exist in finite number covering the set of boundary points mentioned, since this set of boundary points is closed. Denote by $[\overline{K}]$ the augmented set of regions which includes $\overline{G_1}$. Form $\overline{\alpha}_1$ and $\overline{\alpha}_2$ as follows, using only regions that lie in $H_\delta$ and have no point or boundary point in common with $\overline{R}'_{P_1}+P_1C_1B_1E$ or with any region of $G_1$ not in $\overline{G_1}$. Let $\overline{R}_1$ be the region of $G_1$ containing $P_2$, $\overline{R}_2$, a region containing $D_3$, $\overline{\alpha}_1-\overline{R}_1-\overline{R}_2$, a finite set of regions containing $[\overline{K}]$ and all points of $P_2D_1D_2D_3$ not in $[\overline{K}]+\overline{P}_1+\overline{R}_2$, $\overline{\alpha}_2-\overline{R}_1-\overline{R}_2$, a finite set of regions covering all points of $P_2D_3$ (of $P_2D_3A$) not in $\overline{R}_1+\overline{R}_2$. Obviously, $\overline{\alpha}_1$ and $\overline{\alpha}_2$, consisting of regions of this character and satisfying the requirements of Axiom 8, can be obtained. By Axiom 8, there is a region containing $\overline{\alpha}_1+\overline{\alpha}_2$ whose boundary is a subset of the boundary of $\overline{\alpha}_1+\alpha_2$. Let $\overline{R}_{P_2}$ denote this region. Every point on the boundary of $\overline{R}_{P_1}$ can be joined to $E$ by an arc containing no point of the boundary of $\overline{R}_{P_2}$. Then, the boundary of $\overline{R}_{P_1}$ lies in $S-\overline{R}'_{P_2}$. If $\overline{R}_{P_1}$ had a point $P_3$ in common with $\overline{R}_{P_2}$, an arc in $\overline{R}_{P_2}$ joining $P_3$ to $P_2$ would contain a point on the boundary of $\overline{R}_{P_1}$. Thus, it can be shown that $\overline{R}'_{P_1}$ and $\overline{R}'_{P_2}$ are mutually exclusive. Let $G_2$ be the subset of $G$ in the common exterior of $\overline{R}_{P_1}$ and $\overline{R}_{P_2}$. Let $C_4$ and $C_5$ respectively be the last points that the arcs $P_1AP_2$ and $P_1BP_2$ have in common with the boundary of $\overline{R}_{P_1}$. Let $R_1^{(1)}\equiv R_1$ (see above), $R_2^{(1)}\equiv R_2$, $\alpha_1^{(1)}-R_1^{(1)}-R_2^{(1)}\equiv\alpha_1-R_1-R_2$. The boundary points of $\overline{R}_{P_1}$ that belong to $I_J+J$ form a closed bounded set. Hence, there exists a finite set of regions covering this set of boundary points such that no region of this set has a point or boundary point in common with $\overline{R}'_{P_2}+(G_2)'+J_1+P_2D_1B_1E+(\alpha_1-R_1-R_2)'$. Let $R_\beta$ be a region of this finite set that has no point or boundary point on $C_5C_1C_2C_3C_4$. The region $R_\beta$ contains a point $Q_\beta$ of $\overline{R}_{P_1}$. There is in $\overline{R}_{P_1}$ an arc joining $Q_\beta$ to $C$. This arc has a first point $\overline{Q}_\beta$ on $C_5C_1C_2C_3C_4$. Cover each point of every such arc $Q_\beta\overline{Q}_\beta$ not in $R_\beta$ by a region which, with its boundary, lies in $R_{P_1}$ and contains no point or boundary point in common with $\alpha_1-R_1-R_2$. A finite subset, including the regions $R_\beta$ will cover the arcs $Q_\beta\overline{Q}_\beta$.



Let $[\overline{R}]$ denote this finite set of regions. The set $\alpha_2^{(1)}-R_1^{(1)}-R_2^{(1)}$ is to consist of $\alpha_2-R_1-R_2+[\overline{R}]+$ a finite set of regions covering all points of $C_5C_1C_2C_3C_4$ not in $\overline{R}_{P_1}+[\overline{R}]$. This last-mentioned finite set of regions will contain only regions that have no point or boundary point in common with $\overline{R}_{P_2}+G_2$ or with the regions of $\alpha_1^{(1)}-R_1^{(1)}-R_2^{(1)}$. The set $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$, as specified above, can evidently be obtained so as to meet the requirements of Axiom 8. Let $R_{P_1}^{(1)}$ denote the corresponding region. $R_{P_1}^{(1)}$ contains $\overline{R}_{P_1}$ and lies in $S-\overline{R}'_{P_2}$. By a similar process, we can obtain a region $R_{P_2}^{(1)}$ which contains $\overline{R}_{P_2}$ and all boundary points of $\overline{R}_{P_2}$ in $I_J+J$ and all points of $D_4D_3D_2D_1D_5$ not in $\overline{R}_{P_2}$, where $D_4$ and $D_5$ are the last points that arcs $P_2AP_1$ and $P_2BP_1$ have on the boundary of $\overline{R}_{P_2}$. Furthermore, the additional regions used in obtaining $R_{P_2}^{(1)}$ have no point or boundary point in common with $R_{P_1}^{(1)}+G_2$, and every point or boundary point of one of the regions, $R_{P_1}^{(1)}$, $R_{P_2}^{(1)}$, can be joined to $E$ by an arc lying in the exterior of the other region. $G_3$ is now the subset of $G_2$ in the common exterior of $R_{P_1}^{(1)}$ and $R_{P_2}^{(1)}$. Let $T_1$, $V_1$, $T_2$, $V_2$ be points of $J$ in the order $V_1C_5P_1C_4T_1T_2D_4P_2V_2$ such that the arcs $T_1P_1V_1$ and $T_2P_2V_2$ belong to $R_{P_1}^{(1)}$ and $R_{P_2}^{(1)}$ respectively. Evidently the arcs $T_1T_2$ (of $P_1AP_2$) and $V_1V_2$ (of $P_1BP_2$) are in the common exterior of $\overline{R}_{P_1}$ and $\overline{R}_{P_2}$. By the reasoning employed in Theorem 5, which in this case would involve a classification of the points of $I_J$ not in $R_{P_1}^{(1)}+R_{P_2}^{(1)}$, we can obtain an arc $L_1^{(1)}L_2^{(1)}L_3^{(1)}$ such that $\underline{L_1^{(1)}L_2^{(1)}L_3^{(1)}}$ lies in $I_J$ and contains no point of $\overline{R}_{P_1}+\overline{R}_{P_2}$, and $L_1^{(1)}$ and $L_3^{(1)}$ lie on $T_1T_2$ (of $P_1AP_2$) and $V_1V_2$ (of $P_1BP_2$) respectively. If $L_1^{(1)}L_2^{(1)}L_3^{(1)}$ contains no point or boundary point of a region of $G_2$ then we have an arc from a point $L_1^{(1)}$ of $R$ to a point $L_3^{(1)}$ of $S-R'$ which contains no point of the boundary of $R$. This contradiction would exhibit the falsity of the assumption that the subset of the boundary of $R$ which lies in $I_J$ is not connected. Suppose, however, that $L_1^{(1)}L_2^{(1)}L_3^{(1)}$ contains points or boundary points belonging to regions of $G_2$. Denote by $J_3$ the simple closed curve $C_1P_1L_1^{(1)}L_2^{(1)}L_3^{(1)}C_1$. The point $C_2$ belongs to $I_{J_3}$ (Moore, Theorem 25). Suppose that $R_\alpha$ is a region of $G_2$ containing a point or boundary point on $L_1^{(1)}L_2^{(1)}L_3^{(1)}$. In the first case, $R_\alpha$ contains a point $Q_\alpha$ of $I_{J_3}$. In the second case, there is a region $\overline{R}_\alpha$ containing each boundary point of the region $R_\alpha$ which belongs to

$L_1^{(1)} L_2^{(1)} L_3^{(1)}$ such that $\overline{R}_\alpha$ lies in $I_J$ and contains no point of $\overline{R}'_{P_2}$. In either case, there is a point $Q_\alpha$ of $I_{J_3}$ belonging to $R_\alpha$ or $\overline{R}_\alpha$ as the case may be. In $I_{J_3}$ there is an arc $Q_\alpha C_2$ with $\overline{Q}_\alpha$ as the first point on $C_1 C_2 C_3$. Since the set of boundary points of the regions of $G_2$ lying on $L_1^{(1)} L_2^{(1)} L_3^{(1)}$ is closed, there exists a finite set of regions of the type of $\overline{R}_\alpha$ covering all boundary points of regions of $G_2$ that lie on $L_1^{(1)} L_2^{(1)} L_3^{(1)}$. Let $X$ be the set of all regions of $G_2$ that have a point or boundary point on $L_1^{(1)} L_2^{(1)} L_3^{(1)}$ and all regions (finite in number) of the type of $\overline{R}_\alpha$ described above. Let $\overline{X}$ be the set of regions of $G_2$ not contained in $X$ that have a point or boundary point in common with a region of $X$. If a region of $\overline{X}$ has only boundary points in common with the region of $X$, the set of all such boundary points is closed. There exists a finite set of regions covering these boundary points such that each region is in $I_J$ and contains no point or boundary point in common with $\overline{R}_{P_2}$ or any region of $G_2$ not in $X + \overline{X}$. Let $X_1$ be the set of regions consisting of these additional regions together with $X + \overline{X}$. Form $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ as follows: $R_1^{(2)} \equiv R_1^{(1)}$, $R_2^{(2)} \equiv R_2^{(1)}$, $\alpha_1^{(2)} - R_1^{(2)} - R_2^{(2)} \equiv \equiv \alpha_1^{(1)} - R_1^{(1)} - R_2^{(1)}$.

The set $\alpha_2^{(2)} - R_1^{(2)} - R_2^{(2)}$ consists of $\alpha_2^{(1)} - R_1^{(1)} - R_2^{(1)}$, $X_1$ and a finite set of regions lying in $I_{J_3}$ together with their boundaries and covering all points not in $\alpha_2^{(1)} + X_1$ of arcs of the type of $Q_\alpha \overline{Q}_\alpha$. Evidently $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ exist in such wise as to satisfy Axiom 8. Denote by $R_{P_1}^{(2)}$ the corresponding region postulated by the axiom. The region $R_{P_1}^{(2)}$ has no point or boundary point in common with $\overline{R}_{P_2}$. Proceeding as before, with the help of two additional regions containing $R_{P_1}^{(2)}$ and $\overline{R}_{P_2}$ respectively, we obtain an arc $L_1^{(2)} L_2^{(2)} L_3^{(2)}$ such that $\underline{L_1^{(2)} L_2^{(2)} L_3^{(2)}}$ lies in $I_J$ and contains no point of $R_{P_1}^{(2)} + \overline{R}_{P_2}$, and $L_1^{(2)}$ and $L_3^{(2)}$ belong to $P_1 A P_2$ and $P_1 B P_2$ respectively. This process may be continued, but will end after a finite number of steps as the number of regions in $G$ is finite. There is obtained finally an arc $L_1^{(n)} L_2^{(n)} L_3^{(n)}$ joining a point $L_1^{(n)}$ of $P_1 A P_2$ to a point $L_3^{(n)}$ of $P_1 B P_2$ and containing no point of the boundary of $R$. If one of the sets, $\overline{M}_1$ or $\overline{M}_2$, is vacuous, obviously the preceding argument may be modified so as to lead to a contradiction in this case. Hence $M$, the subset of the boundary of $R$ in $I_{\overline{J}}$ is connected. The interior of any simple closed curve related to $R$ as



$J$ is related to $R$ contains a connected subset of the boundary of $R$. The same type of reasoning will show that both $P_1$ and $P_2$ are limit points of the connected set $\overline{M}$, where $\overline{M}$ is the subset of the boundary of $R$ in $I_J$. The set $P_1 + \overline{M} + P_2$ is a closed bounded connected set of points. It can be easily proved that the omission of any point of $\overline{M}$ disconnects the set. Hence $P_1 + \overline{M} + P_2$ is an arc from $P_1$ to $P_2$.

Let $P_1 P_3 P_2$ be the arc which is the subset of the boundary of $R$ in $I_J + J$. Denote by $N$ the subset of the boundary of $R$ which lies in $E_J$. If $Q$ be any point of $N$, by Axioms 2, 3 and 6, there is an arc $F Q K$ such that $F$ and $K$ belong to $P_1 A P_2$ and $P_1 P_3 P_2$ respectively and $\underline{QF}$ and $\underline{QK}$ lie in $R$ and $S - R'$ respectively. By Theorem 27, Moore, which now holds, either $P_2$ lies in the exterior of the simple closed curve $FQKP_1F$ or $P_1$ lies in the exterior of the simple closed curve $FQKP_2F$. If the first alternative is true, it can be shown, by an argument which makes use of the type of reasoning used before, that the subset of $N$ within $FQKP_1F$ consists of points of an arc joining $P_1$ to $Q$. Then for any point $Q$ of $N$, there is an arc joining $Q$ to $P_1$ or $P_2$ and consisting of points of $N$ except for the points $P_1$ or $P_2$.

(1) Suppose that there exists no arc joining a point $Q$ of $N$ to $P_2$ and consisting only of points of the boundary of $R$ and let $\overline{Q}$ be any fixed point of $N$. Let $\overline{F}\,\overline{Q}\,\overline{K} P_1 \overline{F}$, denoted by $\overline{J}$, be a simple closed curve of the nature of $FQKP_1F$ described above. Then, $N$ consists of $P_1 \overline{Q} - P_1$ together with the subset of $N$ in $E_{\overline{J}}$. Let $Q_\delta$ be a point of $N$ in $E_{\overline{J}}$. There exists the corresponding simple closed curve $F_\delta Q_\delta K_\delta P_1 F_\delta$, denoted by $J_\delta$, where $F_\delta$ lies on $P_1 A P_2$ and belongs to $E_{\overline{J}}$. The set $I_{J_\delta}$ contains a subset of $N$ consisting of points of an arc joining $P_1$ to $Q_\delta$. Evidently $\overline{Q}$ belongs to this arc. Thus every point of $N$ in $E_{\overline{J}}$ can be joined to $\overline{Q}$ by an arc consisting of points of $N$. Then $P_1 + N$ is connected. If $P_2$ is a limit point of $P_1 + N$, then $P_1 + N + P_2$ is a closed bounded connected set of points. It can easily be shown that this set is disconnected by the omission of any point other than $P_1$ or $P_2$. Under the circumstances, $P_1 + N + P_2$ is an arc from $P_1$ to $P_2$ and the boundary of $R$ is proved to be a simple closed curve. Suppose, however, in addition to our first assumption, we assume that $P_2$ is not a limit

point of $P_1+N$. Then, $P_1+N$ is closed. Keep $\overline{J}$ fixed. Then, there exists a simple closed curve $J_{\overline{Q}}$ ($J_\delta$ of the preceding discussion) such that $\overline{Q}$ is a subset of $I_{J_{\overline{Q}}}$. If $Q_\mu$ is any point of $N$ in $E_{\overline{J}}$ a simple closed curve $J_{Q_\mu}$ of like character can be found such that $Q_\mu$ is a subset of $I_{J_{Q_\mu}}$. Assign to each $Q_\mu$ a definite simple closed curve of this character. Cover $\overline{Q}$ with a region lying in $I_{J_{\overline{Q}}}$. Cover each point $Q_\mu$ in $E_{\overline{J}}$ with a region lying in the corresponding $I_{J_{Q_\mu}}$. The set consisting of $\overline{Q}$ and the subset of $N$ in $E_{\overline{J}}$ is closed. Then a finite subset of the regions just mentioned will cover the same closed set. But, corresponding to each region of the finite set of

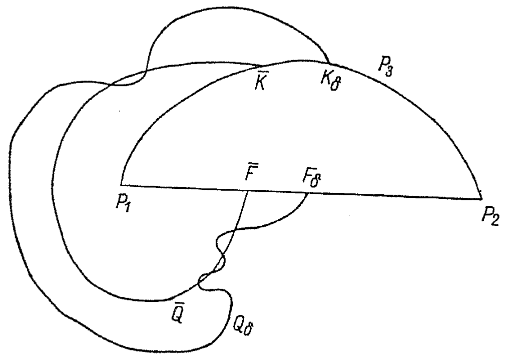


Fig. 3.

regions there is an open set $I_{J_{Q_\mu}}$ containing the region. Then $\overline{Q}+$ $+\{Q_\mu\}$ is covered by a finite number of the open sets $I_{J_{Q_\mu}}$. Let $[O]$ denote this finite set of open sets. Then,

$$[O]\equiv\{I_{J_{Q_\mu^{(1)}}},\ I_{J_{Q_\mu^{(2)}}},\ldots,\ I_{J_{Q_\mu^{(n)}}}\}.$$

By the preceding discussion, $\overline{Q}$ is contained in each $I_{J_{Q_\mu^{(i)}}}$. Let $\overline{Q}_\mu^{(i)}$ be the single point of $N$ on $J_{Q_\mu^{(i)}}$. If there is no point $\overline{Q}_\mu^{(i)}$ which lies in $E_{J_{Q_\mu^{(1)}}}$, then, there is an arc $P_1\,\overline{Q}_\mu^{(1)}$ which contains all points of $N$, and $\overline{Q}_\mu^{(1)}$ does not belong to any set of $[O]$, which fact furnishes a contradiction. Suppose, however, that $\overline{Q}_\mu^{(\beta)}$ is the first point of $\{\overline{Q}_\mu^{(i)}\}$ which lies in $E_{J_{Q_\mu^{(1)}}}$. Then, all points of $N$ in $I_{J_{Q_\mu^{(\beta)}}}$ are



points of an arc joining $P_1$ to the point $\overline{Q}_\mu^{(\beta)}$. If there are no points $\overline{Q}_\mu^{(i)}$ in $E_{J_{Q_\mu^{(\beta)}}}$, we reach a contradiction as before. As the points of $\{\overline{Q}_\mu^{(i)}\}$ are finite in number, we must reach a final contradiction. Thus, if no point of $N$ can be joined to $P_2$ by an arc consisting of points of $N$ except for $P_2$, the point $P_2$ must be a limit point of $N$ and the boundary of $R$ is a simple closed curve. If $P_1$ takes the place of $P_2$ in the preceding argument, we arrive at the same result.

(2) Assume that there are points of $N$ that can be joined to $P_1$ and points of $N$ that can be joined to $P_2$ by arcs which lie in $N$ except for $P_1$ or $P_2$, as the case may be. Let $N_{P_1}$ and $N_{P_2}$ respectively denote these subsets of $N$. If $N_{P_1}$ and $N_{P_2}$ have a common point, or if one set contains a limit point of the other, then, $P_1+$ $+N+P_2$ is a closed bounded connected set which is disconnected by the omission of one point other than $P_1$ or $P_2$. In this case, the boundary of $R$ is shown to be a simple closed curve. If $P_1+N_{P_1}$ and $P_2+N_{P_2}$ are mutually exclusive closed sets, then, by considering $P_1+N_{P_1}$, we are led to a contradiction as in (1). Therefore, the boundary of any region $R$ is a simple closed curve.

**Theorem 8.** (Moore, Theorem 28). *If $P$ is a point of a simple closed curve $J$ and $R_P$, a region about $P$, there exists a simple continuous arc $AXB$ such that* (1) *$A$ and $B$ are on $J$,* (2) *$AXB$ is common to $R_P$ and $I_J$ ($E_J$),* (3) *of the two arcs into which $A$ and $B$ divide $J$, that one which contains $P$ lies in $R_P$.*

Proof. That there exists an arc $A_1X_1B_1$ of the character described such that $A_1X_1B_1$ is common to $I_J$ and $R_P$ follows immediately from Theorem 6. Let $J_{R_P}$ be the simple closed curve which is the boundary of $R_P$ (Theorem 7). There exists an arc $CA_1PB_1D$ of $J$ such that $CA_1PB_1D$ is a subset of $R_P$ and $C$ and $D$ lie on $J$. The points $C$ and $D$ divide $J_{R_P}$ into two arcs $CT_1D$ and $CT_2D$. Let $J_1$ and $J_2$ respectively denote the simple closed curves $CT_1DPC$ and $CT_2DPC$. If $J_3$ is the simple closed curve $A_1X_1B_1PA_1$, then $I_{J_3}$ is a subset of $I_{J_1}$ or $I_{J_2}$ (Moore, Theorem 25). Suppose that $I_{J_3}$ is a subset of $I_{J_1}$. Let $\overline{R}_P$ be a region containing $P$ such that $\overline{R}_P'$ is a subset of $R_P$ and contains no point of $A_1X_1B_1+J-A_1PB_1$. By the first part of the proof, there is an arc of the character described in the theorem relative to $J_2$ and $\overline{R}_P$. If $A_2X_2B_2$ denotes this arc, then $A_2X_2B_2$ is common to $I_{J_2}$ and $\overline{R}_P$. The set $A_2X_2B_2$ cannot belong to $I_J$. For, if $\underline{A_2X_2B_2}$ belonged to $I_J$, it would be a subset of $I_{J_3}$, and $I_{J_3}$ belongs to $I_{J_1}$. Hence $A_2X_2B_2$ is an arc of the character described in the theorem and $\underline{A_2X_2B_2}$ belongs to $E_J$ and $R_P$.

**Theorem 9.** *A simple closed curve is accessible from all sides at every point.*

This theorem can be proved without difficulty by means of Theorem 8 and the reasoning employed by Moore, page 148.

**Remark.** All of the fifty-two theorems of the paper by Moore, not already proved in the preceding, can now be demonstrated precisely as proved by Moore except that the interior of a simple closed curve is to be interpreted as an $I_J$-set of points. The results of the preceding theorems can now be expressed in the following fundamental theorem.

**Theorem 10.** *If $J$ is a simple closed curve, there exist two sets, $I_J$ and $E_J$, such that* (1) *$J$ is the common boundary of $I_J$ and $E_J$,* (2) *any two points of $I_J$ ($E_J$) can be joined by a simple continuous arc lying wholly on $I_J$ ($E_J$),* (3) *$I_J$ is a bounded, and $E_J$, an unbounded, set and* (4) *$J$ is accessible from all sides at every point.*

**Theorem 11.** *Every simple closed curve is the boundary of a set of points that has all the properties of a region.*

Proof. The proof of this theorem consists in showing that all of the axioms hold if a region is interpreted to be an $I_J$-set of points. Theorems 1, 7, 10 show that Axioms 1, 2, 3, 4, 6, 7 are satisfied. Axiom 5, of course, holds.

In Axiom 8, let $I_J$-sets of points replace regions. Denote by $I_{J_1}$, $I_{J_2}$, $\bar{\alpha}_1$, $\bar{\alpha}_2$ the $I_J$-sets of points corresponding to $R_1$, $R_2$, $\alpha_1$, $\alpha_2$ respectively. By Moore, Theorem 41, there exists $I_{J_\mu}$, an $I_J$-set of points corresponding to the region $H$ of Axiom 8. There exists a simple closed curve $\bar{J}$ such that $\bar{J}$ consists of two arcs $P_1T_1P_2$ and $P_1T_2P_2$ where $P_1$, $P_2$, $P_1T_1P_2$, $P_1T_2P_2$ belong to $I_{J_1}$, $I_{J_2}$, $\bar{\alpha}_1$, $\bar{\alpha}_2$ respectively. The set $I_{\bar{J}}$ is a subset of $I_\mu$. There exists a region $H_1$ containing $J_1$ such that $H_1'$ has no point in common with $I_{J_2}+J_2$. The boundary $J_{H_1}$ of $H_1$ is a simple closed curve. $J_{H_1}$ contains no point common to $\bar{\alpha}_1$ and $\bar{\alpha}_2$. It can be shown that there exists an arc $AB$ belonging to $J_{H_1}$ such that $\underline{AB}$ is a subset of $I_{\bar{J}}$ and $A$ and $B$ lie on $\bar{J}$ and belong to $\bar{\alpha}_1 - \overline{I_{J_1}} - I_{J_2}$ and $\bar{\alpha}_2 - I_{J_1} - I_{J_2}$ respectively. The arc $AB$ may be identified with the set $M$ of Axiom 8.



**Theorem 12.** *Every set of points satisfying Axioms I constitutes a space homeomorphic with the Euclidean plane.*

In a paper [1]), „*Concerning a set of postulates for plane analysis situs*", R. L. Moore has proved that every set of points satisfying his $\Sigma_1$ system of axioms is homeomorphic with the Euclidean plane. It can be shown that all the axioms of this set not included in our system are satisfied by a set of points satisfying Axioms I. Furthermore, Axiom 8 holds in the plane.

**Axioms II.** This set of axioms differs from the preceding set in that Axioms 1 and 2 are replaced by the following axioms, which are Axioms 1′ and 2′ of Moore's set $\Sigma_3$.

**Axiom 1′.** *If $P$ is a point, there exists an infinite sequence of regions, $R_1$, $R_2$, $R_3, \ldots,$ such that* (1) *$P$ is the only point that they have in common,* (2) *for every $n$, $R_{n+1}$ is a proper subset of $R_n$,* (3) *if $R$ is a region about $P$, then there exists $n$ such that $R_n'$ is a subset of $R$.*

**Axiom 2′.** *Every two points of a region are the extremities of at least one simple continuous arc that lies wholly in that region.*

**Theorem 10′** (identical with Theorem 10).

Proof. (This proof is based upon Axioms II). Evidently there exist two finite sets of regions, $\bar{\alpha}_1$ and $\bar{\alpha}_2$, covering all points of $J$ and satisfying the requirements of Axiom 8. Let $\bar{G}$ denote $\bar{\alpha}_1+\bar{\alpha}_2$. In accordance with this axiom, there exists a region $\bar{H}$ such that $\bar{H}$ contains $\bar{\alpha}_1+\bar{\alpha}_2$ and the boundary of $\bar{H}$ is a subset of the boundary of $\bar{\alpha}_1+\bar{\alpha}_2$. *All other finite sets of regions similar to $\bar{\alpha}_1$ and $\bar{\alpha}_2$ used hereafter to cover $J$ are to lie with their boundaries in $\bar{G}$.*

(1) By Axiom 1′, there exists a definite sequence of regions for each point of $J$ possessing the properties indicated in the axiom. Cover each point $P$ of $J$ with a region selected from the $P$-sequence of regions postulated by Axiom 1′ after the method of Theorem 1, and obtain the finite sets of regions $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ in accordance with Axiom 8 such that $\alpha_i^{(1)}$ is a subset of $\bar{\alpha}_i$. Let $G_1$ denote $\alpha_1^{(1)}+\alpha_2^{(1)}$. By Axiom 8, there exists a region $H_1$ corresponding to $G_1$. The region $H_1$ and its boundary is a subset of $\bar{H}$. Next cover each point $P$ of $J$ with a region of the $P$-sequence of Axiom 1′ of subscript greater than 2, such that each region with

[1]) Trans. Amer. Math. Soc. 20, 1919.

its boundary lies in $G_1$ and the finite subset of these regions which covers $J$ is composed of two sets $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ in accordance with Axiom 8. The set $\alpha_i^{(2)}$ is to be a subset of $\alpha_i^{(1)}$ as in Theorem 1. Let $G_2$ denote $\alpha_1^{(2)}+\alpha_2^{(2)}$. There is a corresponding region $H_2$ as postulated by Axiom 8. Proceeding in this manner, there is obtained an infinite sequence of regions $H_1, H_2, H_3, \ldots$, each containing $J$, such that $H_i'$ is a subset of $H_{i-1}$, and $H_i'$ is a subset of the fundamental region $\overline{H}$ for every value of $i$. Let $H_\omega$ denote the common part of $H_1, H_2, H_3, \ldots$. The set $H_\omega$ is closed. Precisely as in Theorem 2, it can be shown that $H_\omega$ contains a point $F$ not in $\overline{G}$.

(2) Let $[H_\omega]$ denote the totality of all such sets $H_\omega$ which may be obtained in the manner just indicated. Each $H_\omega$ contains $J$. Hence there is a set of points $\widetilde{H}_\omega$ common to all the sets of $[H_\omega]$. A point $P$ belongs to $\widetilde{H}_\omega$ if and only if there is no set $H_\omega^{(\alpha)}$ of $[H_\omega]$ to which $P$ does not belong. Since $\widetilde{H}_\omega$ is a subset of the fundamental region $\overline{H}$, the set $\widetilde{H}_\omega$ is bounded. Suppose that $Q$, a point of $\widetilde{H}_\omega$ not on $J$, is a boundary point of $\widetilde{H}_\omega$. There exists a region $R_Q$ about $Q$ such that $R_Q'$ contains no point of $J$, and $R_Q$ has within it a point $Q_1$ of $\widetilde{H}_\omega$ and a point $Q_2$ of $S-\widetilde{H}_\omega$. Since $Q_2$ lies in $S-\widetilde{H}_\omega$, there exists a set $H_\omega^{(\beta)}$ of $[H_\omega]$ to which $Q_2$ does not belong. There is a region $H_\delta^{(\beta)}$ of the sequence of regions of which $H_\omega^{(\beta)}$ is the common part such that $Q_2$ belongs to the exterior of $H_\delta^{(\beta)}$. In the exterior of $H_\delta^{(\beta)}$ there is an arc $Q_2E$ joining $Q_2$ to a point $E$ of $S-\overline{H}'$. ($H_\delta^{(\beta)}$ lies in $\overline{H}$). The set $Q_2E+R_Q'$ contains no point of $J$. There exists a set $H_\omega^{(\mu)}$ of $[H_\omega]$ such that $G_1^{(\mu)}$, which consists of the first $\alpha_1$ and $\alpha_2$ used in obtaining $H_\omega^{(\mu)}$, and the boundary of $G_1^{(\mu)}$ contains no point of $Q_2E+R_Q'$. The corresponding region $H_1^{(\mu)}$ according to Axiom 8, has a boundary which is a subset of the boundary of $G_1^{(\mu)}$. Then, each point of $Q_2E+R_Q'$ can be joined to $E$ by an arc which contains no point of the boundary of $H_1^{(\mu)}$. The point $E$ lies in the exterior of $\overline{H}$ and, therefore, in the exterior of $H_1^{(\mu)}$. Hence $Q_1$ does not belong to $H_\omega^{(\mu)}$ and, consequently, not to $\widetilde{H}_\omega$. This contradiction shows that no point of $\widetilde{H}_\omega$ not on $J$ can belong to the boundary of $\widetilde{H}_\omega$. Then, the boundary of $\widetilde{H}_\omega$ is a subset of $J$. As in Theorem 4, it can be shown that every point of $J$ is a boundary point of $\widetilde{H}_\omega$. Since the boundary of $\widetilde{H}_\omega$ belongs to $\widetilde{H}_\omega$, the set $\widetilde{H}_\omega$ is closed.



The set $S-\widetilde{H}_\omega$ is connected. For, let $P_1$ and $P_2$ be two points of $S-\widetilde{H}_\omega$. There exists at least one set $H_\omega^{(\alpha)}$ of $[H_\omega]$ to which $P_1$ does not belong. Hence $P_1$ is exterior to a region $H_\delta^{(\alpha)}$, where $H_\delta^{(\alpha)}$ is a region of the sequence of regions whose common part is $H_\omega^{(\alpha)}$. There is in the exterior of $H_\delta^{(\alpha)}$ an arc $P_1E$ joining $P_1$ to a point $E$ of $S-\overline{H}'$. The arc $P_1E$ contains no point of $J$, which, of course, belongs to $H_\delta^{(\alpha)}$. Similarly, there exists an arc $P_2E$ containing no point of $J$. A subset of $P_1E+P_2E$ is an arc $P_1P_2$ containing no point of $J$, the boundary of $\widetilde{H}_\omega$. Then, any two points of $S-\widetilde{H}_\omega$ can be joined by an arc lying wholly in $S-\widetilde{H}_\omega$.

(3) As stated above, every set $H_\omega$ of $[H_\omega]$ contains at least one point $F$ not belonging to $\overline{G}$. Let $[F]$ be the set of all such points. Since every point of $[F]$ belongs to $\overline{H}$, the set $[F]$ is bounded. No point of $J$, which belongs to $\overline{G}$, can be a limit point of $[F]$. Suppose that no point of $[F]$ belongs to $\widetilde{H}_\omega$. Cover each point $Q$ of $[F]$ and its boundary with a region $R_Q$ such that $R_Q'$ contains no point of $J$. The simple closed curve $J$ is the boundary of $\widetilde{H}_\omega$. Then, $R_Q'$ lies in $S-\widetilde{H}_\omega$. A finite subset of the regions covering $[F]$ and its boundary will cover the same closed set. Since $S-\widetilde{H}_\omega$ is connected, there exists in $S-\widetilde{H}_\omega$ a finite set of arcs joining points of this finite set of regions (one point selected for each region) to a point $E$ of $S-\overline{H}'$. These arcs, together with the finite set of regions and their boundaries, constitute a closed set $N$ lying in $S-\widetilde{H}_\omega$. There exists a set $H_\omega^{(\lambda)}$ of $[H_\omega]$ such that $G_1^{(\lambda)}$ (consisting of the first $\alpha_1$ and $\alpha_2$ used in obtaining $H_\omega^{(\lambda)}$) and the boundary of $G_1^{(\lambda)}$ contain no point of $N$. Then, the corresponding region $H_1^{(\lambda)}$ contains no point of $N$. This follows by an argument previously used. Then, $[F]$ lies in the exterior of $H_1^{(\lambda)}$ and, therefore, in the exterior of $H_\omega^{(\lambda)}$. But $H_\omega^{(\lambda)}$ contains at least one point of $[F]$. This contradiction shows that $[F]$ cannot lie wholly in $S-\widetilde{H}_\omega$. Let $I$ denote $\widetilde{H}_\omega - J$. Then $I_J$ is not vacuous. That $I_J$ is unique and bounded follows as in Theorem 2. Theorems 3—10 can now be proved precisely as in the first part of this paper.

**Theorem 11'** (identical with Theorem 11).

The proof of this theorem, based upon Axioms II, is precisely the proof given for Theorem 11.

The University of Pennsylvania.